Chromaticity of Sylow Graph for the Cyclic Groups

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Abstract— *In our work, we will find the new results on the chromatic uniqueness of Sylow graphs of cyclic group* $\boldsymbol{C_n}$, $\boldsymbol{G_{Syl}C_n}$. *Moreover, we will determine the results about the chromatic number and chromatic polynomial of these graphs. We will discuss all the cases of* $\boldsymbol{n}$, *where* $\boldsymbol{n}$ *is the prime number. Moreover, we prove that if* $\boldsymbol{n = p^{\alpha}}$, $\boldsymbol{n \neq 2^2}$, *and* $\boldsymbol{n = \prod p_i}$, *then* $\boldsymbol{G_{Syl}C_n}$ *is not* $\boldsymbol{\chi - unique}$, *where* $\boldsymbol{|p_i| \geq 2}$ *and* $\boldsymbol{n \neq 2 \times p}$, *otherwise it is* $\boldsymbol{\chi - unique}$.

Keywords*— *Chromatic number; Chromatic polynomial; Chromaticity; Sylow Graph.

I. INTRODUCTION

Duringe than a century that the Four-colocolorlem was discussed, many ideas were proposed that led to solvisolvingis known problem [8]. In 1912, Birkhoff [4] defined a mapping $P(M, \lambda)$ that gives the appropriate number of $\lambda -$ colourings of M. In 1932, the author Whitney [23] is the one who expanded the study of chromatic polynomial (simplly ChP), he transferred the study from maps to graphs. Also he put some basic results for it. In 1968, Read [21] asked two questions:

i- What are the sufficient and necessary conditions for two graphs to share the same ChP; that is, to be chromatically equivalent?

ii- What are the necessary and sufficient conditions for the graph to be chi- unique?

Particularly, Chao and Whitehead Jr.[7] in 1978 explain the chromatically unique of a graph if the graph does not share its ChP with any other graph. Chromaticity problem of graph means the study and discuss the two questions above. There are many authors worked in this area. Dong, Koh, and Teo [11] published a useful book on ChPs and chromaticity of graphs. We refer to the survey papers [10], [9], [19], [18], [21], and [23]. The literature of algebraic graph theory has grown extremely since 1974, many papers have appeared in this area next. Some researchers studied the colouring of the graph of ring, one of them the famous author, I. Beck. He introduced an idea of a zero-divisor graph in 1988 [3]. While he was mainly interested in colorings. In 2013, Sahib and Khalaf [12] introduce many solutions on the chromatic uniqueness of some algebraic graph like the zero-divisor graph $\Pi(z_n)$, in cases, if n is even, $n = p^2$ or $n = p_1 \times p_2$. The chromaticity of $\Pi(z_n)$ is still an open problem for the other cases of n. In 2022, Al-Janabi and Gabor[2] solved this problem for all cases of n. Other researchers studied the properties of the graph for group. We will show about some of them. In 1975-1987, Cavior and Calhoun respectively [5] and [6], compute the subgroups of the dihedral group D_{2n}. In 2013, Ma presented a definition for the cyclic graph of the Dihedral group and Dicyclic group, where the elements of G are the vertices of $\Gamma(G)$, and two different vertices x and y are connected by one edge iff $< x, y >$ is a cyclic subgroup of G [23]. In 2022, Ohood and Hayder presented three types of graphs, which are called Normal, Cyclic and Sylow graph, denoted by G_{Nor}, $G_{Cyc}$and G_{syl} respectively [13], [14], [15], [16], and [17]. In our work, we will discuss the new results about the chromatic uniqueness of Sylow graphs of Cyclic group C_n, $G_{syl}C_n$.

II. Auxiliary RESULTS

- Theorem 1 [11] If the graph G contains K_n as subgraph, then $\chi(G) \geq n$.
- Theorem 2 [22] A group G is said to be cyclic if $\exists a \in G$, such that for every element of G is generated by a.
- Definition 3 [22] A Sylow P-subgroup of a group G is a subgroup of G and p is a prime number, that is a p-group or equivalently, the order of every group element is a power of p, and written as $Syl_p(G)$.
- Definition 4 [16] The Sylow graph of the group G, where G is finite is denoted by G_{Syl}, it is whose two sets, the set $E(G_{Syl}) = \{(a, b) | \langle a, b \rangle \leq G\}$ is an edge set and $V(G_{Syl}) = \{a | \ a \in G\}$ is a vertex set.
- Proposition 5 [16] The matrix degree sequence of sylow graph of the cyclic group $G_{syl}C_n$, where n is positive integer number is given by the following:

1. If $n = p$:

$$\mathcal{M}(G_{syl}C_n) = \begin{pmatrix} p-1 \\ p \end{pmatrix} \cong K_p$$

If $n = p^\alpha$:

$$\mathcal{M}(G_{Syl}C_n) = \begin{pmatrix} p^\alpha - 1 & \varphi(p^\alpha) \\ \varphi(p^\alpha) & p^{\alpha-1} \end{pmatrix}$$

If $n = \prod p_i$:

$$\mathcal{M}(G_{Syl}C_n) = \begin{pmatrix} \sum \varphi(p_i) & \varphi(p_1) & \dots & \varphi(p_j) & 0 \\ 1 & \varphi(p_1) & \dots & \varphi(p_j) & n - (\sum \varphi(p_i) + 1) \end{pmatrix}$$

- Definition 6 [1] We have K_m be a complete-graph of order m, and the order of this graph. Let L_k be the subset of M, for every $k \leq t$ and $t \in N$. The t-clique-join graph (simply, t-CJ-graph) is obtained from t arbitrary graphs $Z_1, Z_2, \dots, Z_t$ on pairwise disjoint vertex sets by joining every vertex in Z_k, where $\forall v \in L_k$, and $Z_k \cap M = \emptyset$. It is defined as $F = F(M, L_1, L_2, \dots, L_t, Z_1, Z_2, \dots, Z_t)$, where $|Z_k| = a_k.\mathbf{I} = \mathbf{I}(M, f_1, f_2, \dots, f_t, Z_1, Z_2 \quad, \dots, Z_t)$ is the set of all such graphs with $|L_1| = f_1, |L_2| = f_2, \dots, |L_t| = f_t$. For example $F \in \mathbf{I}$.
- Corollary 7 [1] We assume that the empty graph it is Z_k such that $|Z_k| = a_k$, where L_k be the clique subgraphs and $|L_k| = f_k$, $L_k \subset M$, $1 \leq k \leq 2$ and $f_1 \neq f_2$. For $G \in \mathbf{I}(M, f_1, f_2, Z_1, Z_2)$ be a 2-CJ-graph, where $|M| = m$, then ChP of G is:

$$P(G, \lambda) = \lambda(\lambda-1)\dots(\lambda - f_1)^{a_1+1}\dots(\lambda - f_2)^{a_2+1}\dots(\lambda - m + 1).$$

- Theorem 8 [1] Suppose that R is the integral-root ChP of a connected graph J, and R has one root only with exponent 2 and there are no roots with the exponent, then J is $\chi -$ unique.
- Theorem 9 [1] If the ChP of an integral-root connected graph F has at least 2 roots with exponent 2, then F is not $\chi -$ unique.
- Theorem 10 [1] Let G be a connected graph with an integral-root ChP P, such that P contains some roots with exponent at least 3, then G it is not $\chi -$ unique.
- Fact [11] Always a complete-graph K_n, and the empty graph $\overline{K_n}$ are $\chi -$ unique.
- Corollary 12 [24] For the graph H has the blocks N_k, $1 \leq k \leq r$, such that, these blocks share the same clique W, then ChP of H is:

$$P(H, \lambda) = \frac{\prod_{k=1}^{r} P(N_k, \lambda)}{s(\lambda)^{r-1}}$$

III. PREPARATION FOR THE MAIN RESULT

Notation 13 We will denote the sylow graph of the cyclic group C_n by $G_{syl}C_n$.

A. *Chromatic Number of Sylow Graph*

- **Lemma 14** We have $G_{syl}C_n$, where $n = p^\alpha$, then there is, $K_{\varphi(p^\alpha)}$, a complete subgraph of $G_{syl}C_n$.

Proof. Let $C_n = \{e, a, a^2, \dots, a^{n-1}\}$, by Proposition 5, then

$$\mathcal{M}(G_{Syl}C_n) = \begin{pmatrix} p^\alpha - 1 & \varphi(p^\alpha) \\ \varphi(p^\alpha) & p^{\alpha-1} \end{pmatrix}$$

Since $\varphi(p^\alpha) = p^\alpha - p^{\alpha-1}$, then we get:

$$\mathcal{M}(G_{Syl}C_n) = \begin{pmatrix} p^\alpha - 1 & p^\alpha - p^{\alpha-1} \\ p^\alpha - p^{\alpha-1} & p^{\alpha-1} \end{pmatrix}$$

So

$$\mathcal{M}(G_{Syl}C_n \backslash p^{\alpha-1}) = \begin{pmatrix} p^\alpha - p^{\alpha-1} - 1 \\ p^\alpha - p^{\alpha-1} \end{pmatrix} = V(K_{p^\alpha - p^{\alpha-1}})$$

$$= V(K_{\varphi(p^\alpha)})$$

- **Lemma 15** We have $G_{syl}C_n$, and $n = \prod p_i$, then there is, K_m, a complete subgraph of $G_{Syl}C_n$, where $m = \max\{p_i\}, i = 1,2, \dots, j$.

Proof. Let $C_n = \{e, a, a^2, \dots, a^{n-1}\}$, by (Proposition 5), we can get:

$$\mathcal{M}(G_{Syl}C_n) = \begin{pmatrix} \sum \varphi(p_i) & \varphi(p_1) & \dots & \varphi(p_j) & 0 \\ 1 & \varphi(p_1) & \dots & \varphi(p_j) & n - (\sum \varphi(p_i) + 1) \end{pmatrix}$$

Since the vertex e is adjacent to $p_1, p_2, \dots, p_j$, such that p_i is a prime number, $i = 1,2, \dots, j$, then $\varphi(p_j) = p_j - 1$, where $p_1 < p_2 < \dots < p_j$. Moreover, there is a set of isolated vertices denoted by S, such that $|S| = n - (\sum \varphi(p_i) + 1)$, then:

$$\mathcal{M}(G_{Syl}C_n \backslash p_1, \dots, p_{j-1}, S) = \begin{pmatrix} p_j - 1 & p_j - 1 \\ 1 & p_j - 1 \end{pmatrix} = \begin{pmatrix} p_j - 1 \\ p_j \end{pmatrix}$$

$$= V(K_{p_j})$$

- **Theorem 16** The chromatic number of the sylow graph of cyclic group, $G_{Syl}C_n$, are given in the following statements:

1. if $n = p$, then $\chi(G_{Syl}C_n) = n$.
2. If $n = p^\alpha$, then $\chi(G_{Syl}C_n) = p^\alpha - p^{\alpha-1} + 1$
3. If $n = \prod p_i$, then $\chi(G_{Syl}C_n) = m$, where $m = \max\{p_i\}, i = 1,2, \dots, j$

Proof.

1. Since $n = p$, so (by Proposition 5) $\mathcal{M}(G_{Syl}C_n) = \begin{pmatrix} p-1 \\ p \end{pmatrix} \cong K_p = K_n$, then (by Theorem 1) $\chi(K_n) = n$, so $\chi(G_{Syl}C_n) = n$.

If $n = p^\alpha$, (by Lemma 14) there is $K_{\varphi(p^\alpha)}$ a complete subgraph of $G_{Syl}C_n$, so (by Theorem 1) $\chi(G_{Syl}C_n) \geq p^\alpha - p^{\alpha-1}$. Since $a^i, i|p$, such that $i = 0,1,2,\dots,n-1$ are adjacent to all vertices of $K_{\varphi(p^\alpha)}$, and they are not adjacent between them, then these vertices $(a^i, i|p, i = 0,1,2,\dots,n-1)$ will take one colour different from the colours of $K_{\varphi(p^\alpha)}$, then the chromatic number of $G_{Syl}C_n$ is $\chi(G_{Syl}C_n) = p^\alpha - p^{\alpha-1} + 1$.

3. If $n = \prod p_i$, then (by Lemma 15) there is K_m a complete subgraph of $G_{Syl}C_n$, where $m = \max\{p_i\}, i = 1,2,\dots,j$, this means K_m is a maximal complete induced subgraph, so (by Theorem 1) $\chi(G_{Syl}C_n) \geq m$. Moreover, by (Proposition 5) we get:

$$\mathcal{M}(G_{Syl}C_n) = \begin{pmatrix} \sum \varphi(p_i) & \varphi(p_1) & \dots & \varphi(p_j) & 0 \\ 1 & \varphi(p_1) & \dots & \varphi(p_j) & n - (\sum \varphi(p_i) + 1) \end{pmatrix}$$

Then for each $p_i, i = 1,2,\dots,j$, there is K_{p_i} a complete subgraph of $G_{syl}C_n$, where p_i is a prime number, such that $V(K_{p_i}) = \{e, a^{i-1}\}, |\{e, a^{i-1}\}| = p_i, i = 1,2,\dots,j$, then the other complete subgraphs K_{p_i} of $G_{syl}C_n$ are adjacent to the vertex $e \in V(K_m)$, then they can take the same colouring of the other vertices of K_m excapt the vertix e. Since the graph $G_{Syl}C_n$ has isolated vertices (let S be the set of isolated vertices, this means $|S| = n - (\sum \varphi(p_i) + 1)$, then the number of colourings will not change, so $\chi(G_{Syl}C_n) = m$.

Example 17 We have the graph $G_{syl}C_8$, where $V(G_{syl}C_8) = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7\}$, and $n = 2^3$, there is (by Theorem 16) a complete subgraph K_4, then $\chi(G_{Syl}C_8) = 5$, see (Figure 1)

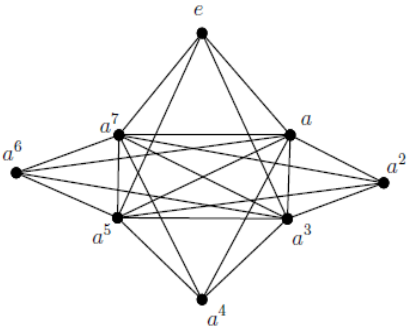

Figure 1: Sylow graph of C_8

- **Example 18** We have the graph $G_{syl}C_6$, where $V(G_{syl}C_6) = \{e, a, a^2, a^3, a^4, a^5\}$, such that $p_1 = 2$, $p_2 = 3$, there is a maximal complete subgraph K_3, so $\chi(G_{syl}C_6) = 3$, see (Figure 2).

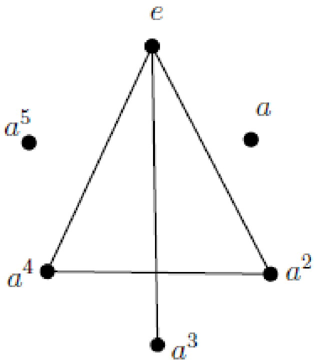

Figure 2: Sylow graph of C_6

B. ChP of Sylow Graph of $\boldsymbol{C_n}$

- **Theorem 19** We have the graph $G_{syl}C_n$, where $n = p$, then the ChP of $G_{syl}C_n$ is:

$$P(G_{syl}C_n, \lambda) = \lambda(\lambda - 1) \dots (\lambda - (n-1)) = P(K_n, \lambda).$$

Proof. By Proposition 5, we get the following matrix degree sequence:

$$\mathcal{M}(G_{syl}C_n) = \begin{pmatrix} p-1 \\ p \end{pmatrix} \cong V(K_p) = V(K_n)$$

Then we get the following:

$$P(K_n, \lambda) = \lambda(\lambda - 1) \dots (\lambda - (n-1))$$

- **Theorem 20** We have the graph $G_{syl}C_n$, where $n = p^\alpha$, then the ChP of $G_{syl}C_n$ is given by the following:

$$P(G_{syl}C_n, \lambda) = \lambda(\lambda - 1) \dots (\lambda - (\varphi(p^\alpha))^{|a^i|}, i|p, i = 0,1,2,\dots,n-1$$

Proof. By Lemma 14, there is $K_{\varphi(p^\alpha)}$ a complete subgraph of $G_{syl}C_n$, and

$$P(K_{\varphi(p^\alpha)}, \lambda) = \lambda(\lambda - 1) \dots (\lambda - (\varphi(p^\alpha) - 1)),$$

by Proposition 5, we get the following:

$$\mathcal{M}(G_{Syl}C_n) = \begin{pmatrix} p^\alpha - 1 & \varphi(p^\alpha) \\ \varphi(p^\alpha) & p^{\alpha-1} \end{pmatrix},$$

this means $\deg(a^i) = \varphi(p^\alpha), i|p, i = 0,1,2,\dots,n-1$, see (Figure 3), then $\forall a^i$ is adjacent to $K_{\varphi(p^\alpha)}$ and they are not adjacent between them, so we can choose the vertex e, it is adjacent to all vertices of $K_{\varphi(p^\alpha)}$, then

$$P(K_{\varphi(p^\alpha)} + e, \lambda) = \lambda(\lambda - 1) \dots (\lambda - (\varphi(p^\alpha))),$$

Now, we can do the same procedure for the other vertices $a^i, i|p, i = 0,1,2,\dots,n-1$, then (by corollary 7) we get:

$$P(G_{syl}C_n,\lambda) = \lambda(\lambda-1)\dots(\lambda-(\varphi(p^\alpha))^{|a^i|}, i|p, i = 0,1,2,\dots,n-1.$$

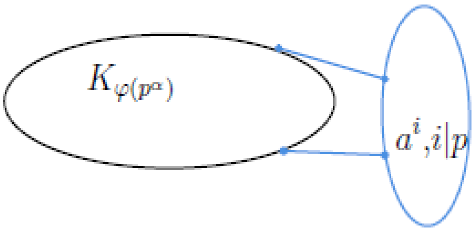

Figure 3: Sylow graph of C_n, $n = p^\alpha$

- **Example 21** Let $G = C_9 = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8\}$, this mean $n = 3^2$. Then (byTheorem 20) the polynomial of this graph is:

$$P(G_{syl}C_9,\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)^3$$

It is clear that in Figure 4, the vertices $e, a^3, a^6 \in G_{syl}C_9$ are adjacent to the complete subgraph K_6 of $G_{syl}C_9$, where $V(K_6) = \{a, a^2, a^4, a^5, a^7, a^8\}$. Moreover, the vertices $e, a^3, a^6 \in G_{syl}C_9$ are not adjacent between them, then we get $K_7 \subset G_{syl}C_9$, such that $V(K_7) = \{e, a, a^2, a^4, a^5, a^7, a^8\}$.

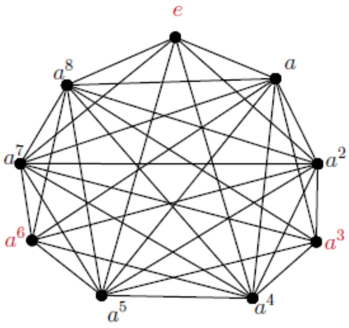

Figure 4: Sylow graph of C_9

- **Theorem 22** We have $G_{syl}C_n$, where $n = \prod p_i$, then the ChP of this graph is given by the following:

$$P(G_{syl}C_n,\lambda) = \lambda^{|S|+1}(\lambda-1)^{|p_i|}\dots(\lambda-(p_1-1))^{|p_i|}(\lambda-p_1)^{|p_i|-1}\dots(\lambda-(p_2-1))^{|p_i|-1}(\lambda-p_2)^{|p_i|-2}\dots(\lambda-(p_{j-1}-1))^2\dots(\lambda-(p_j-1)),$$

where S is the set of isolated vertices, such that $|S| = n - (\sum \varphi(p_i) + 1)$

Proof. By Lemma 15, there is K_m a complete subgraph of $G_{syl}C_n$, where $m = \max\{p_i\}, i = 1,2,\dots,j$, and

$$P(K_m,\lambda) = \lambda(\lambda-1)\dots(\lambda-(m-1))$$

By Proposition 5

$$\mathcal{M}(G_{Syl}C_n) = \begin{pmatrix} \sum_1 \varphi(p_i) & \varphi(p_1) & \dots & \varphi(p_j) & 0 \\ & \varphi(p_1) & \dots & \varphi(p_j) & n - (\sum \varphi(p_i) + 1) \end{pmatrix}$$

Then for each $p_i, i = 1,2,\dots,j$, and p_i is prime number, there is K_{p_i}, it is a complete subgraph of $G_{syl}C_n$ such that $V(K_{p_i}) = \{e, a^{i-1}\}, |\{e, a^{i-1}\}| = p_i, i = 1,2,\dots,j$, this means $\bigcap_{i=1}^{j} K_{p_i} = e$. Now, let the graph $H = \bigcup_{i=1}^{j} K_{p_i}$, then (by Corollary 12), then we get:

$$P(H,\lambda) = \frac{\lambda(\lambda-1)\dots(\lambda-(p_1-1))\lambda(\lambda-1)\dots\left(\lambda-(p_{j-1}-1)\right)\lambda(\lambda-1)\dots\left(\lambda-(p_j-1)\right)}{\lambda^{n-1}}$$

$$P(H,\lambda) = \lambda(\lambda-1)^{|p_i|}\dots(\lambda-(p_1-1))^{|p_i|}(\lambda-p_1)^{|p_i|-1}\dots(\lambda-(p_2-1))^{|p_i|-1}(\lambda-p_2)^{|p_i|-2}\dots(\lambda-(p_{j-1}-1))^2\dots(\lambda-(p_j-1)).$$

Since S is represents the set of isolated vertices, where $|S| = n - (\sum (p_i) + 1)$, and $P(S,\lambda) = \lambda^{|S|}$, and H is subgraph of the graph $G_{syl}C_n$, see (Figure 5) then the ChP of $G_{syl}C_n$ is:

$$P(G_{syl}C_n,\lambda) = \lambda^{|S|+1}(\lambda-1)^{|p_i|}\dots(\lambda-(p_1-1))^{|p_i|}(\lambda-p_1)^{|p_i|-1}\dots(\lambda-(p_2-1))^{|p_i|-1}(\lambda-p_2)^{|p_i|-2}\dots(\lambda-(p_{j-1}-1))^2\dots(\lambda-(p_j-1))$$

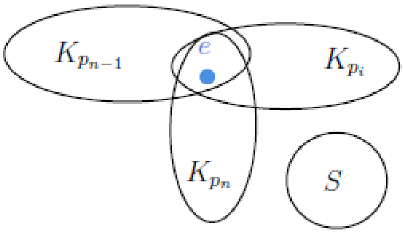

Figure 5: Sylow graph of C_n, $n = \prod p_i$

- **Example 23**

Let $G = C_{15} = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, a^{11}, a^{12}, a^{13}, a^{14}\}$,

where $p_1 = 5$ and $p_2 = 3$. Then by above theorem (Theorem 22) we get the foloeing:

$$P(G_{syl}C_{15},\lambda) = \lambda^9(\lambda-1)^2(\lambda-2)^2(\lambda-3)(\lambda-4)$$

It's clear in (Figure 6), there are K_3 and K_5 which are the subgraphs of $G_{syl}C_{15}$, where $V(K_3) = \{e, a^{10}, a^5\}$ and

$V(K_5) = \{e, a^3, a^{12}, a^6, a^9\}$, also the set of isolated vertices is $S = \{a, a^2, a^4, a^7, a^8, a^{11}, a^{13}, a^{14}\}$

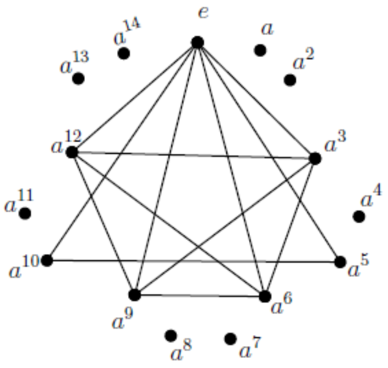

Figure 6: Sylow graph of C_{15}

C. Chromatic Uniqueness of Sylow Graph

- **Theorem 24** Let $G_{Syl}C_n$ be a sylow graph of the cyclic group C_n, if $n = p$, then it is $\chi -$ unique.

Proof. By Theorem 19, then ChP is:

$$P(G_{syl}C_n, \lambda) = \lambda(\lambda - 1) \dots (\lambda - (n - 1)) = P(K_n, \lambda)$$

Then by Fact 11 the sylow graph of the cyclic group $G_{Syl}C_n$, if $n = p$ it is $\chi -$ unique.

- **Theorem 25** If $G_{Syl}C_n$ is a sylow graph of the cyclic group C_n, the following statements are satisfied.

1. if $n = p^\alpha$, then $G_{Syl}C_n$ is not $\chi -$ unique
2. if $n = 2^2$, then $G_{Syl}C_n$ is $\chi -$ unique

Proof.
1. If $n = p^\alpha$, $V(G_{Syl}C_n) = \{e, a, a^2, \dots a^{n-1}\}$, then (by Theorem 20) the ChP is:

$$P\big(G_{syl}C_n, \lambda\big) = \lambda(\lambda - 1) \dots (\lambda - (\varphi(p^\alpha))^{|a^i|}, i|p, i = 0,1,2, \dots, n - 1$$

Since the ChP has roots of multiplicity at least 3, then (by Theorem 10) the sylow graph $G_{Syl}C_n$ if $n = p^\alpha$, it is not $\chi -$ unique.

2. If $n = 2^2$, $V(G_{Syl}C_4) = \{e, a, a^2, a^3\}$, and by using (Theorem 20) we get to:

$$P\big(G_{syl}C_4, \lambda\big) = \lambda(\lambda - 1)(\lambda - 2)^2$$

Since the ChP has one root of multiplicity 2 then by (Theorem 8) the sylow graph $G_{Syl}C_n$ if $n = 2^2$, it is $\chi -$ unique.

- **Theorem 26** Let $G_{Syl}C_n$ be a sylow graph of the cyclic group C_n, the following statements are satisfied.
  1. if $n = \prod p_i$, such that $|p_i| \geq 2$, and $n \neq 2 \times p$, then $G_{Syl}C_n$ is not $\chi -$ unique.

2. if $n = 2 \times p$, then $G_{Syl}C_n$ is $\chi -$ unique.

Proof. 1. if $n = \prod p_i$, such that $|p_i| \geq 2$ and $n \neq 2 \times p$, then by (Theorem 22) the ChP it is

$$P(G_{syl}C_n, \lambda) = \lambda^{|S|+1}(\lambda - 1)^{|p_i|} \dots (\lambda - (p_1 - 1))^{|p_i|}(\lambda - p_1)^{|p_i|-1} \dots (\lambda - (p_2 - 1))^{|p_i|-1}(\lambda - p_2)^{|p_i|-2} \dots (\lambda - (p_{j-1} - 1))^2 \dots (\lambda - (p_j - 1))$$

Since the ChP has at least 2 roots of multiplicity 2, then (by Theorem 9) the sylow graph $G_{Syl}C_n$ it is not $\chi -$ unique.

2. if $n = 2 \times p$, then (by lemma 15) we get two complete subgraphs of $G_{Syl}C_n$, they are K_2 and K_p, such that $V(K_2) = \{e, a\}$ and $V(K_p) = \{e, a^{i-1}\}$, $\left|\{e, a^{i-1}\}\right| = p$, $i = 1,2, \dots, j$, this means $K_2 \cap K_p = e$, and $G_{Syl}C_n = K_2 \cup K_p$, then (by Corollary 12) the ChP of $G_{Syl}C_n$ is:

$$P(G_{Syl}C_n, \lambda) = \frac{\lambda(\lambda - 1) \times \lambda(\lambda - 1) \dots (\lambda - (p - 1))}{\lambda}$$

Then $P(G_{Syl}C_n, \lambda) = \lambda(\lambda - 1)^2 \dots (\lambda - (p - 1)$, since the ChP has one root of multiplicity 2, then (by Theorem 8) the sylow graph $G_{Syl}C_n$ is $\chi -$ unique, where $n = 2 \times p$.

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