Maclaurin Coefficient Estimates for Bi-Univalent Function Classes Generated by a Parameterized Integral Transformation

Amal Mohammed Darweesh
Department of Mathematics,
Faculty of Education for
Women, University of Kufa,
Najaf- Iraq;
amalm.alhareezi@uokufa.edu.iq
Orcid.org/0000-0003-4254-6696

Waggas Galib Atshan
Department of Mathematics,
College of Science,
University of Al-Qadisiyah ,
Diwaniyah –Iraq;
waggas.galib@qu.edu.iq
Orcid.org/0000-0002-7033-8993

Ali Hussein Battor
Department of Mathematics,
Faculty of Education for
Women, University of Kufa,
Najaf- Iraq;
alih.battor@uokufa.edu.iq
Orcid.org/0000-0002-6875-0831

DOI: http://dx.doi.org/10.31642/JoKMC/2018/120203
Received Jul.1, 2025. Accepted for publication Aug. 6, 2025

Abstract— *This paper introduces new subclasses of the bi-univalent function class* $\boldsymbol{\Psi}$, *defined via a newly constructed integral operator denoted by* $\boldsymbol{\mathfrak{J}^{m}_{\alpha,\beta}}$, *acting on analytic functions within the open unit disc* $\mathbb{D}$. *These subclasses are formulated based on multi-parameter operator structures that exhibit rich analytic behavior. We investigate the coefficient bounds associated with functions belonging to these classes, with particular focus on deriving sharp estimates for the first two Taylor–Maclaurin coefficients, namely* $|\boldsymbol{a_2}|$ ***and*** $|\boldsymbol{a_3}|$. *The findings contribute to the structural understanding of bi-univalent functions and offer new directions for application within the framework of geometric function theory.*

Keywords— ***Bi-univalent functions; coefficient estimates; integral operator.***

1. Introduction

Integral, differential, and particularly fractional operators play a fundamental role in modern complex analysis. These operators have proven essential in describing the geometric and analytic properties of analytic functions, including coefficient bounds and distortion phenomena. Fractional calculus extends classical differentiation and integration by allowing the order of these operations to take on non-integer and even complex values, thereby enriching the theoretical framework. Through such operators, new subclasses of starlike and convex functions have been constructed, facilitating a deeper examination of their geometric behavior within the unit disk. Moreover, fractional differential operators are crucial tools in modeling complex dynamical systems and solving fractional differential equations arising in physics and applied sciences. Operators like the Srivastava and Libera fractional integrals have yielded precise results concerning the geometric characteristics and image domains of analytic functions. The utilization of special functions, such as generalized hypergeometric functions, to define fractional operators has further broadened the scope of research in this area. Recently, several noteworthy contributions have emerged, highlighting significant advancements in fractional integral operators within complex analysis [14, 15, 16, 17,18].

Let $\mathcal{A}$ be a class *of* all analytic functions f in *the open unit disk* $\mathbb{U} = \{z: |z| < 1\}$, normalized *by* the conditions $f(0) = 0$ *and* $f'(0) = 1$, of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{1}$$

Let $\mathcal{A}_u$ be *the* class of all functions *in* $\mathcal{A}$ which *are* univalent in $\mathbb{U}$. A function $f \in \mathcal{A}$ is *said* to *be* starlike *if* $f(\mathbb{U})$ is a *starlike* domain *with* respect to the origin *i.e.*, the *line* segment joining *any* point of $f(\mathbb{U})$ *to* the origin *lies* entirely in $f(\mathbb{U})$ *and* a function $f \in \mathcal{A}$ *is said* to be *convex* if $f(\mathbb{U})$ is convex *domain*; i.e., the *line* segment joining *any* two points *in* $f(\mathbb{U})$ lies entirely in $f(\mathbb{U})$. Analytically $f \in \mathcal{A}$ is starlike, denoted by S^*, if and only if $Re\left(\frac{zf'(z)}{f(z)}\right) > 0$, whereas $f \in \mathcal{A}$ is convex, denoted *by* k, if and only if $Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$. The classes $S^*(\tau)$ *and* $k(\tau)$ of starlike and convex functions of order $\tau(0 \leq \tau < 1)$, are respectively characterized by

$$Re\left(\frac{zf'(z)}{f(z)}\right)^{\mathcal{J}^{m}_{\alpha,\beta}f(z)=z}_{>\tau} \qquad (z \in \mathbb{U}), \qquad (2)$$

and

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \tau \qquad (z \in \mathbb{U}), \qquad (3)$$

Lemma 1. Let $f \in \mathcal{A}, for\ \alpha \in N\ and\ \beta \geq 2.$ The integral operator denoted by $\mathcal{J}_{\alpha,\beta}$ defined as following:

$$\mathcal{J}_{\alpha,\beta}: \mathcal{A} \to \mathcal{A},$$

$$\mathcal{J}_{\alpha,\beta}f(z) = \frac{(\ln(\beta))^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty} \mathcal{V}^{\alpha-2}\beta^{-\mathcal{V}} f(z\mathcal{V})\, d\mathcal{V}$$

$$= z + \sum_{n=2}^{\infty}\left(\frac{\Gamma(\alpha+n-1)}{(\ln(\beta))^{n-1}\Gamma(\alpha)}\right) a_n z^n.$$

Proof. We simplify the integral as follows

$$\mathcal{J}_{\alpha,\beta}f(z) = \frac{(\ln(\beta))^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty} \mathcal{V}^{\alpha-2}\beta^{-\mathcal{V}} f(z\mathcal{V})\, d\mathcal{V}$$

$$= \frac{(\ln(\beta))^{\alpha}}{\Gamma(\alpha)}\left[\int_0^{\infty} \mathcal{V}^{\alpha-2} e^{\ln(\beta)^{-\mathcal{V}}}\left(z\mathcal{V} + \sum_{n=2}^{\infty} a_n z^n \mathcal{V}^n\right) d\mathcal{V}\right]$$

$$= \frac{(\ln(\beta))^{\alpha}}{\Gamma(\alpha)}\left[z\int_0^{\infty} \mathcal{V}^{\alpha-1} e^{-\mathcal{V}\ln(\beta)}\, d\mathcal{V} + \sum_{n=2}^{\infty} a_n z^n \int_0^{\infty} \mathcal{V}^{\alpha+n-2} e^{-\mathcal{V}\ln(\beta)}\, d\mathcal{V}\right].$$

Let $\mathcal{V}\ln(\beta) = y, then\ \mathcal{V} = \frac{y}{(\ln(\beta))},\ if\ \mathcal{V} = 0, we\ get\ y = 0, if\ \mathcal{V} = \infty, we\ get\ y = \infty\ and\ d\mathcal{V} = \frac{1}{(\ln(\beta))}dy$.

$$\mathcal{J}_{\alpha,\beta}f(z)$$

$$= \frac{(\ln(\beta))^{\alpha}}{\Gamma(\alpha)}\left[z\int_0^{\infty}\left(\frac{y}{(\ln(\beta))}\right)^{\alpha-1} e^{-y}\frac{1}{(\ln(\beta))}\, dy + \sum_{n=2}^{\infty} a_n z^n \int_0^{\infty}\left(\frac{y}{(\ln(\beta))}\right)^{\alpha+n-2} e^{-y}\frac{1}{(\ln(\beta))}\, dy\right]$$

$$= \frac{(\ln(\beta))^{\alpha}}{\Gamma(\alpha)}\left[\frac{z}{(\ln(\beta))^{\alpha}}\int_0^{\infty} y^{\alpha-1}e^{-y}\, dy + \sum_{n=2}^{\infty}\frac{a_n z^n}{(\ln(\beta))^{\alpha+n-1}}\int_0^{\infty} y^{\alpha+n-2}e^{-y}\, dy\right]$$

$$= z + \sum_{n=2}^{\infty}\left(\frac{\Gamma(\alpha+n-1)}{(\ln(\beta))^{n-1}\Gamma(\alpha)}\right) a_n z^n.$$

In general

$$\mathcal{J}^{m}_{\alpha,\beta}f(z) = z + \sum_{n=2}^{\infty}\left(\mu^{n}_{\alpha,\beta}\right)^{m} a_n z^n,$$

where

$$\left(\mu^{n}_{\alpha,\beta}\right)^{m} = \left(\frac{\Gamma(\alpha+n-1)}{(\ln(\beta))^{n-1}\Gamma(\alpha)}\right)^{m}. \qquad (m \in \mathbb{N}_0)$$

If $m = 0, then\ \mathcal{J}^{0}_{\alpha,\beta}f(z) = f(z)$. Thus

Example 1. Behavior of the operator $\mathcal{J}^{m}_{\alpha,\beta}$ under raw and scaled visualization as follows:

(1) We consider the analytic function

$$f(z) = \frac{z}{1-z} = z + z^2 + z^3 + \cdots,$$

and apply the operator $\mathcal{J}^{1}_{2,3}f(z)$ to obtain

$$\mathcal{J}^{1}_{2,3}f(z) = z + \sum_{n=2}^{\infty}\left(\frac{\Gamma(n+1)}{(\ln 3)^{n-1}}\right) z^n.$$

This operator introduces rapidly growing coefficients, as $\Gamma(n+1)$ grows factorially. For values of $|z|$ near the unit circle, these terms dominate, resulting in an explosive magnitude of the function near $|z| = 1$.

The raw surface plot of $\left|\mathcal{J}^{1}_{2,3}f(z)\right|$ reflects this behavior, showing large spikes along the boundary of the unit disk. Although accurate, such a plot can obscure the inner structure of the transformation within $\mathbb{D}$; see Fig. 1.

(2) To gain better insight into the internal behavior of the operator across the disk, we introduce a logarithmic scaling of the modulus:

$$F(z) := \log\left(1 + \left|\mathcal{J}^{1}_{2,3}f(z)\right|\right).$$

This transformation maintains the analytic structure but compresses the vertical scale. As a result, the plot becomes visually balanced, allowing finer features of the operator's behavior to emerge — especially near the center of the disk, where structural properties of the function are more stable and analytically rich. The logarithmic plot thus serves as a visually enhanced interpretation of the operator's effect; see Fig. 2.

The two plots illustrate different aspects of the operator $\mathcal{J}^{m}_{\alpha,\beta}$. The first reflects the raw magnitude growth, which is important for estimating the distortion induced by the operator, especially in extremal problems. The second offers a smoother analytic landscape, useful for understanding inner dynamics and structure.

Such analysis is of particular interest in fractals and complex dynamics, where rapidly growing iterated structures are common, and understanding their moderated behavior is crucial. However, in our context, we leverage this operator to produce results in geometric function theory, where function classes are studied under analytic constraints, subordination, and coefficient estimates. These visual tools help bridge the abstract operator theory with concrete geometric intuition.

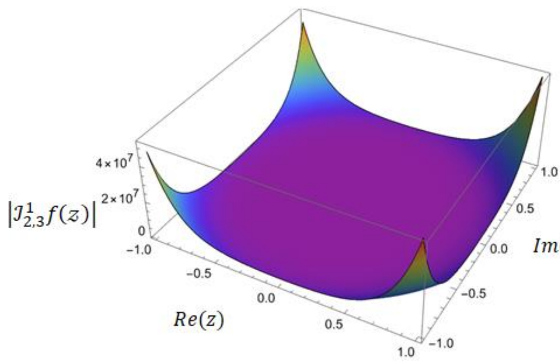

Fig. 1. Plot of $\left|\mathcal{J}^1_{2,3}f(z)\right|$ showing rapid boundary growth.

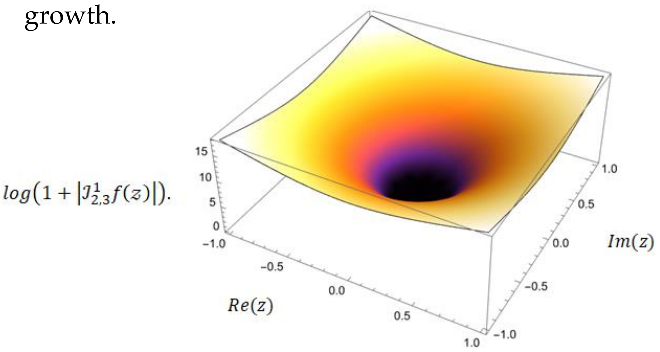

Fig. 1. Plot of $\log\left(1+\left|\mathcal{J}^1_{2,3}f(z)\right|\right)$ revealing internal structure.

The determination of the limits for the coefficients $a_n$is a significant issue in geometric function theory, as these limits provide insights into the geometric characteristics of the functions involved. The bound for the second coefficient $a_2$of functions in $\mathcal{A}_u$provides limits on growth and distortion, in addition to covering theorems. It is well known that the $n-th$ coefficient a_n is bounded by n *for* each $f\in\ \mathcal{A}_u$.

In this concept, we assess the initial coefficients $|a_2|$ *and* $|a_3|$related to the coefficient problem for specific subclasses of bi-univalent functions.

The Koebe one$-$*quarter* theorem [7] proves that the image of $\mathbb{U}$ under every univalent function $f\in\ \mathcal{A}_u$, contains the disk of radius $\frac{1}{4}$. Therefore, every function $f\in\ \mathcal{A}_u$ has an inverse f^{-1}, defined by

$$f^{-1}f(z)=z \qquad (z\in\mathbb{U})$$

and

$$f(f^{-1}(w))=w, \qquad (|w|<r_0(f), \qquad r_0(f)\geq),$$

where

$$f^{-1}(w)=g(w)$$

$$=w+\sum_{n=2}^{\infty}b_nw^n=w-a_2\,w^2+(2a_2{}^2-a_3)w^3$$

$$-(5a_2{}^3-5a_2a_3+a_4)w^4+\cdots. \qquad (5)$$

A function $f\in\mathcal{A}$, is said to be bi-univalent in the *open* unit disk $\mathbb{U}$ if both the functions f *and* f^{-1} are univalent there. Let Ψ denote the class of bi-univalent functions defined in the unit disk $\mathbb{U}$. Examples of functions in the class Ψ are

$$\frac{z}{1-z}, \qquad \log\frac{1}{1-z}, \qquad \log\sqrt{\frac{1+z}{1-z}}.$$

However, the fa liar Keobe function is not a member of Ψ. Other common examples of functions in $\mathbb{U}$ such as

$$\frac{2z-z^2}{2}\ \ and\ \ \frac{z}{1-z^2}$$

are not members of Ψ ethier.

Finding bounds *for* the coefficients of classes of bi-univalent functions dates back to 1967 (see Lewin [9]). Brannan and Taha [4](*see also* [13]) introduced certain subclasses of the bi-univalent function class Ψ similar to the familiar subclasses$S^*(\tau)$ and $k(\tau)$(see [4]). Thus, following Brannan and Taha[4](*see also* [13]), a function$f\in\mathcal{A}$ is in the class$S^*{}_\Psi[\tau]$ of strongly bi-starlike functions of order$\tau(0<\tau\leq 1)$, ifeach of the following conditions are satisfied:

$$f\in\Psi\text{and}\ \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\tau\pi}{2}, \qquad (0<\tau\leq 1,\ z\in\mathbb{U})$$

and

$$\left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right|<\frac{\tau\pi}{2}, \qquad (0<\tau\leq 1, \qquad w\in\mathbb{U}),$$

where g is the extension of f^{-1} *to* $\mathbb{U}$. The classes $S^*{}_\Psi[\tau]$ *and* $k_\Psi(\alpha)$ of bi-starlike functions of order τ, and bi-convex functions of order τ, corresponding (respectively) to the function classes defined by equations (2) and (3), were also introduced analogously. They discovered non-sharp estimates on the first two Taylor-Maclaurin coefficients, $|a_2|$ *and* $|a_3|$, for each of the function classes$S^*{}_\Psi[\tau]$ *and* $k_\Psi(\alpha)$,(see [4,13]).

Motivated by the earlier works of Atshan et al. [1,2], Srivastava et al. [12] and Frasin and Aouf [8](see also [3,5,6,7,9,10,11, 19, 20, 21]). In this study, we provide two new subclasses of the function class Ψ,$C_\Psi(\alpha,\beta,\tau,s,t)$, which generalize the previously stated classes. The novel integral operator$\mathcal{J}_{\alpha,\beta}$of analytic functions involving binomial series in the open unit disk U is used to define this subclass. Furthermore, functions in this new subclass are given upper constraints for the second and third coefficients.

We must remember the following lemma to arrive at our primary conclusions.

Lemma 2. [11] **:**Let $p \in \mathcal{P}$ be family of all functions p *analytic in* $\mathbb{U}$, for which $Re\{p(z)\} > 0$ and have the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, for\ z \in \mathbb{U}, then\ |p_n| \le n,$$

for each n.

2. Limits of the Coefficient for the Function Class $C_\Psi^m(\alpha,\beta,\tau,s,t)$.

Definition 1. A function$f(z)$ given by (1) is said to be in the class $C_\Psi^m(\alpha,\beta,\tau,s,t)$, if the following conditions are satisfied

$$f \in \Psi \text{ and } \left|\arg\left(\frac{(s-t)z^2\left(\mathcal{J}_{\alpha,\beta}f(z)\right)''}{z\left(\mathcal{J}_{\alpha,\beta}f(sz)\right)' - z\left(\mathcal{J}_{\alpha,\beta}f(tz)\right)'} + (s-t)\frac{\mathcal{J}_{\alpha,\beta}f(z)}{z}\right)\right| < \frac{\tau\pi}{2}, (z \in \mathbb{U}) \quad (6)$$

and

$$\left|\arg\left(\frac{(s-t)w^2\left(\mathcal{J}_{\alpha,\beta}g(w)\right)''}{w\left(\mathcal{J}_{\alpha,\beta}g(sw)\right)' - w\left(\mathcal{J}_{\alpha,\beta}g(tw)\right)'} + (s-t)\frac{\mathcal{J}_{\alpha,\beta}g(w)}{w}\right)\right| < \frac{\tau\pi}{2}, \quad (w \in \mathbb{U}) \qquad (7)$$

where $g(w)$ given by (5), $\alpha \in \mathbb{N}, \beta \ge 2,\ 0 < \tau \le 1$ *and* s, t *be a complex number with* $s \ne t$; $|s| \le 1$ *and* $|t| \le 1$.

Now, we being by finding the *estimates* on the coe*ffi*cients $|a_2|$ *and* $|a_3|$, for the functions in the class$C_\Psi^m(\alpha,\beta,\tau,s,t)$.

Theorem 1. Let $f(z)$ given by (1) be in the class$C_\Psi^m(\alpha,\beta,\tau,s,t)$. Then

$$|a_2| \le \frac{2\tau}{\sqrt{\left|\tau(12+2s-2t)\left(\mu_{\alpha,\beta}^3\right)^m - (9\tau s + 7\tau t + 2\tau + s - t - 2)\left(\mu_{\alpha,\beta}^2\right)^{2m}\right|}}, \quad (8)$$

$$|a_3| \le \frac{\tau^2}{|(2+s-t)^2|\left(\mu_{\alpha,\beta}^2\right)^{2m}} + \frac{2\tau}{|6+s-t|\left(\mu_{\alpha,\beta}^3\right)^m}. \quad (9)$$

where $g(w)$ given by (5), $\alpha \in \mathbb{N}, \beta \ge 2,\ 0 < \tau \le 1$ *and* s, t *be a complex number with* $s \ne t$; $|s| \le 1$ *and* $|t| \le 1$.

Proof: By equations (6) and (7), we have

$$\frac{(s-t)z^2\left(\mathcal{J}_{\alpha,\beta}f(z)\right)''}{z\left(\mathcal{J}_{\alpha,\beta}f(sz)\right)' - z\left(\mathcal{J}_{\alpha,\beta}f(tz)\right)'} + (s-t)\frac{\mathcal{J}_{\alpha,\beta}f(z)}{z} = [p(z)]^\tau \quad (10)$$

and

$$\frac{(s-t)w^2\left(\mathcal{J}_{\alpha,\beta}g(w)\right)''}{w\left(\mathcal{J}_{\alpha,\beta}g(sw)\right)' - w\left(\mathcal{J}_{\alpha,\beta}g(tw)\right)'} + (s-t)\frac{\mathcal{J}_{\alpha,\beta}g(w)}{w} = [q(z)]^\tau, \quad (11)$$

where$p(z)$*and* $q(z)$ *in* $\mathcal{P}$and *have the forms*

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \quad (12)$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots. \quad (13)$$

The following relations result from this

$$(2+s-t)\left(\mu_{\alpha,\beta}^2\right)^m a_2 = \tau p_1, \quad (14)$$

$$(6+s-t)\left(\mu_{\alpha,\beta}^3\right)^m a_3 - (4s+4t)\left(\mu_{\alpha,\beta}^2\right)^{2m} {a_2}^2 = \tau p_2 + \frac{\tau(\tau-1)}{2}{p_1}^2 \quad (15)$$

and

$$-(2+s-t)\left(\mu_{\alpha,\beta}^2\right)^m a_2 = \tau q_1, \quad (16)$$

$$\left[(12+2s-2t)\left(\mu_{\alpha,\beta}^3\right)^m - (4s+4t)\left(\mu_{\alpha,\beta}^2\right)^{2m}\right]{a_2}^2 - (6+s-t)\left(\mu_{\alpha,\beta}^3\right)^m a_3 = \tau q_2 + \frac{\tau(\tau-1)}{2}{q_1}^2.$$

By (14) and (16), we get that

$$p_1 = -q_1 \quad (18)$$

and

$$2(2+s-t)^2\left(\mu_{\alpha,\beta}^2\right)^{2m}{a_2}^2 = \tau^2({p_1}^2 + {q_1}^2). \quad (19)$$

Now, adding (15) and (17), we obtain that

$$(12+2s-2t)\left(\mu_{\alpha,\beta}^3\right)^m - 2(4s+4t)\left(\mu_{\alpha,\beta}^2\right)^{2m}{a_2}^2 = \tau(p_2+q_2) + \frac{\tau(\tau-1)}{2}({p_1}^2+{q_1}^2). (20)$$

From (20) and (19), we have

$$\tau\left[(12+2s-2t)\left(\mu_{\alpha,\beta}^3\right)^m - 2(4s+4t)\left(\mu_{\alpha,\beta}^2\right)^{2m}\right]{a_2}^2 = \tau^2(p_2+q_2) + (\tau-1)(2+s-t)^2\left(\mu_{\alpha,\beta}^2\right)^{2m}{a_2}^2. \quad (21)$$

Therefore, we have

$${a_2}^2 = \frac{\tau^2(p_2+q_2)}{\tau(12+2s-2t)\left(\mu_{\alpha,\beta}^3\right)^m - [2\tau(4s+4t) - (\tau-1)(2+s-t)]\left(\mu_{\alpha,\beta}^2\right)^{2m}}.$$

Applying Lemma 2 for the coefficients p_2 *and* q_2, we immediately have

$$|a_2| \leq \frac{2\tau}{\sqrt{\left|\tau(12+2s-2t)\left(\mu^3_{\alpha,\beta}\right)^m - (9\tau s + 7\tau t + 2\tau + s - t - 2)\left(\mu^2_{\alpha,\beta}\right)^{2m}\right|}},$$

This provides the intended approximation of $|a_2|$as stated in (15).

N*e*xt in *ord*er to find *the* bound on $|a_3|$, by subtracting (17) from (15), we get that

$$2(6+s-t)\left(\mu^3_{\alpha,\beta}\right)^m a_3 - (12+2s-2t)\left(\mu^3_{\alpha,\beta}\right)^m {a_2}^2 = \tau(p_2 - q_2) + \frac{\tau(\tau-1)}{2}\left({p_1}^2 - {q_1}^2\right). \quad (22)$$

From (18), (19) and (22), we obtain

$$2(6+s-t)\left(\mu^3_{\alpha,\beta}\right)^m a_3 = (12+2s - 2t)\left(\mu^3_{\alpha,\beta}\right)^m \left[\frac{\tau^2\left({p_1}^2 + {q_1}^2\right)}{2(2+s-t)^2\left(\mu^2_{\alpha,\beta}\right)^{2m}}\right] + \tau(p_2 - q_2),$$

or, equivalently

$$a_3 = \frac{\tau^2\left({p_1}^2 + {q_1}^2\right)}{2(2+s-t)^2\left(\mu^2_{\alpha,\beta}\right)^{2m}} + \frac{\tau(p_2 - q_2)}{2(6+s-t)\left(\mu^2_{\alpha,\beta}\right)^{2m}}.$$

Appling Lemma 2 for the coefficient p_1, q_1, p_2 *and* q_2 we get that

$$|a_3| \leq \frac{\tau^2}{|(2+s-t)^2|\left(\mu^2_{\alpha,\beta}\right)^{2m}} + \frac{2\tau}{|6+s-t|\left(\mu^3_{\alpha,\beta}\right)^m},$$

As stated in (9), we obtain the desired estimate on $|a_3|$.

Setting $m = 0$, in Theorem 1, we get the following result.

Corollary1. Let $f(z)$given by (1) be in the class $C^0_\Psi(\alpha,\beta,\tau,s,t), 0 < \tau \leq 1$. Then

$$|a_2| \leq \frac{2\tau}{\sqrt{|\tau(12+2s-2t) - (9\tau s + 7\tau t + 2\tau + s - t - 2)|}}$$

and

$$|a_3| \leq \frac{\tau^2}{|(2+s-t)^2|} + \frac{2\tau}{|6+s-t|}.$$

Setting $s = 1$ *and* $t = -1$, in Corollary1, we obtain the following outcome.

Corollary2. Let $f(z)$*given* by (1) be in the class $C^0_\Psi(\alpha,\beta,\tau,1,-1), 0 < \tau \leq 1$. Then

$$|a_2| \leq \frac{\tau}{\sqrt{3\tau}},$$

and

$$|a_3| \leq \frac{\tau^2}{16} + \frac{\tau}{4}.$$

Setting $s = 1$ *and* $t = 0$, in Corollary, we obtain the following outcome.

Corollary 3. Let $f(z)$ given by (1) be in the class $C^0_\Psi(\alpha,\beta,\tau,1{,}0), 0 < \tau \leq 1$. Then

$$|a_2| \leq \frac{2\tau}{\sqrt{3\tau - 1}},$$

and

$$|a_3| \leq \frac{\tau^2}{9} + \frac{2\tau}{7}.$$

3. Coefficient Bounds for the Function class $C^m_\Psi(\alpha,\beta,\xi,s,t)$.

Definition 2. A function$f(z)$ given by (1) is said to *be in* the class $C^m_\Psi(\alpha,\beta,\xi,s,t)$,if the following conditions are satisfied

$$f \in \Psi \text{ and } \operatorname{Re}\left(\frac{(s-t)z^2\left(\mathcal{J}_{\alpha,\beta}f(z)\right)''}{z\left(\mathcal{J}_{\alpha,\beta}f(sz)\right)' - z\left(\mathcal{J}_{\alpha,\beta}f(tz)\right)'} + (s-t)\frac{\mathcal{J}_{\alpha,\beta}f(z)}{z}\right) > \xi, \qquad (z \in \mathbb{U}) (23)$$

and

$$\operatorname{Re}\left(\frac{(s-t)w^2\left(\mathcal{J}_{\alpha,\beta}g(w)\right)''}{w\left(\mathcal{J}_{\alpha,\beta}g(sw)\right)' - w\left(\mathcal{J}_{\alpha,\beta}g(tw)\right)'} + (s-t)\frac{\mathcal{J}_{\alpha,\beta}g(w)}{w}\right) > \xi, \quad (w \in \mathbb{U}) \qquad (24)$$

where $g(w)$ given by (5), $\alpha \in \mathbb{N}, \beta \geq 2,\ 0 \leq \xi < 1$ *and* s, t *be a complex number with* $s \neq t$; $|s| \leq 1$ *and* $|t| \leq 1$.

Theorem 2. Let $f(z)$ given by (1) be in the class $C^m_\Psi(\alpha,\beta,\tau,s,t)$. Then

$$|a_2| \leq \sqrt{\frac{(1-\xi)}{\left|(12+2s-2t)\left(\mu^3_{\alpha,\beta}\right)^m - 2(4s+4t)\left(\mu^2_{\alpha,\beta}\right)^{2m}\right|}},$$

$$|a_3| \leq \frac{(1-\xi)^2}{|(2+s-t)^2|\left(\mu^2_{\alpha,\beta}\right)^{2m}} + \frac{2(1-\xi)}{|6+s-t|\left(\mu^3_{\alpha,\beta}\right)^m}, \quad (26)$$

where $g(w)$ given by (5), $\alpha \in \mathbb{N}, \beta \geq 2,\ 0 \leq \xi < 1$ *and* s, t *be a complex number with* $s \neq t$; $|s| \leq 1$ *and* $|t| \leq 1$.

Proof. It follows that, from (23) and (24), there exist p *and* $q \in \mathcal{P}$ such that

$$\frac{(s-t)z^2\left(\mathcal{J}_{\alpha,\beta}f(z)\right)''}{z\left(\mathcal{J}_{\alpha,\beta}f(sz)\right)' - z\left(\mathcal{J}_{\alpha,\beta}f(tz)\right)'} + (s-t)\frac{\mathcal{J}_{\alpha,\beta}f(z)}{z} = \xi + (1-\xi)p(z), \qquad (27)$$

and

$$\frac{(s-t)w^2\left(\mathcal{J}_{\alpha,\beta}g(w)\right)''}{w\left(\mathcal{J}_{\alpha,\beta}g(sw)\right)' - w\left(\mathcal{J}_{\alpha,\beta}g(tw)\right)'} + (s-t)\frac{\mathcal{J}_{\alpha,\beta}g(w)}{w} = \xi + (1-\xi)q(w), \qquad (28)$$

where$p(z)$ and $q(w)$ in $\mathcal{P}$given by (12) and (13). We obtain the following relationships as a result.

$$(2+s-t)\left(\mu^2_{\alpha,\beta}\right)^m a_2 = (1-\xi)p_1, \tag{29}$$

$$(6+s-t)\left(\mu^3_{\alpha,\beta}\right)^m a_3 - (4s+4t)\left(\mu^2_{\alpha,\beta}\right)^{2m} a_2{}^2 = (1-\xi)p_2 \tag{30}$$

and

$$-(2+s-t)\left(\mu^2_{\alpha,\beta}\right)^m a_2 = (1-\xi)q_1, \tag{31}$$

$$\left[(12+2s-2t)\left(\mu^3_{\alpha,\beta}\right)^m - (4s+4t)\left(\mu^2_{\alpha,\beta}\right)^{2m}\right] a_2{}^2 - (6+s-t)\left(\mu^3_{\alpha,\beta}\right)^m a_3 = (1-\xi)q_2. \tag{32}$$

By (29) and (31), we get that

$$p_1 = -q_1 \tag{33}$$

and

$$2(2+s-t)^2\left(\mu^2_{\alpha,\beta}\right)^{2m} a_2{}^2 = (1-\xi)^2\left(p_1{}^2 + q_1{}^2\right). \tag{34}$$

Now, adding (30) and (32), we obtain that

$$(12+2s-2t)\left(\mu^3_{\alpha,\beta}\right)^m - 2(4s+4t)\left(\mu^2_{\alpha,\beta}\right)^{2m} a_2{}^2 = (1-\xi)(p_2+q_2). \tag{35}$$

Therefore, we have

$$a_2{}^2 = \frac{(1-\xi)(p_2+q_2)}{\left|(12+2s-2t)\left(\mu^3_{\alpha,\beta}\right)^m - 2(4s+4t)\left(\mu^2_{\alpha,\beta}\right)^{2m}\right|},$$

Applying Lemma 2 for the coefficients p_2 and q_2, we immediately have

$$|a_2{}^2| \le \frac{4(1-\xi)}{\left|(12+2s-2t)\left(\mu^3_{\alpha,\beta}\right)^m - 2(4s+4t)\left(\mu^2_{\alpha,\beta}\right)^{2m}\right|},$$

w It provides us with the intended estimate of $|a_2|$as stated in (25).

Next, by deducting (32) from (30), we can determine the bound on $|a_3|$. This yields that

$$2(6+s-t)\left(\mu^3_{\alpha,\beta}\right)^m a_3 - (12+2s-2t)\left(\mu^3_{\alpha,\beta}\right)^m a_2{}^2 = (1-\xi)(p_2-q_2). \tag{36}$$

By (33), (34) and (36), we obtain

$$2(6+s-t)\left(\mu^3_{\alpha,\beta}\right)^m a_3 = (12+2s - 2t)\left(\mu^3_{\alpha,\beta}\right)^m \left[\frac{(1-\xi)^2\left(p_1{}^2+q_1{}^2\right)}{2(2+s-t)^2\left(\mu^2_{\alpha,\beta}\right)^{2m}}\right] + (1-\xi)(p_2-q_2),$$

or, equivalently

$$a_3 = \frac{(1-\xi)^2\left(p_1{}^2+q_1{}^2\right)}{2(2+s-t)^2\left(\mu^2_{\alpha,\beta}\right)^{2m}} + \frac{(1-\xi)(p_2-q_2)}{2(6+s-t)\left(\mu^2_{\alpha,\beta}\right)^{2m}}.$$

Appling Lemma 2 for the coefficient p_1, q_1, p_2 and q_2 we get that

$$|a_3| \le \frac{(1-\xi)^2}{|(2+s-t)^2|\left(\mu^2_{\alpha,\beta}\right)^{2m}} + \frac{2(1-\xi)}{|6+s-t|\left(\mu^3_{\alpha,\beta}\right)^m},$$

As stated in (26), we obtain the desired estimate on $|a_3|$.

Setting $m = 0$, in Theorem2, we obtain the following outcome.

Corollary 4. Let$f(z)$given by (1) be in the class$C^0_\Psi(\alpha,\beta,\xi,s,t), 0 \le \xi < 1$. Then

$$|a_2| \le \sqrt{\frac{4(1-\xi)}{|(12+2s-2t) - 2(4s+4t)|}},$$

and

$$|a_3| \le \frac{4(1-\xi)^2}{|(2+s-t)^2|} + \frac{2(1-\xi)}{|6+s-t|}.$$

Setting $s = 1$ and $t = -1$, in Corollary 4, we get the following result.

Corollary 5. Let $f(z)$ given by (1) be in the class $C^0_\Psi(\alpha,\beta,\xi,s,t), 0 \le \xi < 1$. Then

$$|a_2| \le \sqrt{\frac{(1-\xi)}{4}},$$

and

$$|a_3| \le \frac{(1-\xi)^2}{16} + \frac{(1-\xi)}{8}.$$

4. Conclusion

In this work, we introduced new subclasses of bi-univalent functions defined via a novel parameterized integral operator. We established precise estimates for the second and third Taylor–Maclaurin coefficients, highlighting the analytic impact of the operator's parameters. Theoretical insights were further supported by complex 3D visualizations, illustrating the geometric deformation induced by the operator. These findings offer potential applications in geometric function theory and open new directions for future investigations involving fractional and integral transformations.

REFERENCES

[1] W. G. Atshan, I. A. Rahman, and A. A. Lupas, Some results of new subclasses for bi-univalent functions using quasi-subordination, *Symmetry* **13**(9) (2021).

[2] W. G. Atshan and R. A. Hiress, Coefficient estimates for subclasses of bi-univalent functions, *Journal of Al-Qadisiyah for Computer Science and Mathematics* **10**(3) (2018).

[3] D. A. Brannan, J. Clunie, and W. E. Kirwan, Coefficient estimates for a class of starlike functions, *Canadian Journal of Mathematics* **22** (1970), 476–485.

[4] D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, in Mathematical Analysis and Its Applications, *Pergamon Press, Turkey*, 1988, 53–60.

[5] S. Bulut, Coefficient estimates for a class of analytic and bi-univalent functions, *Novi Sad Journal of Mathematics* **43** (2013), 59–65.

[6] N. E. Cho, O. S. Kwon, and S. Owa, Certain subclasses of Sakaguchi functions, *Southeast Asian Bulletin of Mathematics* **17** (1993), 121–126.

[7] P. L. Duren, Univalent Functions, Grundlehren der Mathematischen Wissenschaften, Vol. 259, Springer, New York, Berlin, Heidelberg, Tokyo, 1983.

[8] B. A. Frasin and M. K. Aouf, New subclasses of bi-univalent functions, *Applied Mathematics Letters* **24** (2011), 1569–1573.

[9] M. Lewin, On a coefficient problem for bi-univalent functions, *Proceedings of the American Mathematical Society* **18** (1967), 63–68.

[10] G. Murugusundaramoorthy, N. Magesh, and V. Prameela, Coefficient bounds for certain subclasses of bi-univalent functions, *Abstract and Applied Analysis* **2013** (2013), Article ID 573017.

[11] C. Pommerenke, Univalent Functions, Vandenhoeck and Ruprecht, Göttingen, Germany, 1975.

[12] H. M. Srivastava, A. K. Mishra, and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, *Applied Mathematics Letters* **23** (2010), 1188–1192.

[13] T. S. Taha, Topics in Univalent Function Theory, Ph.D. Thesis, University of London, London, UK, 1981.

[14] A. S. Tayyah and W. G. Atshan, New results on (r, k, μ)-Riemann–Liouville fractional operators in complex domain with applications, *Fractal and Fractional* **8**(3) (2024), Article 165.

[15] A. S. Tayyah and W. G. Atshan, Differential subordination and superordination results for ppp-valent analytic functions associated with (r, k)-Srivastava fractional integral calculus, *MethodsX* **13** (2024), Article 103079.

[16] A. S. Tayyah, W. G. Atshan, and G. I. Oros, Third-order differential subordination results for meromorphic functions associated with the inverse of the Legendre Chi function via the Mittag-Leffler identity, *Mathematics* **13**(13) (2025), Article 2089.

[17] I. Aldawish and R. W. Ibrahim, Studies on a new K-symbol analytic functions generated by a modified K-symbol Riemann–Liouville fractional calculus, *MethodsX* **11** (2023), Article 102398.

[18] I. Aldawish and R. W. Ibrahim, Fractional-order modeling of sediment transport and coastal erosion mitigation in shorelines under extreme climate conditions: A case study in Iraq, *Computation* **13**(5) (2025), Article 104.

[19] A. S. Tayyah and W. G. Atshan, A class of bi-Bazilevič and bi-pseudo-starlike functions involving Tremblay fractional derivative operator, *Problems of Analysis. Issues of Analysis* **14**(32)(2) (2025), 145–161.

[20] A. S. Tayyah and W. G. Atshan, Starlikeness and bi-starlikeness associated with a new Carathéodory function, *Journal of Mathematical Sciences* (2025).

[21] Q. A. Shakir, A. S. Tayyah, D. Breaz, L.-I. Cotîrlă, E. Rapeanu, and F. M. Sakar, Upper bounds of the third Hankel determinant for bi-univalent functions in crescent-shaped domains, *Symmetry* **16** (2024), Article 1281.