Properties of Asymptotic Behaviour Regarding the resolution of linear neutral difference equations of the second order with positive and negative coefficients

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Abstract— *In this research concerns the asymptotic behavior properties for resolve second order of linear Neutral Difference equations with Positive and Negative Coefficients. Some necessary and Sufficient obtained for all solution non oscillatory tend to zero or infinity n→∞ . Example are given to illustrated the ordinary results.*

Keywords*— *Non Oscillatory, Asymptotic behavior, Neutral Difference Equation.

I. INTRODUCTION

The study of neutral difference equations with Positive and Negative Coefficients has been recently considered the attention of many authors all over the world for last several years, see [1 - 6] Non oscillation solution of first and second order for difference and differential equation. formula. In this study, researchers developed new adequate conditions for oscillation and the equation's asymptotic behavior solution[7].

The neutral difference equation has received attention of many authors in the last several years. Berezansky (2014)[9], studied the oscillatory behavior of the solution of the difference equation and obtained some sufficient conditions for the oscillatory of above equation.

$$\Delta x(n) + \sum_{i=1}^{m} b_i(n) x(\tau_i(n)) = 0$$

and obtained some sufficient conditions for the oscillatory of above equation.

Bohner.M, EI-Morshedy. H.A, Grace. S. R, Sager. I. (2019) established some new criteria for oscillation with a sublinear neutral tem of the second-order linear difference equation .

In this work, Necessary while sufficient are obtained in order for each equation solution to

$$\boldsymbol{\Delta^2(g_n - b_n g_{n-u}) + a_n(g_{n-e}) - c_n g_{n-t} = Z_n \quad \ldots \quad (1a)}$$

$$\boldsymbol{\Delta^2(g_n + b_n g_{n-u}) + a_n(g_{n-e}) + c_n g_{n-t} = Z_n \quad \ldots \quad (1b)}$$

As we go, we assume that the following assumption are content:

(B_1) $0 < b_n < b < \infty,$ b *is constsnt*

(B_2) $\sum_{j=n_1}^{\infty} \sum_{i=g+e-t}^{g-1} c_i < \infty$

(B_3) There exists an equation $\{Z_n\}$ such that:

$$\Delta^2 Z_n = Z_n \text{and} \lim_{n\to\infty} Z_n = 0$$

Through an equation's solution (1a-1b) We refer to a series of g_n This makes the equation work for each Large n. a solution It is stated that y is nonoscillatory if it's positive or negative, eventually in any other case, it is known as oscillatory [1] .

II. MAIN RESULT

The next result provides sufficient conditions for the oscillation of all solution equation (1a-1b) for simplicity set:

$$Z_n = g_n - b_n g_{n-u} \qquad \ldots.(2a)$$

$$Z_n = g_n + b_n g_{n-u} \qquad \ldots.(2b)$$

let the sequence V_n be defined as :

$$V_n = \sum_{i=n}^{\infty} \sum_{i=j-e+t}^{j-1} a_i g_{i-t} - Z_n, \qquad e > t \qquad \ldots(3)$$

And $\Delta^2 T_n$=-$(a_n(g_n) + a_{n-e+t}(g_{n-e}))$, $\qquad e > t$

and the sequence B_2 holds be defined as

$$V_n = g_n - b_n g_{n-u} - \sum_{i=n}^{\infty} \sum_{i=j-e+t}^{j-1} a_i g_{i-t} - Z_n, \quad e > t \qquad \ldots(4a)$$

$$V_n = g_n + b_n g_{n-u} + \sum_{i=n}^{\infty} \sum_{i=j-e+t}^{j-1} a_i g_{i-t} - Z_n, \quad e > t \qquad \ldots(4b)$$

The following theorem based on theorem $([14], pp, 182)$

Theorem 2.1 ($[7], pp, 182$)

Suppose that $\{g_n\}$ be a nonnegative let t be a positive integer and a sequence of real numbers.

Assume non oscillatory characteristics of second- order nonlinear neutral difference equations with positive and negative coefficients during solution

$$\lim_{n\to\infty} \sum_{i=n-t}^{n-1} b_i > \frac{t^{t+1}}{(t+1)^{t+1}}$$

Then

(i). The difference inequality $g_{n+1} - g_n + b_n g_{n-t} \leq 0$, $n = 0, 1, 2, 3 \ldots ....$

Negative solutions are not possible in the long run.

(ii). The difference inequality $g_{n+1} - g_n + b_n g_{n-t} \geq 0$, $n = 0, 1, 2, 3 \ldots ....$

Possible solutions are not negative in the long run.

Theorem 2.2

$([7], Lemma\ 2, 1 - Lemma\ 2.2, pp. 477 - 478)$

(i). Assume that $0 < b_n \leq \propto < 1$, for a positive constant, $n \geq n_0$. Let's go g_n been a solution to a functional inequality that is nonoscillatory $g_n[g_n - b_n g_{n-u}] < 0$. In a neighborhood of infinity, where $u > 0$, thus

$$\lim_{n\to\infty} g_n = 0 .$$

(ii). Assume that the $1 < \propto \leq b_n$ for positive constant $\varepsilon, n \geq n_0$. Let g_n be a solution to a functional inequality that is nonoscillatory $g_n[g_n - b_n g_{n+u}] > 0$. In a neighborhood of infinity, where $u > 0$, thus

$$\lim_{n\to\infty} g_n = 0.$$

Theorem 2.3 Suppose that $a_n \leq c_{n-e+t}$, if $(H_1) - (H_3)$ hold, in addition to

$$\liminf_{n\to\infty} \sum_{i=n(e-b-u)}^{j+b} \sum_{i=j}^{j+b} \frac{|a_i - c_{i-e+t}|}{b_{i-e+u}} > \frac{(e-b-u)^{e-b-u+1}}{(e-b-u+1)^{e-b-u+1}} \qquad \ldots \quad (5)$$

$e > b + u$. Next, each nonoscillatory equation solution (1a), goes to

infinity like $n \to \infty$.

Proof. Let's g_n bean ultimately Positive Solution of the (1a), For $n \geq n_0$, we can

presume that a positive integer exists. n_0. And

$g_{n-u} > 0$, $g_{n-t} > 0$, $g_{n-e} > 0$, for $n \geq n_0 \geq N$ set

S_n, R_n, V_n as in (2), (3) and (4) then (1) it become

$$\Delta^2 V_n - (a_n - c_{n-e+t})(g_{n-1}) = 0 \qquad \ldots(6)$$

Hence $\Delta^2 V_n \leq 0$,

thus there exist $n_1 \geq n_0$ such that

$$\Delta V_n < 0 \ or \ \Delta V_n > 0 \ for \ n \geq n_1 \geq n_0$$

Now let $\Delta V_n < 0\ for\ n \geq n_1$.

Thus implies that $\Delta V_n < 0\ for\ n \geq n_2 \geq n_1$

$and\ \Delta V_n \to -\infty\ as\ n \to \infty$.

We claim that

$g_n \to \infty\ as\ n \to \infty$

Otherwise there exist $n_2 \geq n_1 and\ \ \propto > 0$

that $g_n \leq \propto$,so (H_3) is hold then (3) is become

$$R_n \leq \propto \sum_{j=n}^{\infty} \sum_{i=j-e+t}^{j-1} a_i \qquad ... \quad (7)$$

Substitution (6) in (4a) we obtained

$$V_n \geq -b_n g_{n-u} - \propto \sum_{j=n}^{\infty} \sum_{i=j-e+t}^{j-1} q_i - Z_n$$

$$V_n > -b_n g_{n-u} + \varepsilon$$

Implies that

$$V_n \geq -b_n g_{n-u} \qquad ... \quad (8)$$

This is contradiction.

Then $g_n \to \infty\ as\ n \to \infty$.

Now

$\Delta V_n > 0\ \ for\ n \geq n_1\ , then\ V_n > 0\ or\ V_n < 0$ for $n \geq n_2 \geq n_1$

First , suppose that $V_n > 0$ from(4a) we can conclude

$V_n \leq S_n - Z_n\ \ then\ V_n < S_n + \varepsilon$ for ε>0

$hence\ g_n S_n > 0$,

through Theorem 2.2 –ii we obtain

$\lim_{n\to\infty} g_n = 0$, then $\lim_{n\to\infty} S_n = \lim_{n\to\infty} V_n = 0$,

it is contradiction because V_n is the positive number rising

The next step if $V_n < 0$. By using the summation

for $n\ to\ n+b, b > e$ for both sides of (4a) yield .

$$\Delta V_{n+b+1} - \Delta V_n = \sum_{i=n}^{n+b} (c_i - a_{i-e+t})(g_{i-e})$$

$$\Delta V_n \leq \sum_{i=n}^{n+b} (c_i - a_{i-e+t})(g_{i-e}) \qquad .... \quad (9)$$

So by (H_3) is hold then (8) become

$$\boldsymbol{\Delta V_n \leq \sum_{i=n}^{n+b} (c_i - a_{i-e+t})(g_{i-e})} \qquad ...(\mathbf{10})$$

Now (7) can be written in the $-\frac{1}{b_n} V_n \leq g_{n-e}$ **Hence**

$$\frac{-1}{b_{n-e+u}} V_{n-e+u} \leq g_{n-1}$$

Substituting the last inequality in (9) we obtain

$$-\Delta V_n \leq -\sum_{i=n}^{n+b} \frac{(c_i - a_{i-e+t})}{b_{n-e+u}} V_{i-e+u}$$

$$-\Delta V_n \leq \left(\sum_{i=n}^{n+b} \frac{|c_i - a_{i-e+t}|}{b_{i-e+u}} \right) V_{n+b-e+u}$$

$$\Delta V_n + \left((\sum_{i=n}^{n+b} \frac{|c_i - a_{i-e+t}|}{b_{i-e+u}} \right) V_{n+b-e+u} \geq 0$$

The last inequality cannot eventually have a negative solution, which is contradicted by theorem 2.1–ii and by virtue of (4a).

Theorem 2.4 Let that $c_n \geq a_{n-e+t}$, and $(H_1) - (H_3)$ are placed on hold, then every

equation with a nonoscillatory bounded solution (1.a) converge with zero as $n \to \infty$.

Proof: Allow g_n is eventually positive, solution of the (1a) For $n \geq n_0$,

We can presume that a positive integer $n_0$exists, and

$$g_{n-u} > 0\ , g_{n-t} > 0\ , g_{n-e} > 0\ ,$$

for $n \ge n_0 \ge N,$

since from (5) we have

$$\Delta^2 V_n \le 0\ for\ n \ge n_0,$$

thus there exist $n_1 \ge n_0$ such that

$\Delta V_n > 0$ or $\Delta \mathrm{V} < 0$ for $n \ge n_1 \ge n_0$

Next suppose $\Delta V_n > 0$ .for $n \ge n_1$

Thus implies that $V_n < 0\ or\ V_n > 0$ for $n \ge n_2 \ge n_1$

and $V_n \to \infty\ as\ n \to \infty$ *So* for (4a) we get

$V_n \le g_n - Z_n,$

it implies that $g_n \to \infty\ as\ n \to \infty$,

which is a contradiction , hence $\Delta V_n > 0$ is not possible . Now If

$\Delta V_n > 0$ for $n \ge n_1$,

Then

$V_n > 0$ or$V_n < 0$ for $n \ge n_2 \ge n_1$

First, let's say that $V_n < 0$.

Next, there exists $n_3 \ge n_2$, ρ > 0

like that $V_n \le -\rho < 0, for$

$n \ge n_2$. Because of g_n has boundaries, then $\lim_{n\to\infty} \sup g_n = \gamma$<∞,

and a subsequence is present.

$\{k_j\}_{j=1}^{\infty}$ like that

$$g_{\propto j} \to \infty\ as\ k_j \to \infty\ , and$$

$$g_{kj} = \max\{g_k : k_0 \le \varepsilon \le k_j\},$$

$$\lim_{n\to\infty} \sup g_{\propto j} \le \gamma$$

Since g_n Then, it is enclosed by (6) And (4a) We are given

$$g_{kj} \le -\rho + b_{sj} g_{sj-u} + \gamma \sum_{j=n}^{\infty} \sum_{i=j-e+t}^{j-1} a_i + Z_{sj}$$

$$< -\rho + b_{sj} g_{sj} + \varepsilon\ , \varepsilon > 0$$

so

$$g_{sj} \le -\rho + \lim_{n\to\infty} \sup g_{\propto j}\ \ as\ \mathrm{j} \to \infty$$

We are given form Lastly, the inequality $\alpha \le -\rho + \alpha$

which contradicts itself, finally

if $V_n > 0\ , from\ (4a)$ We are given $V_n \le S_n - Z_n,$

suggests that $0 < V_n \le S_n$

for large enough n Thus, $g_n S_n > 0,$

as well as by Theorem 2.2-ii , Thus, it follows that

$\lim_{n\to\infty} g_n = 0,$

Theorem 2.5 Suppose that $c_{n+t-e} - a_{n-e+t} \le 0$,if $(H_1) - (H_3)$

If this is true, then all of the nonoscillatory bounded solutions to (1a) converge to zero as n approaches zero.

Proof. Let g_b be positive solution eventually of the

(1a) For $n \ge n_0$,

We can presume that a positive integer exists.

n_0 , And $g_{n-u} > 0\ , g_{n-t} > 0\ , g_{n-e} > 0$, For

$n \ge n_0, \ge \mathrm{N},$ because (5)

We've $\Delta^2 V_n \le 0$, For $n \ge n_0$

Therefore, there are $n_1 > n_0$ like that

$$\Delta V_n > 0 \text{ or } \Delta \mathrm{V} < 0 \text{ for } n \ge n_1 \ge n_0$$

Next suppose that

$\Delta V_n < 0,$ for $n \ge n_1$.

Thus implies that

$V_n > 0$ or$V_n > 0$ for $n \geq\ n_2 \geq\ n_1$

and

$V_n \to -\infty\ as\ n \to -\infty\ So\ for\ (4)$ we get

$$V_n \geq g_n - Z_n$$

It implies that $g_n \to \infty\ as\ n \to \infty$, which is contradiction,

hence $\Delta V_n < 0$ is not possible .

Now If $\Delta V_n < 0\ for\ n \geq n_1$,

Then

$$V_n < 0\ orV_n > 0\ \ for\ n \geq\ n_2 \geq\ n_1$$

First , suppose that $V_n > 0.$

then $there\ exist\ \ n_3 > n_2$ ρ < 0 so that

$V_n \geq -\rho > 0$, For $n \geq n_2$

Given that g_n has a bound, then $\lim_{n\to\infty} \inf g_n = \gamma$, $\gamma < \infty,$ and

there exist a subsequence

$\{k_j\}_{j=1}^{\infty}$ such that $g_{kj} \to \infty\ as\ k_j \to \infty,$

$and\ g_{kj} = \max\{g_k : k_0 \leq \varepsilon \leq k_j\}$,

$$\lim_{n\to\infty} \inf g_{\propto j} \leq \gamma$$

Since g_n is then bounded by (6) And (4a) We are given

$$g_{kj} \geq -\rho - b_{sj} g_{sj} - \gamma \sum_{j=n}^{\infty} \sum_{i=j-e+t}^{j-1} a_i + Z_{sj}$$

$$> -\rho + b_{sj} g_{sj} - \varepsilon, \qquad \varepsilon > 0$$

so $g_{sj} \geq -\rho - \lim_{n\to\infty} \inf g_{\propto j}\ as\ j \to \infty,$

The final inequality is obtained. α

$\alpha \geq -\rho - \alpha$

It is a contradiction ,finally if $V_n < 0$ from (4a) We are given

$V_n \geq S_n - Z_n,$ suggests that

$S_n \leq V_n < 0$ for large enough n , So

$$g_n S_n < 0$$

and by Theorem 2.3-is, it follows that

$$\lim_{n\to\infty} g_n = 0$$

Examples: 3. 3

3. 3. 1 Examine the **difference equation:**

$$\Delta^2 \left((\boldsymbol{g_n}) - \left(\mathbf{1} + \left(\tfrac{1}{2}\right)^n \boldsymbol{g_{n-1}} \right) + \tfrac{\mathbf{3}}{\mathbf{2}} \left(\left(\tfrac{\mathbf{1}}{\mathbf{2}}\right)^{\boldsymbol{n}} \right) \boldsymbol{g_{n-2}} - \left(\tfrac{\mathbf{1}}{\mathbf{2}}\right)^{\boldsymbol{n}} \mathrm{g}_{n-3} \right) =$$

$$2^n \left(1 + \left(\frac{1}{2}\right)^{n+3} \right) \frac{1}{2}$$

$b_n = 1 + \left(\tfrac{1}{2}\right)^n$ Where u =1, t=2, e=3, $a_n = \tfrac{\mathbf{3}}{\mathbf{2}} \left(\left(\tfrac{1}{2}\right)^{\boldsymbol{n}} \right)$, $\boldsymbol{c_n} = \left(\tfrac{\mathbf{1}}{\mathbf{2}}\right)^{\boldsymbol{n}}$, **b=2**

The following conditions of theorem 2.3 can be found to be true:

$$b_n = \left(1 + \left(\tfrac{1}{2}\right)^n \right) = 1\ ,\ as\ n \to \infty$$

$$\sum_{j=n_1}^{\infty} \sum_{i=j-e+t}^{j-1} a_i < \sum_{j=n_1}^{\infty} \left(\frac{3}{2} \left(\frac{\mathbf{1}}{\mathbf{2}}\right)^{\boldsymbol{n}} \right) < \frac{\mathbf{3}}{\mathbf{2}} \sum_{j=n_1}^{\infty} \left(\frac{1}{2}\right)^j < 0, n \to \infty c_n - a_{n-e+k=} 0$$

$$\boldsymbol{c_n} - \boldsymbol{a_{n-e+k=}} \left(\left(\tfrac{\mathbf{1}}{\mathbf{2}}\right)^{\boldsymbol{n}} \right) - \tfrac{\mathbf{3}}{\mathbf{2}} \left(\tfrac{\mathbf{1}}{\mathbf{2}}\right)^{\boldsymbol{n}} = -1 < 0$$

$$\lim_{n\to\infty} \inf \sum_{j=n-(e-b-u)}^{n-1} \sum_{i=j}^{j+b} \frac{|c_i - a_{i-e+t}|}{b_{i-e+u}} > \frac{1}{3}$$

Therefore, any solution to (11) leads to infinity in accordance with theorem $2.3.n \to \infty$, for example $g_n = 2^{n-1}$ is a solution of that kind.

III. CONSIDER THE DIFFERENCE EQUATION:

$$\Delta^2 \left((\mathbf{3^n}) - \left(\mathbf{1} + \left(\frac{1}{3}\right)^n \mathbf{3^{n-1}} \right) + \mathbf{5} \left(\frac{\mathbf{1}}{\mathbf{3}}\right)^{\boldsymbol{n}} \mathbf{3^{n-2}} - \left(\frac{\mathbf{1}}{\mathbf{3}}\right)^{\boldsymbol{n}} 3^{n-3} \right)$$

$$= 3^{n+1}\left(-1+\left(\frac{1}{3}\right)^{n+1}\right)$$

Where u=1, t=2, e=3, b=3, $a_n = 5\left(\frac{1}{3}\right)^{\boldsymbol{n}}, c_n = \left(\frac{1}{3}\right)^{\boldsymbol{n}}$

The following are all of the requirements of theorem 2.3 hold:

$$b_n = \left(1+\left(\tfrac{1}{3}\right)^n\right) = 1\,,\ as\ n \to \infty$$

$$\sum_{j=n_1}^{\infty}\sum_{i=j-e+t}^{j-1} a_i < \sum_{j=n_1}^{\infty}\left(\mathbf{5}\left(\frac{1}{3}\right)^j\right) < \infty$$

$$\left(\left(\frac{\mathbf{1}}{\mathbf{3}}\right)^{\boldsymbol{n}}\right) - \mathbf{5}\left(\frac{\mathbf{1}}{\mathbf{3}}\right)^{\boldsymbol{n}} = - < 0, n \to \infty$$

$$\liminf_{n\to\infty} \sum_{j=n-(e-b-u)}^{n-1}\sum_{i=j}^{j+b}\frac{|c_i - a_{i-e+t}|}{b_{i-e+u}} > \frac{-5}{12}$$

Theorem 2.4 states that every solution to (11) goes to infinity as n→∞. For example,

$g_n = 3^n$ is one such solution.

IV. CONCLUSION

In this paper, we study the asymptotic behavior to ensure the converge for all solution to zero or diverge for $n \to \infty$ our obtained some necessary and sufficient of granite every solution second order linear difference equations with positive and negative coefficients.

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