Forward-Backward System for Delay Doubly Stochastic Differential Equations

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Abstract**—***The research aims to study a general category of forward-backward double delay differential equations system (FBDDSDEs), with a focus on solutions and their characteristics. The existence and uniqueness of solution to the front and rear equation of the system is proven according to the conditions of Lipchitz, which confirms the existence of a maximum solution under the conditions of unilateral continuity.*

Keywords— Ito's integral, Forward Doubly stochastic differential equations, Backward Doubly stochastic differential equations, Delay of the random variable, Maximal solution.

I. INTRODUCTION

In this research, we present a new type of random differential equation system, which are known as a random differential equation system with double forward and backward delays (FBDSDEs). It is defined as follows: for $\zeta \in \mathrm{F}^2(\Omega, \Gamma_0, \mathrm{P}; \mathrm{R}), \mathrm{t} \in [0, \mathrm{T}]$,

$$\begin{cases} \Psi(t) = \zeta + \int_0^t \Gamma\big(\varrho, \Psi(\varrho), \Phi(\varrho)\big)d\varrho + \int_0^t \acute{\Gamma}\,(\varrho, \\ \quad \big(\varrho, \Psi(\varrho), \Phi(\varrho)\big)dB(\varrho) + \int_0^t \Phi(\varrho)dW(\varrho) \\ \Upsilon(t) = h(x(T) + \int_t^T g\big(\varrho, \gamma(\varrho), \Lambda(\varrho)\big)d\varrho + \int_0^T \acute{g}(\big(\varrho, \gamma(\varrho), \Lambda(\varrho)\big)dB(\varrho) \\ \quad + \int_t^T \Lambda(\varrho)dW(\varrho) \end{cases} \tag{1}$$

where W and B are two mutually independent standard Brownian motions, with the processes Ψ, Φ, γ, and Λ defined on $R^k, R^d, R^{k\times r}$ and $R^{d\times r}$ respectively. Also, the function, $\Gamma, \mathrm{g}, \acute{\Gamma}, \acute{g}$, and h are defined on $R^k, R^d, R^{k\times r}, R^{d\times r}$, and R^d, respectively.

Bao et al. [1] a study was conducted on differential equations using the advanced way, in addition to the methods of finding digital solutions for them. This study includes error analysis and determining the rate of rapprochement the rate of rapprochement by rapprochement by proposing digital algorithms. Qingfeng and Yufeng [2] the discussion was linked to two differential equations in an advanced manner, where the concept of the bridge was used. This process has shown that both equals have a unique solution. This, the establishment of the appropriate bridge allows access to a variety of uniquely solvable equation. Zhu and Shi [3], in the context of the same monotonous assumptions, an average model- the double random difference was proposed equations showed that unique solutions are measurable by taking advantage of the continuity feature. Zhu et al. [4] a study was presented on a model of the dual and rear, frontal and background equations associated with the Brown and Poisson operation, where the presence and uniqueness of measurable solutions have been proven and discriminatory based on the variables. Bao et al [5] it was suggested a model linking the complications filter problem, as well as a solution to the opposite differential equations using the Markov is deployment process. Abdul Rahman [6] an important study that dealt with the advanced and backward differential equations associated with Poisson jumps, confirmed that there are unique solutions to this type of equations with the continuity of time. Ksendal and Sulem [7] multiple maximum principle models have been proposed for optimal control of backward stochastic differential equations (BSDEs), especially in the presence of random jumps. These models aim to provide a clear picture of optimal control while minimizing risks using G-expectations. Ji and Wei [8] a recent study focuses on advanced differential equations, specifically on how they can be used to perfectly describe stochastic systems. The main goal of this study was to improve the performance and increase utility in financial markets. Zhang and Shi [9] in the context of model control, the maximum principle is discussed through the use of a comprehensive forward-backward differential equations model. This model aims to analyze the system's behavior by integrating both forward and backward differential equations, which allows for the calculation of conditions under specific assumptions.

II. BASIC ASSUMPTIONS

Suppose that $\{W(t); 0 \leq t \leq T\}$ and $\{\mathrm{B}(t); 0 \leq t \leq T\}$ are two mutually independent standard b −dimensioal Brownian motions defined on the complete probability spaces $(\Omega_1, \mathrm{F}_1, \mathrm{P}_1)$ and $(\Omega_2, \mathrm{F}_2, \mathrm{P}_2)$ respectively, and for all finite time horizons $T < \infty$, let Q be the class of P −null sets of F.

We consider $\Omega = \Omega_1 \times \Omega_2, \mathrm{F} = \mathrm{F}_1 \times \mathrm{F}_2$, and $\mathrm{P} = \mathrm{P}_1 \times \mathrm{P}_2$. For each $t \in [0, T]$, we define $\mathrm{F}_t = \mathrm{F}_t^W\ S\ \mathrm{F}_{t,T}^p \vee N$, for any process ϖ_t *then* $\mathrm{F}_t^{\varpi} = \mathrm{F}_{0,t}^{\varpi}, \mathrm{F}_{o,t}^{\varpi} = \sigma\{\varpi_z - \varpi_o; o \le z \le t\}$. Note that the family of $\sigma -$fields $\{\mathrm{F}_t, t \in [0, T]\}$ is neither increasing nor decreasing, and it is not a filtration. For a Euclidean space D, we denote $C^2(0, T; D)$ as the space of jointly measurable processes $\{X(t), t \in [0, T]\}$ taking values in D, for each $t \in [0, T]$ then $X(t)$ is $\mathrm{F}_t -$ measurable such that $E[\int_0^T |X(t)|^2\, dt] < \infty$. We define the following space:

Let $C_1^2(\Omega, \mathrm{F}, \mathrm{P}; R^k)$be the space of F-measurable random variable X such that $\|X\|_{C_1^2} = (E|X(t)|^2)^{\frac{1}{2}} < \infty$.

Let $C_2^2(\Omega, \mathrm{F}, \mathrm{P}; R^d)$ be the continuous space $\{\mathrm{F}_s\}_{t \le s \le T}$-adapted process X such that $\|X\|_{C_2^2} = [E(sup_{t \le s \le T}|X(s)|^2)]^{\frac{1}{2}} < \infty$.

Let $C_3^2(\Omega, \mathrm{F}, \mathrm{P}; R^{k \times d})$ be the space of $\{\mathrm{F}_s\}_{t \le s \le T}$-progressively measurable processes X such that $\|X\|_{C_3^2} = [E \int_0^T |X\,(s)|^2 ds]^{\frac{1}{2}} < \infty$.

Let $MSC = C_4^2([0, T], C_1^2(\Omega, \mathrm{F}, \mathrm{P}; R^k))$ be the class of all mean-square continuous- second order stochastic processes.

III. NUMERICAL ASSUMPTIONS

We consider the forward- backward stochastic, differential equation with doubly delay, given as follows:

$$\begin{cases} d(\Psi(t) = \Gamma\big(\varrho, \Psi(\varrho), \Phi(\varrho), \Psi_\varrho, \Phi_\varrho\big)d\varrho + \acute{\Gamma}\big(\varrho, \Psi(\varrho), \Phi(\varrho), \Psi_\varrho, \Phi_\varrho\big)dB(\varrho) \\ +\Phi(\varrho)dW(\varrho), \\ 0 \le \varrho \le t, \Psi_0 = \zeta(\varrho_0), -t \le \varrho \le 0 \\ d\big(\gamma(t)\big) = g\big(\varrho, \gamma(\varrho), \Lambda(\varrho), \gamma_\varrho, \Lambda_\varrho\big)d\varrho + \acute{g}\big((\varrho, \gamma(\varrho), \Lambda(\varrho), \gamma_\varrho, \Lambda_\varrho\big)dB(\varrho) \\ +\Lambda(\varrho)dW(\varrho) \\ , 0 \le \varrho \le T \\ \gamma_T = h(X(T), -T \le t \le 0 \end{cases} \tag{2}$$

where Γ, g, and $\acute{\Gamma}$ are Borel-measurable functions that depends on the past values of the solution $\Psi_\varrho = (\Psi(\varrho + \theta)), -T \le \theta \le 0$, $\gamma_\varrho = (\gamma(\varrho + \theta)), -T \le \theta \le 0$, and $\Phi_\varrho = (\Phi(\varrho + \theta)), -T \le \theta \le 0$. We define the functions $\Gamma, g, \acute{\Gamma}, \acute{g}$ and h as to be continuous

$$\Gamma: \Omega \times [0, T] \times R^k \times R^{k \times r} \times Q_{-T}^2 R^k \times Q_{-T}^2 R^{r \times k} \to R^k,$$

$$g: \Omega \times [0, T] \times R^k \times R^d \times Q_{-T}^2 R^k \times Q_{-T}^2 R^d \to R^k,$$

$$\acute{\Gamma}: \Omega \times [0, T] \times R^k \times R^{k \times r} \times Q_{-T}^2 R^k \times Q_{-T}^2 R^{k \times r} \to R^{k \times r},$$

$$\acute{g}: \Omega \times [0, T] \times R^d \times R^{d \times r} \times Q_{-T}^2 R^d \times Q_{-T}^2 R^{d \times r} \to R^{d \times r},$$

$$\Lambda: \Omega \times R^k \to R^k$$

We use the Euclidean norms in R^k and R^d, and define $|\Phi| = \{trans(\Phi\Phi^T)\}^{\frac{1}{2}}$, where $\Phi \in R^{k \times d}$. Thus, $R^{k \times d}$ is treated as a Hilbert space. Let be jointly measurable and satisfy the following assumptions:

A1. For every $t \in [0, T], (\Psi, \Phi, \gamma, \Lambda) \in R^k \times R^{k \times r} \times R^d \times R^{d \times r}, (\Psi, \Phi, \gamma, \Lambda) \in R^k \times R^{k \times r} \times R^d \times R^{d \times r}$ and $N_1, N_2, N_3, N_4 > 0$ and a finite measure α on $[-\varrho, 0]$ such that

$|\Gamma\big(t, \kappa^1, \Psi^1, \gamma^1, \Phi^1, \kappa_\varrho^1, \Psi_\varrho^1, \gamma_\varrho^1, \Phi_\varrho^1\big) - \Gamma\big(t, \kappa^2, \Psi^2, \gamma^2, \Phi^2, \kappa_\varrho^2, \Psi_\varrho^2, \gamma_\varrho^2, \Phi_\varrho^2\big) \le N_1\, |\Psi^1 - \Psi^2| + |\gamma^1 - \gamma^2| + |\Phi^1 - \Phi^2| + \mathrm{R}_1(\int_{-\mathrm{T}}^0 |\Psi^1(\mathrm{t} + \theta) - \Psi^2(\mathrm{t} + \theta)|^2 \alpha d\theta + \int_{-\mathrm{T}}^0 |\gamma^1\,(\mathrm{t} + \theta) - \gamma^2(\mathrm{t} + \theta)|^2 \alpha d\theta + \int_{-\mathrm{T}}^0 |\Phi^1(\mathrm{t} + \theta) - \Phi^2(\mathrm{t} + \theta)|^2 \alpha d\theta,$

$|g\big(t, \kappa^1, \Psi^1, \gamma^1, \Lambda^1, \kappa_\varrho^1, \Psi_\varrho^1, \gamma_\varrho^1, \Lambda_\varrho^1\big) - \Gamma\big(t, \kappa^2, \Psi^2, \gamma^2, \Lambda^2, \kappa_\varrho^2, \Psi_\varrho^2, \gamma_\varrho^2, \Lambda_\varrho^2\big) \le N_2|\Psi^1 - \Psi^2| + |\gamma^1 - \gamma^2| + |\Lambda^1 - \Lambda^2| + \mathrm{R}_1(\int_{-\mathrm{T}}^0 |\Psi^1(\mathrm{t} + \theta) - \Psi^2(\mathrm{t} + \theta)|^2 \alpha d\theta + \int_{-\mathrm{T}}^0 |\gamma^1\,(\mathrm{t} + \theta) - \gamma^2(\mathrm{t} + \theta)|^2 \alpha d\theta + \int_{-\mathrm{T}}^0 |\Lambda^1(\mathrm{t} + \theta) - \Lambda^2(\mathrm{t} + \theta)|^2 \alpha d\theta,$

$|\acute{\Gamma}\big(t, \Psi^1, \Phi^1, \Psi_\varrho^1, \Phi_\varrho^1\big) - \big(t, \Psi^2, \Phi^2, \Psi_\varrho^1, \Phi_\varrho^1\big)|^2 \le N_3(|\Psi^1 - \Psi^2|^2 + ||\Phi^1 - \Phi^2|^2 + \mathrm{R}_3(\int_{-\mathrm{T}}^0 |\Psi^1\,(\mathrm{t} + \theta) - \Psi^2(\mathrm{t} + \theta)|^2 \alpha d\theta + \int_{-\mathrm{T}}^0 |\Phi^1(\mathrm{t} + \theta) - \Phi^2(\mathrm{t} + \theta)|^2 \alpha d\theta).$

$|\acute{g}\big(t, \gamma^1, \Lambda^1, \gamma_\varrho^1, \Lambda_\varrho^1\big) - \big(t, \gamma^2, \Lambda^2, \gamma_\varrho^1, \Lambda_\varrho^1\big)|^2 \le N_4(|\gamma^1 - \gamma^2|^2 + ||\Lambda^1 - \Lambda^2|^2 + \mathrm{R}_4(\int_{-\mathrm{T}}^0 |\gamma^1\,(\mathrm{t} + \theta) - \gamma^2(\mathrm{t} + \theta)|^2 \alpha d\theta + \int_{-\mathrm{T}}^0 |\Lambda^1(\mathrm{t} + \theta) - \Lambda^2(\mathrm{t} + \theta)|^2 \alpha d\theta).$

A2. For every $t \in [0, T], (\Psi, \Phi, \gamma, \Lambda) \in R^k \times R^{k \times r} \times R^d \times R^{d \times r}$and $k_1, k_2, k_3, k_4, k_5 > 0$ such that

$|\Gamma(t, \Psi, \gamma, \Phi, \Psi_\varrho, \gamma_\varrho, \Phi_\varrho)| \le |\Psi| + |\gamma| + |\Phi| + \mathrm{R}_1(\int_{-\mathrm{T}}^0 |\Psi\,(\mathrm{t} + \theta)| \alpha d\theta + \int_{-T}^0 |\gamma(t + \theta) \alpha d\theta + \int_{-\mathrm{T}}^0 |\Phi(\mathrm{t} + \theta) \alpha d\theta),$

$\big|g\big(t, \Psi, \gamma, \Lambda, \Phi, \Psi_\varrho, \gamma_\varrho, \Lambda_\varrho\big)\big| \le |\Psi| + |\gamma| + |\Lambda| + \mathrm{R}_1(\int_{-\mathrm{T}}^0 |\Psi\,(\mathrm{t} + \theta)| \alpha d\theta + \int_{-T}^0 |\gamma(t + \theta) \alpha d\theta + \int_{-T}^0 |\Lambda(t + \theta) \alpha d\theta,$

$|\acute{\Gamma}(t, \Psi, \Phi, \Psi_\varrho, \Phi_\varrho)|^2 \le k_3(1 + |\Psi|^2 + |\Phi|^2 + \mathrm{R}_3(\int_{-\mathrm{T}}^0 |\Psi\,(\mathrm{t} + \theta)|^2 \alpha d\theta + \int_{-\mathrm{T}}^0 |\Phi\,(\mathrm{t} + \theta)|^2 \alpha d\theta),$

$|\acute{g}(t, \gamma, \Lambda, \gamma_\varrho, \Lambda_\varrho)|^2 \le k_4(1 + |\gamma|^2 + |\Lambda|^2 + \mathrm{R}_3(\int_{-\mathrm{T}}^0 |\gamma\,(\mathrm{t} + \theta)|^2 \alpha d\theta + \int_{-\mathrm{T}}^0 |\Lambda\,(\mathrm{t} + \theta)|^2 \alpha d\theta),$

For all $-\varrho \le s \le t \le 0$ such that $E|(t) - \zeta(s)|^2 \le k_4(t - s)$ and $|\mathrm{h(X)}|^2 \le k_5(1 + |\Psi|^2)$.

A3.

$E(\int_0^T |\Gamma(\varrho, \zeta_0\,, 0,0,0,0)|^2 d\varrho) < \infty,$

$E(\int_0^T |g(\varrho, \zeta_0\,, 0,0,0,0)|^2 d\varrho) < \infty,$

$E(\int_0^T |\acute{\Gamma}(\varrho, 0,0,0,0)|^2 d\varrho) < \infty,$

$E(\int_0^T |\acute{g}(\varrho, 0,0,0,0)|^2 d\varrho) < \infty.$

IV. NUMERICAL ASSUMPTIONS

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We consider a system of forward -Backward delay stochastic differential equations with delay, given as follows:

$$\begin{cases} d\Psi(t) = \zeta + \Gamma(t,\Psi(t),\gamma(t),\Phi(t),\Psi_t,\gamma_t,\Phi_t)dt + \\ \acute{\Gamma}(t,\Psi(t),\Phi(t),\Psi_t,\Phi_t)dB(t) + \Phi(t)dW(t), \\ d\gamma(t) = h(\Psi(T)) + g(t,\Psi(t),\gamma(t),\Lambda(t),\Psi_t,\gamma_t,\Lambda_t)dt + \\ \acute{g}(t,\gamma(t),\Lambda(t),\gamma_t,\Lambda_t)dB(t) + \Lambda(t)dW(t), \end{cases} \quad (3)$$

where $0 \le t \le T$,

We consider numerical formula for the FBDSDDEs on the interval $[t_j, t_{j+1}]$, given as follows:

$$\Psi(t_j) = \zeta + \int_{t_j}^{t_{j+1}} \Gamma(\varrho,\Psi(\varrho),\gamma(\varrho),\Phi(\varrho),\Psi_\varrho,\gamma_\varrho,\Phi_\varrho)d\varrho + \int_{t_j}^{t_{j+1}} \acute{\Gamma}(\varrho,\Psi(\varrho),\Phi(\varrho),\Psi_\varrho,\Phi_\varrho)dB(\varrho) + \int_{t_j}^{t_{j+1}} \Phi(\varrho)dW(\varrho)$$

$$\gamma(t_j) = \gamma(t_{j+1}) + \int_{t_j}^{t_{j+1}} g(\varrho,\Psi(\varrho),\gamma(\varrho),\Lambda(\varrho),\Psi_\varrho\gamma_\varrho,\Lambda_\varrho)d\varrho + \int_{t_j}^{t_{j+1}} \acute{g}(\varrho,\gamma(\varrho),\Lambda(\varrho),\gamma_\varrho,\Lambda_\varrho)dB(\varrho) + \int_{t_j}^{t_{j+1}} \Lambda(\varrho)dW(\varrho) \quad (4)$$

Let $0 \le t \le T$, let $0 = t_0 < t_1 < \cdots < t_m = T, m \ge 1$ be a partition of interval $[0,T]$, we denote $\tau = \Delta t_{j+1=t_{j+1}} - t_j = \frac{T}{m}, 1 \le j \le m, \Delta B(t_{j+1}) = B(t_{j+1}) - B(t_j), \Delta W(t_{j+1}) = W(t_{j+1}) - W(t_j)$ and $\Delta t = max\Delta t_{j+1}$ for $j = 0,1,\dots,m-1, m \ge 1$. Therefore, we consider the following approximation:

$$\begin{cases} \Psi(t) = \zeta + \int_0^t \Gamma(\varrho,\Psi_j^m(\varrho),\gamma_j^m(\varrho),\Phi_j^m(\varrho),{\Psi_j^m}_\varrho,{\gamma_j^m}_\varrho,{\Phi_j^m}_\varrho)d\varrho \\ + \int_0^t \acute{\Gamma}\left(\varrho,\Psi_j^m(\varrho),\Phi_j^m(\varrho),{\Psi_j^m}_\varrho,{\Phi_j^m}_\varrho\right)dB(\varrho) + \int_0^t \Phi_j^m(\varrho)dW(\varrho) \\ \gamma(t) = h(\Psi(T)) + \int_t^T g(\varrho,\Psi_j^m(\varrho),\gamma_j^m(\varrho),\Lambda_j^m(\varrho),{\Psi_j^m}_\varrho,{\gamma_j^m}_\varrho,{\Lambda_j^m}_\varrho)d\varrho \\ + \int_t^T \acute{g}(\varrho,\gamma_j^m(\varrho),\Lambda_j^m(\varrho),{\gamma_j^m}_\varrho,{\Lambda_j^m}_\varrho)dB(\varrho) + \int_t^T \Lambda_j^m(\varrho)dW(\varrho), \end{cases} \quad (5)$$

where $0 \le t \le T, j = 1,\dots m$

Lemma 1. Suppose that assumptions(A1-A3) are fulfilled. Then FBDSDDEs has an unique solution $(\Psi,\Phi,\gamma,\Lambda) \in C_2^2(\Omega,F,P;R^k) \times C_3^2(\Omega,F,P;R^{k\times r}) \times C_2^2(\Omega,F,P;R^d) \times (C_3^2(\Omega,F,P;R^{d\times r})$, and there exists a constant $D > 0$ such that $||\Psi|| + ||\Phi|| + ||\gamma|| + ||\Lambda|| \le D$.

Proof. We consider FDSDDE

$$\Psi(\varrho) = \zeta + \int_0^t \Gamma(\varrho,\Psi(\varrho),\gamma(\varrho),\Phi(\varrho),\Psi_\varrho,\gamma_\varrho,\Phi_\varrho)d\varrho + \int_0^t \acute{\Gamma}((\varrho,\Psi(\varrho),\Phi(\varrho),\Psi_\varrho,,\Phi_\varrho)dB(\varrho) + \int_0^t \Phi(\varrho)dW(\varrho) \quad (6)$$

where $t \in [0,T]$. Let us define the mapping

$\Theta(S,\Lambda) = (\Psi,\Phi): C_2^2([0,T]), R^k \times C_3^2([0,T]), R^{k\times r}) \to C_2^2([0.T]), R^k \times C_3^2([0,T]), R^{k\times r}$

For arbitrary $(S^j,\Lambda^j) \in C_2^2([0.T]), R^k \times C_3^2([0,T]), R^{k\times r})$, we have $(\Psi^j,\Phi^j) = \Theta((S^j,\Lambda^j), j = 1,2$. Let $(\Psi^*,\Phi^*) = (\Psi^1 - \Psi^2, \Phi^1 - \Phi^2)$. By applying Ito's formula to $|\Psi(t)|^2$, yields $E[|\Psi^*(t)|^2] + E\left[\int_0^t |\Psi^*(\varrho)|^2 d\varrho\right] + \left[\int_0^t |\Phi(\varrho)|^2 d\varrho \le 2E[\int_0^t \Psi^*(\varrho)\Gamma^*(\varrho)d\varrho\right] + E\left[\int_0^t |\acute{\Gamma}^*(\varrho)|^2 d\varrho\right]$,

$\Gamma^*(\varrho) = \Gamma(\varrho,\Psi^1(\varrho),\gamma^1(\varrho),\Phi^1(\varrho),\Psi_\varrho^1,\gamma_\varrho^1,\Phi_\varrho^1) - \Gamma(\varrho,\Psi^2(\varrho),\gamma^2(\varrho),\Phi^2(\varrho),\Psi_\varrho^2,\gamma_\varrho^2,\Phi_\varrho^2)$,

$\acute{\Gamma}^*(\varrho) = \acute{\Gamma}(\varrho,\Psi^1(\varrho),\Phi^1(\varrho),,\Psi_\varrho^1,\Phi_\varrho^1) - \acute{\Gamma}(\varrho,\Psi^1(\varrho),\Phi^1(\varrho),,\Psi_\varrho^1,\Phi_\varrho^1)$

From inequality $2ab \le \frac{1}{r}a^2 + rb^2, r > 0$,

there exists constants $r_1,r_2,r_3,r_4 > 0$ such that $2E\left[\int_0^t \Psi^*(\varrho)\Gamma^*(\varrho)d\varrho\right] \le 2N_1E\left[\int_0^t \Psi^*(\varrho)|^2 d\varrho\right] + 2N_1E[\int_0^t |\Psi^*(\varrho)||\gamma^*(\varrho)|d\varrho + 2N_1E\left[\int_0^t |\Psi^*(\varrho)\|\acute{\Phi}(\varrho)|d\varrho\right] \le E\frac{N_1}{r_1}E[\int_0^t |\Psi^*(\varrho)|^2 d\varrho + N_1r_1\left[\int_0^t |\Psi^*(\varrho)|^2 d\varrho\right] + \frac{N_1}{r_2}E\left[\int_0^t |\Psi^*(\varrho)|^2 d\varrho\right] + N_1r_2E\left[\int_0^t |\Phi^*(\varrho)|^2 d\varrho\right] + 2N_1E\left[\int_0^t |\Psi^*(\varrho)|^2 d\varrho\right] + \frac{N_1}{r_3}E\left[\int_0^t |\Psi^*(\varrho)|^2 d\varrho\right] + N_1r_3E\left[\int_0^t |\gamma^*(\varrho)|^2 d\varrho\right] + \frac{N_1}{r_4}E\left[\int_0^t |\Psi^*(\varrho)|^2 d\varrho\right] + N_1r_4E\left[\int_0^t |\Phi^*(\varrho)|^2 d\varrho\right] = \left(\frac{N_1}{r_1} + \frac{N_1}{r_2} + 2N_1 + \frac{N_1}{r_3} + \frac{N_1}{r_4}\right)E\left[\int_0^t |\Psi^*(\varrho)|^2 d\varrho\right] + N_1r_3E\left[\int_0^t |\gamma^*(\varrho)|^2 d\varrho\right] + N_1r_4E\left[\int_0^t |\Phi^*(\varrho)|^2 d\varrho\right] + N_1r_1E\left[\int_0^t |\Psi^*(\varrho)|^2 d\varrho\right] + N_1r_2E\left[\int_0^t |\Phi^*(\varrho)|^2 d\varrho\right]$

and

$E\left[\int_0^t |\acute{\Gamma}^*(\varrho)|^2 d\varrho\right] + \le N_3E\left[\int_0^t |\Psi^*(\varrho)|^2 d\varrho\right] + N_3E\left[\int_0^t |\Phi^*(\varrho)|^2 d\varrho\right]$.

$E[\int_0^t N_3(|S^*|^2 + |\Lambda^*|^2 + R_3(\int_0^t \int_{-t}^0 |S^*(t+\theta)|^2 \alpha d\theta + \int_0^t \int_{-t}^0 |\Lambda^*(t+\theta)|^2 \alpha d\theta + \le N_3E\left[\int_0^t |\acute{\Psi}(\varrho)|^2 d\varrho\right] + N_3E\left[\int_0^t |\acute{\Phi}(\varrho)|^2 d\varrho\right]$

By changing of integration order argument, we obtain

$$\int_0^t \int_{-t}^0 |S^*(\varrho+\theta)|^2 \alpha(d\theta)d\varrho = \int_{-t}^0 \int_0^t |S^*(\varrho+\theta)|^2 d\varrho\alpha(d\theta) = \int_{-t}^0 \int_\theta^{t+\theta} |S^*(\varrho)|^2 d\varrho\alpha(d\theta) \le \beta \int_0^t |S^*(\varrho)|^2 d\varrho \quad (7)$$

and

$\int_0^t \int_{-t}^0 |\Lambda^*(\varrho+\theta)|^2 \alpha(d\theta) d\varrho = \int_{-t}^0 \int_0^t |\Lambda^*(\varrho+\theta)|^2 \, d\varrho\alpha(d\theta) = \int_{-t}^0 \int_\theta^{t+\theta} |\Lambda^*(\varrho)|^2 \, dt\alpha d\theta \ \leq \beta \ \int_0^t |\Lambda^*(\varrho)|^2 d\varrho$ (8)

where $\beta = \int_{-T}^0 \alpha(d\theta)$. From (3.6) and (3.7), we have

$2E\left[\int_0^t \Psi^*(\varrho)\Gamma^*(\varrho)d\varrho\right] + E[\int_0^t N_1(|S^*(\varrho)|^2 + |\Lambda^*(\varrho)|^2)d\varrho] + R_1\beta \, E\int_0^t |S^*(\varrho)|^2 \, d\varrho + R_1\beta \, E\int_0^t |\Lambda^*(\varrho)|^2 d\varrho + \leq \ N_3 E\left[\int_0^t |\acute{\Psi}(\varrho)|^2 d\varrho\right] + N_3 E\left[\int_0^t |\acute{\Phi}(\varrho)|^2 d\varrho\right]$

Let $\pi = \frac{N_1}{r_1} + \frac{N_1}{r_2} + R_1\beta + 2N_1 + \frac{N_1}{r_3} + \frac{N_1}{r_4} + N_3, \mu = N_1 r_4 + N_3$, we have $(1-\pi)E\left[\int_0^t |\Psi^*(\varrho)|^2 d\varrho\right] + (1-\mu)E\left[\int_0^t |\Phi^*(\varrho)|^2 d\varrho\right] \leq N_1 r_3 E\left[\int_0^t |\gamma^*(\varrho)|^2 d\varrho\right] + N_1 r_1 E\left[\int_0^t |\acute{\Psi}(\varrho)|^2 d\varrho\right] + N_1 r_2 E[\int_0^t |\acute{\Phi}(\varrho)|^2 d\varrho]$.

Now, let us define a norm which is equivalent to the space $C_3^2((\Omega, F, P; R^{k\times d})$,

$||\Psi,\Phi|| =)E[\int_0^t (\Psi(\varrho)|^2 + (\Phi(\varrho)|^2)d\varrho])^{\frac{1}{2}}$

and $E[\int_0^t |\gamma^*(\varrho)|^2 d\varrho]^{\frac{1}{2}} < \infty$.

Therefore, the mapping Θ is a contraction map on $C_2^2([0.T], R^k) \times C_3^2([0.T], R^{k\times r})$. By the contraction mapping theorem, there exists a unique fixed point $(\Psi,\Phi) \in C_2^2([0.T], R^k) \times C_3^2([0.T], R^{k\times r})$ such that $\Theta(\Psi,\Phi) = (\Psi,\Phi)$. Thus, there is a unique solution of equation (3.6). Now, we must prove that there exists a constant $D_1 > 0$ such that we set

$$||\Psi||_{C_2^2([0.T],R^k)} + ||\Phi||_{G_2^2([0.T],R^{k\times r})} \leq D_1. \quad (9)$$

$|\Psi(t)|^2 + \int_0^t |\Psi(\varrho)|^2 \, d\varrho + \int_0^t |\Phi(\varrho)|^2 \, d\varrho = |\zeta|^2 + 2\int_0^t \Psi(\varrho)\Gamma(\varrho, \Psi(\varrho), \gamma(\varrho), \Phi(\varrho), \Psi_\varrho, \gamma_\varrho, \Phi_\varrho)d\varrho$

$+2 \ \int_0^t \Psi(\varrho)\acute{\Gamma}(\varrho, \Psi(\varrho), \Phi(\varrho), \Psi_\varrho, \Phi_\varrho)dB(\varrho) + 2 \ \int_0^t \Psi(\varrho)\Phi(\varrho), \Psi_\varrho, \Phi_\varrho dW(\varrho) + \int_0^t |\acute{\Gamma}(\varrho, \Psi(\varrho), \Phi(\varrho), \Psi_\varrho, \Phi_\varrho)|^2 d\varrho.$

By taking the expectation, there exist constants $N_1, k_3, r_1, r_2, r_3, r_4, r_5 > 0$ such that

$E\left|\Psi(t)|^2 + E\left[\int_0^t |\Psi(\varrho)\varrho|^2 d\varrho\right] + E[\int_0^t |\Phi(\varrho)\varrho|^2 d\varrho] = E|\zeta|^2 + 2E[\int_0^t \Psi(\varrho)\Gamma(\varrho, \Psi(\varrho), \gamma(\varrho), \Phi(\varrho), \Psi_\varrho, \gamma_\varrho, \Phi_\varrho\right) d\varrho\right]$

$+E[\int_0^t |\acute{\Gamma}(\varrho, \Psi(\varrho), \Phi(\varrho), \Psi_\varrho, \Phi_\varrho)|^2 d\varrho \leq E|\zeta|^2 + 2N_1 E\left[\int_0^t \Psi(\varrho)\Gamma(\varrho, \zeta_0, 0{,}0{,}0{,}0{,}0{,}0\right) d\varrho]$

$2N_1 E\left[\int_0^t |\Psi(\varrho)|(E|\Psi(\varrho)|^2)^{\frac{1}{2}} d\varrho\right] + 2N_1 E\left[\int_0^t |\Psi(\varrho)|(E|\Phi(\varrho)|^2)^{\frac{1}{2}} d\varrho\right] + 2N_1 E\left[\int_0^t |\Psi(\varrho)|^2 \, d\varrho\right]$

$+2N_1 E[\int_0^t |\Psi(\varrho)||\gamma(\varrho)|d\varrho] + 2N_1 E[\int_0^t |\Psi(\varrho)||\Phi(\varrho)|d\varrho] + N_3 E[\int_0^t |\Gamma^*(\varrho, 0{,}0{,}0{,}0)|^2 d\varrho] + N_3 \, E[\int_0^t |\Psi(\varrho)|^2 d\varrho + N_3 E[\int_0^t |\Phi(\varrho)|^2 d\varrho \leq |\zeta|^2$

$+\frac{2N_1}{r_1} E\left[\int_0^t |\Psi(\varrho)|^2 d\varrho\right] + 2N_1 r_1 \, E\left[\int_0^t |\Gamma(\varrho, \zeta_0, 0{,}0{,}0{,}0{,}0{,}0\right) |^2 d\varrho\right] + \frac{2N_1}{r_2} E[\int_0^t |\Psi(\varrho)|^2 d\varrho] + 2N_1 \, r_2 E \int_0^t |\Psi(\varrho)|^2 d\varrho] + \frac{2N_1}{r_3} E[\int_0^t |\Psi(\varrho)|^2 d\varrho]$

$+2N_1 r_3 E[\int_0^t |\Phi(\varrho)|^2 d\varrho] + 2N_1 E[\int_0^t |\Psi(\varrho)|^2 d\varrho] + \frac{2N_1}{r_4} E[\int_0^t |\Phi(\varrho)|^2 d\varrho] + 2N_1 r_4 E[\int_0^t |\gamma(\varrho)|^2 d\varrho]$

$\frac{2N_1}{r_5} E[\int_0^t |\Psi(\varrho)|^2 d\varrho] + 2N_1 r_5 E\,[\int_0^t |\Phi(\varrho)|^2 d\varrho] + N_3 \, E\left[\int_0^t |\acute{\Gamma}(\varrho, 0{,}0{,}0{,}0)|^2 \, d\varrho\right] + N_3 E[\int_0^t |\Psi(\varrho)|^2 d\varrho] + N_3 E[\int_0^t |\Phi(\varrho)|^2 d\varrho] = E|\zeta|^2 + (\frac{2N_1}{r_1} + \frac{2N_1}{r_2} + 2N_1 r_2 + \frac{2N_1}{r_3} + 2N_1 + \frac{2N_1}{r_4} + \frac{2N_1}{r_5} + N_3)E[\int_0^t |\Psi(\varrho)|^2 d\varrho]$

$+(2N_1 r_3 + 2N_1 r_5 + N_3)E\left[\int_0^t |\Phi(\varrho)|^2 d\varrho\right] + 2N_1 r_4 \, E\int_0^t |\gamma(\varrho)|^2 d\varrho] + 2N_1 r_1 \, E\left[\int_0^t |\Gamma(\varrho, \zeta_0, 0{,}0{,}0{,}0{,}0{,}0\right) |^2 d\varrho\right] + N_3 E\left[\int_0^t |\acute{\Gamma}(\varrho, \zeta_0, 0{,}0{,}0{,}0{,}0{,}0\right) |^2 d\varrho$.

More simplified

$E|\Psi(t)|^2 + R_1 E[\int_0^t |\Psi(\varrho)|^2 d\varrho] + R_2 E[\int_0^t |\Phi(\varrho)|^2 d\varrho] \leq E|\zeta|^2 + 2N_1 r_4 E[\int_0^t |\gamma(\varrho)|^2 d\varrho] +$

$2N_1 r_1 E\left[\int_0^t |\Gamma(\varrho, \zeta_0, 0{,}0{,}0{,}0{,}0{,}0\right) |^2 d\varrho + N_3 E\left[\int_0^t |\acute{\Gamma}(\varrho, \zeta_0, 0{,}0{,}0{,}0{,}0{,}0\right) |^2 d\varrho] \leq D_1,$

where D_1 depends on

$$R_1, R_2, E|\zeta|^2, E\left[\int_0^t |\Gamma(\varrho, \zeta_0, 0{,}0{,}0{,}0{,}0{,}0\right) |^2 d\varrho,$$

$\left[\int_0^t |\acute{\Gamma}(\varrho, \zeta_0, 0{,}0{,}0{,}0{,}0{,}0\right) |^2 d\varrho],$

and $E\left[\int_0^t |\gamma(\varrho)|^2 \, d\varrho\right]$. Applying the Burkholder-Davis-Gund inequality and Young 's inequality, using the same technique above, for every $t \in [0,T]$, such that

$E[sup_{z\in[0,t]} |\Psi(z)|^2 \leq T\{E|\zeta|^2 + 2N_1 r_4 E[\int_0^t |\gamma(\varrho)|^2 d\varrho] \, 2N_1 r_1 E\left[\int_0^t |\Gamma(\varrho, \zeta_0, 0{,}0{,}0{,}0{,}0{,}0\right) |^2 d\varrho + N_3 E\left[\int_0^t |\acute{\Gamma}(\varrho, \zeta_0, 0{,}0{,}0{,}0{,}0{,}0\right) |^2 d\varrho] \leq D_1.$

This means that inequality (7) is fulfilled. Now, we consider the backward FDSDDE

$$\Upsilon(t) = h(x(T) + \int_t^T g(\varrho,\gamma(\varrho),\Lambda(\varrho),\gamma_\varrho,\Lambda_\varrho)d\varrho + \int_0^T \acute{g}((\varrho,\gamma(\varrho),\Lambda(\varrho),\gamma_\varrho,\Lambda_\varrho)dB(\varrho) + \int_t^T \Lambda(\varrho)dw(\varrho) \quad (10)$$

Where $0 \le t \le T$, we first conclude from the proof of the existence and uniqueness of the solution (Ψ,Φ) for the forward FDSDDE (3.7). Once we Know (Ψ,Φ), the backward FDSDDE becomes a classical equation. Therefore, using the same proof technique for FDSDDE, we can prove that the backward FDSDDE has a unique solution (γ,Λ). Then there exists $D_2 > 0$ such that

$$\|\gamma\|_{C_2^2} + \|\Psi\|_{C_3^2} \le D_2 \quad (11)$$

Let $D = D_1 + D_2$, and by combining (6) and (8), we have

$$\|\Psi\|_{C_2^2} + \|\Phi\|_{C_3^2} + \|\gamma\|_{C_2^2} + \|\Lambda\|_{C_3^2} \le M$$

MAXIMAL SOLUTION

The preferred spelling of the word "acknowledgment" in America is without an "e" after the "g." Avoid the stilted expression "one of us (R. B. G.) thanks ...". Instead, try "R. B. G. thanks...". Put sponsor acknowledgments in the unnumbered footnote on the first page.

Definition 1. [27]

Let $X(t)$ be a solution of a stochastic differential equation. $X(t)$ is a maximal solution if for every $\Psi(t)$ such that $E(\Psi^2)) \le E(X^2))$.

Definition 2. [27] Let $\Psi,\gamma \in C_2^2([0,T]),R^k)$ such that $\|\Psi(t)\|^2 < \|\gamma(t)\|^2$.

A function Γ:

$C_2^2([0,T]),\Omega,F,\mathrm{P},R^k) \to C_2^2([0,T]),\Omega,F,\mathrm{P},R^k)$

is stochastic increasing if $\|\Gamma(t,\Psi(t)\|^2 < \|\Gamma(t,\gamma(t)\|^2$.

Definition 3. [27] Let $\Psi,\gamma \in C_2^2([0,T]),R^k)$

such that $\|\Psi(t)\|^2 < \|\gamma(t)\|^2$.

A function $\Gamma: C_2^2([0,T]),\Omega,F,\mathrm{P},R^k) \to$

$C_2^2([0,T]),\Omega,F,\mathrm{P},R^k)$ is stochastic decreasing if

$\|\Gamma(t,\Psi(t)\|^2 > \|\Gamma(t,\gamma(t)\|^2$.

Lemma 2. Let $\Gamma^j = \Gamma^j(t,\kappa,\Psi,\gamma,\Phi), = 1,2$ be two functions satisfying (A1,i). Let (Ψ^1,Φ^1) and (Ψ^2,Φ^2) be the functions of the forward equation of the FBDSDDEs. If $\zeta^1 \le \zeta^2$ and $\Gamma^1 \le \Gamma^2$, then $\Psi^1(t) \le \Psi^2(t)$ for every $t \in [0,T]$.

Proof.

Without loss of generality, let $b = r = 1$ and Γ^1 satisfy A1 and A4. We define $\acute{\zeta} = \zeta^1 - \zeta^2$, then $\acute{\zeta} = 0, \left(\acute{\Psi}(t),\acute{\Phi}(t)\right) = \left(\Psi^1(t) - \Psi^2(t),\Phi^1(t) - \Phi^2(t)\right), t \in [0,T]$. From Ito's formula and the inequality $2ab \le \frac{1}{r}a^2 + rb^2, r > 0$, there is

$$E[(\acute{\Psi}(t))^2] + E\left[\int_0^t \Theta_{\{\acute{\Psi}_{(\varrho)\ge 0}\}}\left|\acute{\Phi}(\varrho)\right|^2 d\varrho\right] = E[(\acute{\zeta})^2]$$

$$+2E[\int_0^t \acute{\Psi}(\varrho)(\Gamma^1(\varrho,\Psi^1(\varrho),\gamma^1(\varrho),\Phi^1(\varrho),\Psi_\varrho^1,\gamma_\varrho^1,\Phi_\varrho^1)$$
$$-\Gamma^2(\varrho,\Psi^2(\gamma^2(\varrho),\Phi^2(\varrho),\Psi_\varrho^1,\gamma_\varrho^1,\Phi_\varrho^1)d\varrho]$$
$$+E\left[\int_0^t \Theta_{\{\acute{\Psi}_{(\varrho)\ge 0}\}}\left|\begin{matrix}\acute{\Gamma}(\varrho,\Psi^1(\varrho),\Phi^1(\varrho),\Psi_\varrho^1,\Phi_\varrho^1) - \\ \acute{\Gamma}(\varrho,\Psi^2(\varrho),\Phi^2(\varrho),\Psi_\varrho^2,\Phi_\varrho^2)d\varrho\end{matrix}\right|\right]$$

$$\le E[(\acute{\zeta})^2] + E[\int_0^t \acute{\Psi}(\varrho)(\Gamma^1(\varrho,\Psi^1(\varrho),\gamma^1(\varrho),\Phi^1(\varrho),\Psi_\varrho^1,\gamma_\varrho^1,\Phi_\varrho^1)$$
$$-\Gamma^1(\varrho,\Psi^1(\varrho),\gamma^1(\varrho),\Phi^1(\varrho),\Psi_\varrho^1,\gamma_\varrho^1,\Phi_\varrho^1) + \Gamma^1(\varrho,\Psi^1(\varrho),\gamma^1(\varrho),\Phi^1(\varrho),\Psi_\varrho^1,\gamma_\varrho^1,\Phi_\varrho^1)$$
$$-\Gamma^1(\varrho,\Psi^2(\varrho),\gamma^2(\varrho),\Phi^2(\varrho),\Psi_\varrho^2,\gamma_\varrho^2,\Phi_\varrho^2) + \Gamma^1(\varrho,\Psi^2(\varrho),\gamma^2(\varrho),\Phi^2(\varrho),\Psi_\varrho^2,\gamma_\varrho^2,\Phi_\varrho^2)$$
$$-\Gamma^2(\varrho,\Psi^2(\varrho),\gamma^2(\varrho),\Phi^2(\varrho),\Psi_\varrho^2,\gamma_\varrho^2,\Phi_\varrho^2))\, d\varrho]$$
$$+E\left[\int_0^t \Theta_{\{\Psi^*_{(\varrho)\ge 0}\}}\left|\begin{matrix}\Gamma^1(\varrho,\Psi^1(\varrho),\Phi^1(\varrho),\Psi_\varrho^1,\Phi_\varrho^1) - \\ \Gamma^1(\varrho,\Psi^2(\varrho),\Phi^2(\varrho),\Psi_\varrho^2,\Phi_\varrho^2)|^2 d\varrho\end{matrix}\right.\right]$$

$$\le E\left[\int_0^t(\acute{\Psi}(\varrho)\left(\left(2N_5\left(E\left|\acute{\Psi}(\varrho)\right|^2\right)^{\frac{1}{2}}\right) + 2N_1\left|\acute{\Psi}(\varrho)\right| + 2N_1|\acute{\gamma}(\varrho)| + 2N_1\left|\acute{\Phi}(\varrho)\right|\right)\right)d\varrho\right]$$
$$+N_3E[\int_0^t \Theta_{\{\acute{\Psi}_{(\varrho)\ge 0}\}}(\,|\Psi(\varrho)|^2 + |\Phi(\varrho)|^2\,)d\varrho]$$

$$\le E[\int_0^t(\,2N_5\,(\acute{\Psi}(\varrho))^2 + 2N_1(\acute{\Psi}(\varrho))^2 + \frac{N_1}{r}(\acute{\Psi}(\varrho))^2 + 2N_1|\acute{\gamma}(\varrho)|^2 + \frac{N_1}{r}(\acute{\Psi}(\varrho))^2$$
$$+2N_1|\acute{\Phi}(\varrho)|^2)d\varrho] + N_3E[\int_0^t|\acute{\Psi}(\varrho)|^2 d\varrho] + N_3E[\int_0^t \Theta_{\{\acute{\Psi}(\varrho)\ge 0\}}|\acute{\Phi}(\varrho)|^2 d\varrho]$$

$$\le \left(2N_5 + 2N_1 + \frac{N_1}{r} + \frac{N_1}{r} + N_3\right)E\left[\int_0^t(\acute{\Psi}(\varrho))^2 d\varrho\right] + N_1rE\left[\int_0^t|\acute{\gamma}(\varrho)|^2 d\varrho\right] + (N_1r + N_3)E[\int_0^t \Theta_{\{\acute{\Psi}(\varrho)\ge 0\}}\left(\left|\acute{\Phi}(\varrho)\right|^2 d\varrho\right]$$

$$\le D\left(2N_5 + 2N_1 + \frac{2N_1}{r} + N_3\right)E\left[\int_0^t(\acute{\Psi}(\varrho))^2 d\varrho\right] + (N_1r + N_3)E[\int_0^t \Theta_{\{\acute{\Psi}(\varrho)\ge 0\}}\left(\left|\acute{\Phi}(\varrho)\right|^2 d\varrho\right]$$

where $D > 0$. Since $E\left[\int_0^t \acute{\gamma}(\varrho)|^2 d\varrho\right] < \infty$. From Gronwall is inequality, we have $E[(\acute{\Psi}(\varrho))^2] = 0, t \in [0,T]$, therefore, we have $\Psi^1(t) \le \Psi^2(t)$, P. a.s.

Lemma 3. Let $g^j = g^j(t,\kappa,\Psi,\gamma,\Lambda) = 1,2$ two functions satisfying (A1,ii). Let (γ^1,Λ^1) and (γ^2,Λ^2) be the solution of backward of FBDSDDEs system. If $\Psi^1(T) \le \Psi^2(T), g^1 \le g^2$, then $\gamma^1(t) \le \gamma^2(t)$ for every $t \in [0,T]$.

Proof. Without loss of generalized, let $d = r = 1$ and g^1 satisfy A1 and A5. We define $\left(\acute{\gamma}(t), \acute{\Lambda}(t)\right) = \left(\gamma^1(t) - \gamma^2(t), \Lambda^1(t) - \Lambda^2(t)\right)$, $t \in [0,T]$. From Ito's formula and the inequality $2ab \leq \frac{1}{r}a^2 + rb^2, r > 0$, there is

$$E[(\acute{\gamma}(t))^2] + E[\int_t^T \Theta_{\{\acute{\gamma}(\varrho)\geq 0\}}|\acute{\Lambda}(\varrho)|^2 d\varrho] \leq 2E[\int_t^T \acute{\gamma}(\varrho)\left(g^1(\varrho, \Psi^1(\varrho), \gamma^1(\varrho), \Lambda^1(\varrho), \Psi^1_\varrho, \gamma^1_\varrho, \Lambda^1_\varrho) - g^2(\varrho, \Psi^2(\varrho), \gamma^2(\varrho), \Lambda^2(\varrho), \Psi^2_\varrho, \gamma^2_\varrho, \Lambda^2_\varrho)\right) d\varrho + E[\int_t^T \Theta_{\{\acute{\gamma}\geq 0\}}|\acute{g}\left(\varrho, \Psi^1(\varrho), \Lambda^1(\varrho), \Psi^1_\varrho, \Lambda^1_\varrho\right) - \acute{g}(\varrho, \Psi^2(\varrho), \Lambda^2(\varrho), \Psi^2_\varrho, \Lambda^2_\varrho) d\varrho \leq$$

$$E[\int_t^T \acute{\gamma}(\varrho)(g^1(\varrho, \Psi^1(\varrho), \gamma^1(\varrho), \Lambda^1(\varrho), \Psi^1_\varrho, \gamma^1_\varrho, \Lambda^1_\varrho) - g^1(\varrho, \Psi^1(\varrho), \gamma^1(\varrho), \Lambda^1(\varrho), \Psi^1_\varrho, \gamma^1_\varrho, \Lambda^1_\varrho) + g^1(\varrho, \Psi^1(\varrho), \gamma^1(\varrho), \Lambda^1(\varrho), \Psi^1_\varrho, \gamma^1_\varrho, \Lambda^1_\varrho) -$$

$$g^1(\varrho, \Psi^2(\varrho), \gamma^2(\varrho), \Lambda^2(\varrho), \Psi^2_\varrho, \gamma^2_\varrho, \Lambda^2_\varrho) + g^1((\varrho, \Psi^2(\varrho), \gamma^2(\varrho), \Lambda^2(\varrho), \Psi^2_\varrho, \gamma^2_\varrho, \Lambda^2_\varrho) - g^2(\varrho, \Psi^2(\varrho), \gamma^2(\varrho), \Lambda^2(\varrho), \Psi^2_\varrho, \gamma^2_\varrho, \Lambda^2_\varrho) d\varrho + E[\int_t^T \Theta_{\{\acute{\gamma}(\varrho)\geq 0\}}|g^*\left(\varrho, \gamma^1(\varrho), \Lambda^1(\varrho), \gamma^1_\varrho, \Lambda^1_\varrho\right) - g^*(\varrho, \gamma^2(\varrho), \Lambda^2(\varrho), \gamma^2_\varrho, \Lambda^2_\varrho) d\varrho \leq$$

$$2E\left[\int_t^T (\acute{\gamma}(\varrho)\left(\left(N_6\left(E\left|\acute{\gamma}(\varrho)|^2\right)^{\frac{1}{2}}\right) + N_2\left|\acute{\Psi}(\varrho)\right| + N_2|\acute{\gamma}(\varrho)| + N_2\left|\acute{\Lambda}(\varrho)\right|\right)\right) d\varrho\right]$$

$$+N_4 E[|\acute{\gamma}(\varrho)|^2 + N_4 E[\int_t^T \Theta_{\{\acute{\gamma}(\varrho)\geq 0\}}|\acute{\Lambda}(\varrho)|^2 d\varrho] \leq E[\int_t^T (2N_6(\acute{\gamma}(\varrho))^2 + \frac{N_2}{r}|\acute{\gamma}(\varrho)|^2 + N_2 r|\acute{\Psi}(\varrho)|^2 + 2N_2 r\left|\acute{\gamma}(\varrho)|^2 + \frac{N_2}{r}\right|\acute{\gamma}(\varrho)|^2 + N_2 r|\acute{\Lambda}(\varrho)|^2 + N_4 E\left[\int_t^T |\acute{\gamma}(\varrho)|^2 d\varrho\right] + N_4 E[\int_t^T \Theta_{\{\acute{\gamma}(\varrho)\geq 0\}}|\acute{\Lambda}(\varrho)|^2 d\varrho] \leq \left(2N_6 + \frac{N_2}{r} + 2N_2 + \frac{N_2}{r} + N_4\right) E\left[\int_t^T |\acute{\gamma}(\varrho)|^2 d\varrho\right] + N_2 r E[\int_t^T |\acute{\Psi}(\varrho)|^2 d\varrho\Big]$$

$$+(N_2 r + N_4)E\left[\int_t^T \Theta_{\{\acute{\gamma}(\varrho)\geq 0\}}|\acute{\Lambda}(\varrho)|^2 d\varrho\right] \leq N\left(2N_6 + \frac{N_2}{r} + 2N_2 + \frac{2N_2}{r} + 2N_2 + N_4\right) E\left[\int_t^T |\acute{\gamma}(\varrho)|^2 d\varrho\right] + (2N_2 + N_4)E\left[\int_t^T \Theta_{\{\acute{\gamma}(\varrho)\geq 0\}}|\acute{\Lambda}(\varrho)|^2 d\varrho\right],$$

where $N > 0$. Since $E[\int_t^T |\acute{\Psi}(\varrho)|^2 d\varrho] < \infty$.From Gronwall is inequality, we have $E[(\acute{\gamma}(t))^2] = 0, t \in [0,T]$. Therefore, we have $\gamma^1(t) \leq \gamma^2(t)$, p.a.s.

Theorem 1. Suppose that $\Gamma, \acute{\Gamma}, g$, and $\acute{g}$ are stochastically increasing function. Under assumptions (A1-A5), the general system of FBDSDDEs has a maximal solution$(\Psi, \Phi, \gamma, \Lambda)$.

Proof. Let $\pi > 0$. We consider the forward equation of the FBDSDDEs system as follows:

$$\Psi_\pi(t) = \Psi(0) + \int_0^t \Gamma_\pi(\varrho, \Psi^m_\pi(\varrho), \gamma^m_\pi(\varrho), \Phi^m_\pi(\varrho), \Psi^m_{\pi\,\varrho}, \gamma^m_{\pi\,\varrho}, \Phi^m_{\pi\,\varrho}) d\varrho + \int_0^t \acute{\Gamma}_\pi(\varrho, \Psi^m_\pi(\varrho), \Phi^m_\pi(\varrho), \Psi^m_{\pi\,\varrho}, \Psi^m_{\pi\,\varrho}) dB(\varrho) + \int_0^t \Phi^m_\pi(\varrho)\, dW(\varrho).$$

Let is put

$$\Gamma_\pi\left(\varrho, \Psi^m_\pi(\varrho), \gamma^m_\pi(\varrho), \Phi^m_\pi(\varrho), \Psi^m_{\pi\,\varrho}, \gamma^m_{\pi\,\varrho}, \Phi^m_{\pi\,\varrho}\right) = \Gamma\left(\varrho, \Psi^m_\pi(\varrho), \gamma^m_\pi(\varrho), \Phi^m_\pi(\varrho), \Psi^m_{\pi\,\varrho}, \gamma^m_{\pi\,\varrho}, \Phi^m_{\pi\,\varrho}\right) + \pi$$

$$\acute{\Gamma}_\pi\left(\varrho, \Psi^m_\pi(\varrho), \Phi^m_\pi(\varrho), \Psi^m_{\pi\,\varrho}, \Psi^m_{\pi\,\varrho}\right) = \acute{\Gamma}\left(\varrho, \Psi^m_\pi(\varrho), \Phi^m_\pi(\varrho), \Psi^m_{\pi\,\varrho}, \Psi^m_{\pi\,\varrho}\right) + \pi.$$

Therefore, if $\pi_1$and π_2 are chosen such that $0 < \pi_2 < \pi_1 < \pi$, we deduce

$$\Psi_{\pi_1}(t) = \Psi(0) + \int_0^t (\Gamma(\varrho, \Psi^m_{\pi_1}(\varrho), \gamma^m_{\pi_1}(\varrho), \Phi^m_{\pi_1}(\varrho), \Psi^m_{\pi_1\varrho}, \gamma^m_{\pi_1\varrho}, \Phi^m_{\pi_1\varrho}) + \pi_1) d\varrho$$

$$+ \int_0^t (\acute{\Gamma}\left(\varrho, \Psi^m_{\pi_1}(\varrho), \Phi^m_{\pi_1}(\varrho), \Psi^m_{\pi_1\varrho}, \Phi^m_{\pi_1\varrho}\right) dB(\varrho) + \int_0^t \Phi^m_{\pi_1}(\varrho) dW(\varrho).$$

By lemma 3, we have

$$\left|\left|\Psi_{\pi_2}(t)\right|\right|^2 = ||\Psi(0) + \int_0^t (\Gamma(\varrho, \Psi^m_{\pi_2}(\varrho), \gamma^m_{\pi_2}(\varrho), \Phi^m_{\pi_2}(\varrho), \Psi^m_{\pi_2\varrho}, \gamma^m_{\pi_2\varrho}, \Phi^m_{\pi_2\varrho}\Big) + \pi_2) d\varrho$$

$$\int_0^t (\acute{\Gamma}(\varrho, \Psi^m_{\pi_2}(\varrho), \Phi^m_{\pi_2}(\varrho), \Psi^m_{\pi_2\varrho}, \Phi^m_{\pi_2\varrho}) + \pi_2) dB(\varrho) + \int_0^t \Phi^m_{\pi_2}(\varrho) dW(\varrho)||^2 \leq ||\Psi_{\pi_1}(t) + \int_0^t (\Gamma(\varrho, \Psi^m_{\pi_1}(\varrho), \gamma^m_{\pi_1}(\varrho), \Phi^m_{\pi_1}(\varrho), \Psi^m_{\pi_1\varrho}, \gamma^m_{\pi_1\varrho}, \Phi^m_{\pi_1\varrho}) + \pi_1) d\varrho$$

$$+ \int_0^t (\acute{\Gamma}\left(\varrho, \Psi^m_{\pi_1}(\varrho), \Phi^m_{\pi_1}(\varrho), \Psi^m_{\pi_1\varrho}, \Phi^m_{\pi_1\varrho} + \pi_1\right) dB(\varrho) + \int_0^t \Phi^m_{\pi_2}(\varrho) dW(\varrho)$$

$$-\Big[||\Psi(0) + \int_0^t (\Gamma(\varrho, \Psi^m_{\pi_2}(\varrho), \gamma^m_{\pi_2}(\varrho), \Phi^m_{\pi_2}(\varrho), \Psi^m_{\pi_2\varrho}, \gamma^m_{\pi_2\varrho}, \Phi^m_{\pi_2\varrho}\Big) + \pi_2) d\varrho + \int_0^t (\acute{\Gamma}\left(\varrho, \Psi^m_{\pi_2}(\varrho), \Phi^m_{\pi_2}(\varrho), \Psi^m_{\pi_2\varrho}, \Phi^m_{\pi_2\varrho}, \pi_2\right) dB(\varrho) + \int_0^t \Phi^m_{\pi_1}(\varrho) dW(\varrho)]||^2 \leq$$

$$\left|\left|\Psi_{\pi_1}(t)\right|\right|^2 + \int_0^t || (\Gamma\left(\varrho, \Psi^m_{\pi_1}(\varrho), \gamma^m_{\pi_1}(\varrho), \Phi^m_{\pi_1}(\varrho), \Psi^m_{\pi_1\varrho}, \gamma^m_{\pi_1\varrho}, \Phi^m_{\pi_1\varrho}\right) - (\Gamma(\varrho, \Psi^m_{\pi_2}(\varrho), \gamma^m_{\pi_2}(\varrho), \Phi^m_{\pi_2}(\varrho), \Psi^m_{\pi_2\varrho}, \gamma^m_{\pi_2\varrho}, \Phi^m_{\pi_2\varrho})||^2 d\varrho$$

$\int_0^t ||\acute{\Gamma}(\varrho, \Psi_{\pi_1}^m(\varrho), \Phi_{\pi_1}^m(\varrho), \Psi_{\pi_{1_\varrho}}^m, \Phi_{\pi_{1_\varrho}}^m) - \Gamma\acute{}(\varrho, \Psi_{\pi_2}^m(\varrho), \Phi_{\pi_2}^m(\varrho), \Psi_{\pi_{2_\varrho}}^m, \Phi_{\pi_{2_\varrho}}^m||^2 +$

$\int_0^t ||\Phi_{\pi_2}^m(\varrho) - \Phi_{\pi_1}^m(\varrho)||^2 d\varrho + 2\int_0^t |\pi_1 - \pi_2| d\varrho.$

Because π_1 and π_2 are chosen, it follows that$|\pi_1 - \pi_2| \to 0$. Also, because Γ and $\acute{\Gamma}$ are stochastically increasing functions, we have

$||\Gamma\left(\varrho, \Psi_{\pi_1}^m(\varrho), \gamma_{\pi_1}^m(\varrho), \Phi_{\pi_1}^m(\varrho), \Psi_{\pi_{1_\varrho}}^m, \gamma_{\pi_{1_\varrho}}^m, \Phi_{\pi_{1_\varrho}}^m\right) -$

$\Gamma\left(\varrho, \Psi_{\pi_2}^m(\varrho), \gamma_{\pi_2}^m(\varrho), \Phi_{\pi_2}^m(\varrho), \Psi_{\pi_{2_\varrho}}^m, \gamma_{\pi_{2_\varrho}}^m, \Phi_{\pi_{2_\varrho}}^m\right)||^2 \to 0,$

$||\acute{\Gamma}\left(\varrho, \Psi_{\pi_1}^m(\varrho), \Phi_{\pi_1}^m(\varrho),, \Psi_{\pi_{1_\varrho}}^m, \Phi_{\pi_{1_\varrho}}^m\right) -$
$\acute{\Gamma}\left(\varrho, \Psi_{\pi_2}^m(\varrho), \Phi_{\pi_2}^m(\varrho),, \Psi_{\pi_{2_\varrho}}^m, \Phi_{\pi_{2_\varrho}}^m\right)||^2 \to 0.$

Following the same technique above, we have

$\int_0^t ||\Psi_{\pi_2}^m(\varrho) - \Psi_{\pi_1}^m(\varrho)||^2 d\varrho \to 0$ as $m \to \infty$

Hence, $||\Psi_{\pi_1}(t)||^2 \leq ||\Psi_{\pi_2}(t)||^2$. For $\pi_m \leq \pi_{m-1} \leq \cdots \leq \pi_2 \leq \pi_1 \leq \pi$, we have

$||\Psi_{\pi_m}(t)||^2 \leq ||\Psi_{\pi_{m-1}}(t)||^2 \leq \cdots \leq ||\Psi_{\pi_2}(t)||^2 \leq ||\Psi_{\pi_1}(t)||^2 \leq ||\Psi_\pi(t)||^2.$

Now, let $\pi > 0$. We consider the backward equation of FBDSDDEs system as follows:

$\gamma_\gamma(t) = \gamma(T) + \int_t^T g_\gamma\left(\varrho, \Psi_\gamma^m(\varrho), \gamma_\gamma^m(\varrho), \Lambda_\gamma^m(\varrho),, \Psi_{\gamma_\varrho}^m, \gamma_{\gamma_\varrho}^m, \Lambda_{\gamma_\varrho}^m\right) d\varrho +$

$\int_t^T \acute{g}_\gamma\left(\varrho, \gamma_\gamma^m(\varrho), \Lambda_\gamma^m(\varrho), \gamma_{\gamma_\varrho}^m, \Lambda_{\gamma_\varrho}^m\right) dB(\varrho) + \int_t^T \Lambda_\gamma^m(\varrho) dW(\varrho).$

For every chosen small real number γ_1 and γ_2 such that $0 < \gamma_2 < \gamma_1 < \gamma$, we deduce

$\gamma_\gamma(t) = \gamma(T) + \int_t^T \left(g\left(\varrho, \Psi_{\gamma_1}^m(\varrho), \gamma_{\gamma_1}^m(\varrho), \Lambda_{\gamma_1}^m(\varrho), \Psi_{\gamma_{1_\varrho}}^m, \gamma_{\gamma_{1_\varrho}}^m, \Lambda_{\gamma_{1_\varrho}}^m\right) + \gamma_1\right) d\varrho + \int_t^T (\acute{g}(\gamma_{\gamma_1}^m(\varrho), \Lambda_{\gamma_1}^m(\varrho), \gamma_{\gamma_{1_\varrho}}^m, \Lambda_{\gamma_{1_\varrho}}^m) + \gamma_1) dB(\varrho) + \int_t^T \Lambda_{\gamma_1}^m(\varrho) dW(\varrho).$

By lemma 3, we have

$\left\|\gamma_{\gamma_2}\right\|^2 = \left\|\gamma(T) + \int_t^T (g(\varrho, \Psi_{\gamma_2}^m(\varrho), \gamma_{\gamma_2}^m(\varrho), \Lambda_{\gamma_2}^m(\varrho), \Psi_{\gamma_{2_\varrho}}^m, \gamma_{\gamma_{2_\varrho}}^m, \Lambda_{\gamma_{2_\varrho}}^m\right) + \gamma_2\Big) d\varrho + \int_t^T (\acute{g}(\varrho, \gamma_{\gamma_2}^m(\varrho), \Lambda_{\gamma_2}^m(\varrho), \gamma_{\gamma_{2_\varrho}}^m, \Lambda_{\gamma_{2_\varrho}}^m) + \gamma_2) dB(\varrho) + \int_t^T \Lambda_{\gamma_2}^m(\varrho) dW(\varrho)\|^2 \leq \gamma_{\gamma_1}(t) +$

$\int_t^T \left(g\left(\varrho, \Psi_{\gamma_1}^m(\varrho), \gamma_{\gamma_1}^m(\varrho), \Lambda_{\gamma_1}^m(\varrho), \Psi_{\gamma_{1_\varrho}}^m, \gamma_{\gamma_{1_\varrho}}^m, \Lambda_{\gamma_{1_\varrho}}^m\right) + \gamma_1\right) d\varrho + \int_t^T (\acute{g}(\gamma_{\gamma_1}^m(\varrho), \Lambda_{\gamma_1}^m(\varrho), \gamma_{\gamma_{1_\varrho}}^m, \Lambda_{\gamma_{1_\varrho}}^m) + \gamma_1) dB(\varrho) + \int_t^T \Lambda_{\gamma_2}^m(\varrho) dW(\varrho)\|^2 -$

$\Big[\int_t^T (g(\varrho, \Psi_{\gamma_2}^m(\varrho), \gamma_{\gamma_2}^m(\varrho), \Lambda_{\gamma_2}^m(\varrho), \Psi_{\gamma_{2_\varrho}}^m, \gamma_{\gamma_{2_\varrho}}^m, \Lambda_{\gamma_{2_\varrho}}^m\Big) + \gamma_2) d\varrho + \int_t^T (\acute{g}(\varrho, \gamma_{\gamma_2}^m(\varrho), \Lambda_{\gamma_2}^m(\varrho), \gamma_{\gamma_{2_\varrho}}^m, \Lambda_{\gamma_{2_\varrho}}^m) + \gamma_2) dB(\varrho) + \int_t^T \Lambda_{\gamma_1}^m(\varrho) dW(\varrho)\|^2 \leq \gamma_{\gamma_1}(t) +$

$\int_t^T \|g(\varrho, \Psi_{\gamma_1}^m(\varrho), \gamma_{\gamma_1}^m(\varrho), \Lambda_{\gamma_1}^m(\varrho), \Psi_{\gamma_{1_\varrho}}^m, \gamma_{\gamma_{1_\varrho}}^m, \Lambda_{\gamma_{1_\varrho}}^m) - g\left(\varrho, \Psi_{\gamma_2}^m(\varrho), \gamma_{\gamma_2}^m(\varrho), \Lambda_{\gamma_2}^m(\varrho), \Psi_{\gamma_{2_\varrho}}^m, \gamma_{\gamma_{2_\varrho}}^m, \Lambda_{\gamma_{2_\varrho}}^m\right)\|^2 d\varrho + \int_t^T \|\acute{g}(\varrho, \gamma_{\gamma_1}^m(\varrho)\, \Lambda_{\gamma_1}^m(\varrho), \gamma_{\gamma_{1_\varrho}}^m, \Lambda_{\gamma_{1_\varrho}}^m) - g^*(\varrho, \gamma_{\gamma_2}^m(\varrho) \Lambda_{\gamma_2}^m(\varrho), \gamma_{\gamma_{2_\varrho}}^m, \Lambda_{\gamma_{2_\varrho}}^m)\|^2 d\varrho +$

$\int_t^T \|\Lambda_{\gamma_2}^m(\varrho) -, \Lambda_{\gamma_2}^m(\varrho)\|^2 d\varrho + 2\int_t^T |\gamma_1 - \gamma_2| d\varrho.$

Because $\gamma_1$and γ_2 are chosen, it follows that $||\gamma_1 - \gamma_2| \to 0$. Also, because $g_1$and $\acute{g}_2$ are stochastically increasing functions, we have

$\|g(\varrho, \Psi_{\gamma_1}^m(\varrho), \gamma_{\gamma_1}^m(\varrho), \Lambda_{\gamma_1}^m(\varrho), \Psi_{\gamma_{1_\varrho}}^m, \gamma_{\gamma_{1_\varrho}}^m, \Lambda_{\gamma_{1_\varrho}}^m) - g(\varrho, \Psi_{\gamma_2}^m(\varrho), \gamma_{\gamma_2}^m(\varrho), \Lambda_{\gamma_2}^m(\varrho), \Psi_{\gamma_{2_\varrho}}^m, \gamma_{\gamma_{2_\varrho}}^m, \Lambda_{\gamma_{2_\varrho}}^m)\|^2 \to 0,$

$\|\acute{g}(\varrho, \Psi_{\gamma_1}^m(\varrho), \gamma_{\gamma_1}^m(\varrho), \Lambda_{\gamma_1}^m(\varrho), \Psi_{\gamma_{1_\varrho}}^m, \gamma_{\gamma_{1_\varrho}}^m, \Lambda_{\gamma_{1_\varrho}}^m) - \acute{g}(\varrho, \Psi_{\gamma_2}^m(\varrho), \gamma_{\gamma_2}^m(\varrho), \Lambda_{\gamma_2}^m(\varrho), \Psi_{\gamma_{2_\varrho}}^m, \gamma_{\gamma_{2_\varrho}}^m, \Lambda_{\gamma_{2_\varrho}}^m)\|^2 \to 0.$ we have $\int_t^T \|\Lambda_{\gamma_2}^m(\varrho) -, \Lambda_{\gamma_2}^m(\varrho)\|^2 d\varrho \to 0$, as $m \to \infty$. Hence $\|\gamma_{\gamma_1}^m(t)\|^2 \leq \|\gamma_{\gamma_2}^m(t)\|^2$. For $\gamma_m \leq \gamma_{m-1} \leq \cdots \leq \gamma_2 \leq \gamma_1 \leq \gamma$, we deduce

$\|\gamma_{\gamma_m}(t)\|^2 \leq \|\gamma_{\gamma_{m-1}}(t)\|^2 \leq \cdots \leq \|\gamma_{\gamma_2}(t)\|^2 \leq \|\gamma_{\gamma_1}(t)\|^2 \leq \|\gamma_{\gamma_\gamma}(t)\|^2.$

Therefore, there exists a decreasing sequence π_m such that $\pi \to 0$ as $m \to \infty$ with $lim_{n\to\infty}\Psi_{\pi_m}(t) \in MSC$.

Also, there exists a decreasing sequence γ_m such that $\gamma \to 0$ as $m \to \infty$, with $0lim_{n\to\infty}\gamma_{\gamma_m}(t) \in MSC$. Since $\Gamma, \acute{\Gamma}, g$ and $\acute{g}$ are continuous functions, by applying the Lebesgue convergence theorem, we have $a(t) = lim_{n\to\infty}\Psi_{\pi_m}(t)$ and $b(t) = lim_{n\to\infty}\gamma_{\gamma_m}(t)$. Suppose that $\Psi(t)$ is any solution of the forward part of the FBDDSDEs system. Therefore,$\|\Psi_\pi(t) - \Psi(t)\|^2 = \pi$, then $, \|\Psi_\pi(t)\Psi\|^2 - \|\Psi(t)\|^2 \geq \pi$, we have $, \|\Psi_\pi(t) \geq \Psi(t)\|^2$ as $\pi \to 0$. Since the maximal solution is unique, we have $\Psi_\pi(t)$ tends to $a(t)$ as $\pi \to 0$. Suppose that $\gamma(t)$ is any solution of the backward part of the FBDDSDEs system.

Therefore, $\|\gamma_\gamma(t) - \gamma(t)\|^2 = \gamma$, then , $\|\gamma_\gamma(t) - \gamma(t)\|^2 \geq \gamma$, we have , $\|\gamma_\gamma(t) \geq \gamma(t)\|^2$,

As $\gamma \to 0$, and the maximal solution is unique, we have $\gamma_\gamma(t)$ tends to $b(t)$ as $\gamma \to 0$.

Theorem 2. Under the hypotheses (Hyp1 – Hyp6), the FBDSDDEs system (3.1) has a unique solution $(\gamma, \Lambda, \Psi, F)$

Proof: Let $(\gamma^1, \Lambda^1, \Psi^1, F^1)$ and $(\gamma^2, \Lambda^2, \Psi^2, F^2)$ be two solutions of FBDSDDEs system (3.1).

We set $(\gamma^*, \Lambda^*, \Psi^*, F^*) = (\gamma^1 - \gamma^2, \Lambda^1 - \Lambda^2, \Psi^1 - \Psi^2, F^1 - F^2)$ $0 \le t \le T, j = 1, \dots, m.$

Applying Itô's formula to γ^* and $\Psi^*$and using the inequality $xy \le \frac{1}{2b}x^2 + \frac{b}{2}y^2,\ b > 0$, we have:

$$E[\gamma^{*m}(T) l \gamma^{*m}(T)]$$

$$\le E\left[\int_0^T \Psi^{*m}(\varrho)\left(\Gamma\left(\varrho, \gamma^{2,m}(\varrho), \Lambda^{2,m}(\varrho), \gamma_\varrho^{2,m}, \Lambda_\varrho^{2,m}\right) - \Gamma\left(\varrho, \gamma^{1,m}(\varrho), \Lambda^{1,m}(\varrho), \gamma_\varrho^{1,m}, \Lambda_\varrho^{1,m}\right)\right) d\varrho\right]$$

$$+E\left[\int_0^T F^{*m}(\varrho)\left(g\left(\varrho, \gamma^{2,m}(\varrho), \gamma_\varrho^{2,m}\right) - g\left(\varrho, \gamma^{1,m}(\varrho), \gamma_\varrho^{1,m}\right)\right) d\varrho\right]$$

$$+E\left[\int_0^T \gamma^{*m}(\varrho)\left(\acute{\Gamma}\left(\varrho, \Psi^{2,m}(\varrho), \mathrm{F}^{2,m}(\varrho), \Psi_\varrho^{2,m}, \mathrm{F}_\varrho^{2,m}\right) - \acute{\Gamma}\left(\varrho, \Psi^{1,m}(\varrho), \mathrm{F}^{1,m}(\varrho), \Psi_\varrho^{1,m}, \mathrm{F}_\varrho^{1,m}\right)\right) d\varrho\varrho\right]$$

$$+E\left[\int_0^T \Lambda^{*m}(\varrho)\left(\acute{g}\left(\varrho, \Psi^{2,m}(\varrho), \Psi_\varrho^{2,m}\right) - \acute{g}\left(\varrho, \Psi^{1,m}(\varrho), \Psi_\varrho^{1,m}\right)\right) d\varrho\right]$$

$$\le \frac{1}{2b} E\left[\int_0^T |\Psi^{*m}(\varrho)|^2 d\varrho\right]$$

$$+\frac{b}{2} E\left[\int_0^T \left|\Gamma\left(\varrho, \gamma^{2,m}(\varrho), \Lambda^{2,m}(\varrho), \gamma_\varrho^{2,m}, \Lambda_\varrho^{2,m}\right) - \Gamma\left(\varrho, \gamma^{1,m}(\varrho), \Lambda^{1,m}(\varrho), \gamma_\varrho^{1,m}, \Lambda_\varrho^{1,m}\right)\right|^2 d\varrho\right]$$

$$+\frac{1}{2b} E\left[\int_0^T |F^{*m}(\varrho)|^2 d\varrho\right]$$

$$+\frac{b}{2} E\left[\int_0^T \left|g\left(\varrho, \gamma^{2,m}(\varrho), \gamma_\varrho^{2,m}\right) - g\left(\varrho, \gamma^{1,m}(\varrho), \gamma_\varrho^{1,m}\right)\right|^2 d\varrho\right]$$

$$-\frac{1}{2b} E\left[\int_0^T |\gamma^{*m}(\varrho)|^2 d\varrho\right]$$

$$-\frac{b}{2} E\left[\int_0^T \left|\acute{\Gamma}\left(\varrho, \Psi^{2,m}(\varrho), \mathrm{F}^{2,m}(\varrho), \Psi_\varrho^{2,m}, \mathrm{F}_\varrho^{2,m}\right) - \acute{\Gamma}\left(\varrho, \Psi^{1,m}(\varrho), \mathrm{F}^{1,m}(\varrho), \Psi_\varrho^{1,m}, \Gamma_t^{1,m}\right)\right|^2 dt\right]$$

$$+\frac{1}{2b} E\left[\int_0^T \left|\Lambda^{*m}(\varrho)\right|^2 d\varrho\right]$$

$$+\frac{b}{2} E\left[\int_0^T \left|g'\left(\varrho, \Psi^{2,m}(\varrho), \Psi_\varrho^{2,m}\right) - g'\left(\varrho, \Psi^{1,m}(\varrho), \Psi_\varrho^{1,m}\right)\right|^2 d\varrho\right]$$

$$\le \left(\frac{db}{2} + \frac{bR_2\kappa}{2} + \frac{bd_3}{2} + \frac{bR_3}{2} - \frac{1}{2b}\right) E\left[\int_0^T \left|\gamma^{*m}(\varrho)\right|^2 d\varrho\right]$$

$$+\left(\frac{1}{2b} - \frac{bd_2}{2} - \frac{bR_2\kappa}{2} + \frac{bd_4}{2} + \frac{bR_4}{2}\right) E\left[\int_0^T \left|\Psi^{*m}(\varrho)\right|^2 d\varrho\right]$$

$$+\left(\frac{bC_1}{2} + \frac{bR_1\kappa}{2} + \frac{1}{2b}\right) E\left[\int_0^T \left|\Lambda^{*m}(\varrho)\right|^2 d\varrho\right] + \left(\frac{1}{2b} - \frac{bd_2}{2} - \frac{bR_2\kappa}{2}\right) E\left[\int_0^T \left|F^{*m}(\varrho)\right|^2 d\varrho\right]$$

By Cauchy sequence, we have

$$E\left[\int_0^T |\gamma^{*m}(\varrho)|^2 d\varrho + \int_0^T |\Psi^{*m}(\varrho)|^2 d\varrho + \int_0^T |\Lambda^{*m}(\varrho)|^2 d\varrho + \int_0^T |F^{*m}(\varrho)|^2 d\varrho\right] \to 0 \text{ as } m \to \infty.$$

That means

$$E\left[\int_0^T \left|\left(\gamma^1(\varrho), \Lambda^1(\varrho), \Psi^1(\varrho), \mathrm{F}^1(\varrho)\right) - \left(\gamma^2(\varrho), \Lambda^2(\varrho), \Psi^2(\varrho), \mathrm{F}^2(\varrho)\right)\right|^2 d\varrho\right] = 0$$

Hence

$$\left(\gamma^1(\varrho), \Lambda^1(\varrho), \Psi^1(\varrho), \mathrm{F}^1(\varrho)\right) = \left(\gamma^2(\varrho), \Lambda^2(\varrho), \Psi^2(\varrho), \mathrm{F}^2(\varrho)\right)$$

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