Some Interesting Properties of a Novel Subclass of Multivalent Function with Positive Coefficients

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Abstract— *In this paper, we introduce a new class of multivalent functions defined by $A(p,\gamma,\omega)$ where $A(p)$ is a subclass of analytic and multivalent functions $W(p)$ in the open unit disc $U=\{z:|z|<1\}$. Moreover, we consider and prove theorems explain Some of the geometric properties for such new class was $A(p,\gamma,\omega)$, such as, coefficient estimates, growth and distortion, extreme points, radii of starlikeness, convexity and close-to-convexity as well as the convolution properties for the class $A(p,\gamma,\omega)$.*

Keywords— Multivalent function; coefficient estimates; Hadamard product; growth theorem.

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1.Introduction

Let $W(p)$ be denote the class of functions of the form:

$$f(z)=z^p+\sum_{k=p+1}^{\infty}a_k z^k,(z\in U,p\in\mathbb{N}=\{1,2,3,\ldots\}) \qquad (1)$$

which are analytic and multivalent in the open unit disc $U=\{z:|z|<1\}$.

Let $A(p)$ denotes a subclass of $W(p)$ of functions of the form:

$$f(z)=z^p+\sum_{k=p+1}^{\infty}a_k z^k \ (a_k\geq 0,z\in U,\ p\in\mathbb{N},). \qquad (2)$$

The convolution [6] (Hadamard) product of two power series for the function $f(z)$ given by (1) and $g(z)$ given by

$$g(z)=z^p+\sum_{k=p+1}^{\infty}b_k z^k\,, \qquad (z\in U,p\in\mathbb{N}=\{1,2,3,\ldots\})$$

Can be defined by:

$$(f*g)(z)=z^p+\sum_{k=p+1}^{\infty}a_kb_kz^k.\ (z\in U,\ p\in\mathbb{N}) \qquad (3)$$

A function $f(z)\in A(p)$is said to be multivalent starlike of order
$\delta,$ multivalent convex of order δ *and* multivalent closed $-$ to $-$
convex of order $\delta,(0\le\delta<p\,,z\in U)$ [3], respectively if $Re\left\{\frac{zf'(z)}{f(z)}\right\}>\delta\,,Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\delta$ and $Re\left\{\frac{f'(z)}{z^{p-1}}\right\}>\delta.$

In the next definition, we given the condition for the function f belongs in the class $A(p,\gamma,\omega)$.

Definition: A function $f\in A(p)$ belongs in the class $A(p,\gamma,\omega)$, if it's satisfies the following condition:

$$\left|\frac{\frac{z[w(z)]''}{f''(z)}-pz}{\frac{\omega z[w(z)]''}{f''(z)}-(\gamma-\omega)}\right|<1\,,\quad\left(\frac{1}{2}\le\omega<1,0<\gamma\le\frac{1}{2}\right) \qquad (4)$$

where $w(z)=zf'(z)$.

Such type of study was carried out by various authors for another classes, like, Khairnar and More [5], AL-khafaji et al. [1], Aouf and Mostafa [2], Raina and Srivastava [7] and Dziok and Srivastava [4].

In this paper we introduces a new class $A(p,\gamma,\omega)$, of multivalent functions in the open unit disc. Coefficient estimates, growth and distortion theorems, radii of close-to-convexity, starlikeness and convexity, extreme points and the Hadamard product for functions in the class $A(p,\gamma,\omega)$ are obtained.

2. Geometric properties for $A(p,\gamma,\omega)$.

In this section, we introduce theorems with their proofs to discuss some of the geometric properties for such class $A(p,\gamma,\omega)$.

2.1. Coefficient estimates.

A sufficient and necessary condition to the function $f(z)$ to be in the class $A(p,\gamma,\omega)$ will discuss in the following theorem.

Theorem 2.1.1. A function f in (2) belongs to the class $A(p,\gamma,\omega)$ if and only if

$$\sum_{k=p+1}^{\infty}k(k-1)[k-p-\omega(k+1)+\gamma]a_k\le p(p-1)[\omega(p+1)-\gamma], \qquad (5)$$

where $\left(p\ge1\,,\frac{1}{2}\le\omega<1,0<\gamma\le\frac{1}{2}\right)$.

The result is sharp for the function

$$f(z)=z^p+\frac{p(p-1)[\omega(p+1)-\gamma]}{k(k-1)[k-p-\omega(k+1)+\gamma]}z^k\,. \qquad (6)$$

Proof: Suppose that $f\in A(p,\gamma,\omega)$, then by (4), we have:

$$\left|\frac{z\left[pz^p+\sum_{k=p+1}^{\infty}ka_kz^k\right]''-pz\left[z^p+\sum_{k=p+1}^{\infty}a_kz^k\right]''}{\omega z\left[pz^p+\sum_{k=p+1}^{\infty}ka_kz^k\right]''-(\gamma-\omega)z\left[z^p+\sum_{k=p+1}^{\infty}a_kz^k\right]''}\right|<1$$

$$=\left|\frac{p^2(p-1)z^{p-1}+\sum_{k=p+1}^{\infty}k^2(k-1)a_kz^{k-1}-p^2(p-1)z^{p-1}-\sum_{k=p+1}^{\infty}pk(k-1)a_kz^{k-1}}{\omega p^2(p-1)z^{p-1}+\sum_{k=p+1}^{\infty}\omega k^2(k-1)a_kz^{k-1}-(\gamma-\omega)[p(p-1)z^{p-1}-\sum_{k=p+1}^{\infty}k(k-1)a_kz^{k-1}]}\right|$$

$$=\left|\frac{\sum_{k=p+1}^{\infty}k(k-1)(k-p)a_kz^{k-1}}{p(p-1)[\omega(p+1)-\gamma]z^{p-1}+\sum_{k=p+1}^{\infty}k(k-1)[\omega(k+1)-\gamma]a_kz^{k-1}}\right|$$

Since $|Re(z)|\le|z|$ for all z, we have

$$Re\left\{\frac{\sum_{k=p+1}^{\infty}k(k-1)(k-p)a_kz^{k-1}}{p(p-1)[\omega(p+1)-\gamma]z^{p-1}+\sum_{k=p+1}^{\infty}k(k-1)[\omega(k+1)-\gamma]a_kz^{k-1}}\right\}\le1.$$

Choosing the value of z on the real axis and letting $z\to1^-$ through values, we get:

$$\sum_{k=p+1}^{\infty}k(k-1)(k-p)a_k\le p(p-1)[\omega(p+1)-\gamma]+\sum_{k=p+1}^{\infty}k(k-1)[\omega(k+1)-\gamma]a_k].$$

Hence

$$\sum_{k=p+1}^{\infty}k(k-1)[k-p-\omega(k+1)+\gamma]a_k\le p(p-1)[\omega(p+1)-\gamma].$$

Conversely, assume that (5) holds $|z|=r,r<1,$ then

$$|z[w(z)]'' - pzf''(z)| - |\omega z[w(z)]'' - (\gamma-\omega)f''(z)|$$
$$= \left| z\left[pz^p + \sum_{k=p+1}^{\infty} ka_k z^k\right]'' - pz\left[z^p + \sum_{k=p+1}^{\infty} a_k z^k\right]''\right| - \left|\omega z\left[pz^p + \sum_{k=p+1}^{\infty} ka_k z^k\right]'' - (\gamma-\omega)z\left[z^p + \sum_{k=p+1}^{\infty} a_k z^k\right]''\right|$$

$$= \left|p^2(p-1)z^{p-1} + \sum_{k=p+1}^{\infty} k^2(k-1)a_k z^{k-1} - p^2(p-1)z^{p-1} + \sum_{k=p+1}^{\infty} pk(k-1)a_k z^{k-1}\right| - \left|\omega p^2(p-1)z^{p-1} + \sum_{k=p+1}^{\infty} \omega k^2(k-1)a_k z^{k-1} - (\gamma-\omega)[p(p-1)z^{p-1} - \sum_{k=p+1}^{\infty} k(k-1)a_k z^{k-1}]\right|$$

$$= \left|\sum_{k=p+1}^{\infty} k(k-1)(k-p)a_k z^{k-1}\right| - \left|p(p-1)[\omega(p+1)-\gamma]z^{p-1} + \sum_{k=p+1}^{\infty} k(k-1)[\omega(k+1)-\gamma]a_k z^{k-1}\right|$$

$$\leq \sum_{k=p+1}^{\infty} k(k-1)(k-p)a_k|z|^{k-1} - p(p-1)[\omega(p+1)-\gamma]|z|^{p-1} - \sum_{k=p+1}^{\infty} k(k-1)[\omega(k+1)-\gamma]a_k|z|^{k-1}$$

$$= \sum_{k=p+1}^{\infty} k(k-1)(k-p)a_k r^{k-1} - p(p-1)[\omega(p+1)-\gamma]r^{p-1} - \sum_{k=p+1}^{\infty} k(k-1)[\omega(k+1)-\gamma]a_k r^{k-1}$$

$$< \sum_{k=p+1}^{\infty} k(k-1)(k-p)a_k - p(p-1)[\omega(p+1)-\gamma] - \sum_{k=p+1}^{\infty} k(k-1)[\omega(k+1)-\gamma]a_k .$$

Since (5) holds. So we have:

$$\sum_{k=p+1}^{\infty} k(k-1)[k-p-\omega(k+1)+\gamma]a_k - p(p-1)[\omega(p+1)-\gamma] \leq 0 .$$

Thus, $f \in A(p,\gamma,\omega)$ and the theorem is established ■

Note that, the sharpness follows if we choose the function $f(z)$ as

$$f(z) = z^p + \frac{p(p-1)[\omega(p+1)-\gamma]}{k(k-1)[k-p-\omega(k+1)+\gamma]}z^k , \quad where(k = p+1, p+2, \dots).$$

Corollary 2.1,1. Let $f \in A(p,\gamma,\omega)$. Then

$$a_k \leq \frac{p(p-1)[\omega(p+1)-\gamma]}{k(k-1)[k-p-\omega(k+1)+\gamma]}.$$

$$where(k = p+1, p+2, \dots) \tag{7}$$

2.2. **Growth and Distortion**.

A lower and upper bound of $|f(z)|$ and $|f'(z)|$ will be considered by the following theorems respectively, where the bounds for the function $f(z)$ of the form

$$f(z) = z^p + \frac{(p-1)[\omega(p+1)-\gamma]}{(p+1)[1-\omega(p+2)+\gamma]}z^{p+1} .$$

Theorem 2.2.1. If the function $f \in A(p,\gamma,\omega)$ that defined in (2), then

$$r^p - r^{p+1}\frac{(p-1)[\omega(p+1)-\gamma]}{(p+1)[1-\omega(p+2)+\gamma]} \leq |f(z)| \leq r^p + r^{p+1}\frac{(p-1)[\omega(p+1)-\gamma]}{(p+1)[1-\omega(p+2)+\gamma]},$$

for $0 < |z| = r, r < 1$.

Proof: Since $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$, then

$$|f(z)| = \left| z^p + \sum_{k=p+1}^{\infty} a_k z^k \right| \leq |z|^p + |z|^{p+1} \sum_{k=p+1}^{\infty} a_k .$$

From Theorem (2.1.1), we get:

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{(p-1)[\omega(p+1)-\gamma]}{(p+1)[1-\omega(p+2)+\gamma]}.$$

Then

$$|f(z)| \leq r^p + r^{p+1} \frac{(p-1)[\omega(p+1)-\gamma]}{(p+1)[1-\omega(p+2)+\gamma]},$$

and

$$|f(z)| \geq |z|^p - |z|^{p+1} \sum_{k=p+1}^{\infty} a_k = r^p - r^{p+1} \sum_{k=p+1}^{\infty} a_k .$$

Hence

$$|f(z)| \geq r^p - r^{p+1} \frac{(p-1)[\omega(p+1)-\gamma]}{(p+1)[1-\omega(p+2)+\gamma]}.$$

So the proof is complete ■

Theorem 2.2.2. If the function $f \in A(p,\gamma,\omega)$ that defined in (2), then

$$pr^{p-1} - r^p \frac{(p-1)[\omega(p+1)-\gamma]}{[1-\omega(p+2)+\gamma]} \leq |f'(z)| \leq pr^{p-1} + r^p \frac{(p-1)[\omega(p+1)-\gamma]}{[1-\omega(p+2)+\gamma]},$$

for $0 < |z| = r, r < 1$.

Proof: since $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$, then

$$|f'(z)| = \left| pz^{p-1} + \sum_{k=p+1}^{\infty} k a_k z^{k-1} \right| \leq p|z|^{p-1} + |z|^p \sum_{k=p+1}^{\infty} k a_k .$$

From Theorem (2.1.1), we have

$$\sum_{k=p+1}^{\infty} k a_k \leq \frac{(p-1)[\omega(p+1)-\gamma]}{[1-\omega(p+2)+\gamma]}.$$

Thus

$$|f'(z)| \leq pr^{p-1} + r^p \frac{(p-1)[\omega(p+1)-\gamma]}{[1-\omega(p+2)+\gamma]},$$

and
$|f'(z)| \geq p|z|^{p-1} - |z|^p \sum_{k=p+1}^{\infty} k a_k$

$$|f'(z)| \geq pr^{p-1} - r^p \sum_{k=p+1}^{\infty} k a_k$$

$$|f'(z)| \geq pr^{p-1} - r^p \frac{(p-1)[\omega(p+1)-\gamma]}{[1-\omega(p+2)+\gamma]} \; ■$$

2.3 Radii of Starlikeness, Convexity and Close-to-Convexity.

The following theorems explain the radii of starlikeness, convexity and close-to-convexity.

Theorem 2.3.1. If the function $f(z) \in A(p,\gamma,\omega)$ that defined in (2). Then it is multivalent starlike of order δ $(0 \leq \delta < p)$ in the disc $|z| < r_1$, where

$$r_1(p,\gamma,\omega,\delta) = \inf_k \left[\frac{k(p-\delta)(k-1)[k-p-\omega(k+1)+\gamma]}{p(k-\delta)(p-1)[\omega(p+1)-\gamma]} \right]^{\frac{1}{k-p}},$$

$$(k \geq p+1).$$

The result is sharp for the external function $f(z)$ given by (6).

Proof: It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \delta, \quad (0 \leq \delta < p),$$

for $|z| < r_1(p,\gamma,\omega,\delta)$.

We have

$$\left| \frac{zf'(z)}{f(z)} - p \right| = \left| \frac{z\left[pz^{p-1} + \sum_{k=p+1}^{\infty} k a_k z^{k-1}\right] - p\left[z^p + \sum_{k=p+1}^{\infty} a_k z^k\right]}{z^p + \sum_{k=p+1}^{\infty} a_k z^k} \right| \leq \frac{\left[\sum_{k=p+1}^{\infty} (k-p) a_k |z|^{k-p}\right]}{\left[1 - \sum_{k=p+1}^{\infty} a_k |z|^{k-p}\right]}.$$

Thus

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p - \delta.$$

If $\sum_{k=p+1}^{\infty} \frac{(k-\delta)a_k|z|^{k-p}}{(p-\delta)} \leq 1$. (8)

Therefore by Corollary (2.1.1), inequality (8) is true if :

$$\frac{(k-\delta)|z|^{k-p}}{(p-\delta)} \leq \frac{k(k-1)[k-p-\omega(k+1)+\gamma]}{p(p-1)[\omega(p+1)-\gamma]},$$

equivalently if :

$$|z| \leq \left[\frac{k(p-\delta)(k-1)[k-p-\omega(k+1)+\gamma]}{p(k-\delta)(p-1)[\omega(p+1)-\gamma]}\right]^{\frac{1}{k-p}} \quad (9)$$

The theorem follows from (9) ■

Theorem 2.3.2. If the function $f(z) \in A(p,\gamma,\omega)$ that defined in (2). Then $f(z)$ is multivalent convex of order δ $(0 \leq \delta < p)$ in the disc $|z| < r_2$, where

$r_2(p,\gamma,\omega,\delta) = \inf_k \left[\frac{(p-\delta)(k-1)[k-p-\omega(k+1)+\gamma]}{p(k-\delta)(p-1)[\omega(p+1)-\gamma]}\right]^{\frac{1}{k-p}}$, $(k \geq p + 1)$.

The result is sharp for the external function $f(z)$ given by (6).

Proof: It is sufficient to show that

$\left|1 + \frac{zf''(z)}{f'(z)} - p\right| \leq p - \delta$, $(0 \leq \delta < p)$, for

$|z| < r_2(p,\gamma,\omega,\delta)$.

We have

$$\left|1 + \frac{zf''(z)}{f'(z)} - p\right| \leq \frac{\sum_{k=p+1}^{\infty} k(k-p)a_k|z|^{k-p}}{1-\sum_{k=p+1}^{\infty} ka_k|z|^{k-p}}.$$

Thus $\left|1 + \frac{zf''(z)}{f'(z)} - p\right| \leq p - \delta$,

if $\sum_{k=p+1}^{\infty} \frac{k(k-\delta)a_k|z|^{k-p}}{(p-\delta)} \leq 1$.

Therefore by Corollary (2.1.1), last inequality is true if :

$$\frac{k(k-\delta)|z|^{k-p}}{(p-\delta)} \leq \frac{k(k-1)[k-p-\omega(k+1)+\gamma]}{p(p-1)[\omega(p+1)-\gamma]},$$

equivalently if

$$|z| \leq \left[\frac{(p-\delta)(k-1)[k-p-\omega(k+1)+\gamma]}{p(k-\delta)(p-1)[\omega(p+1)-\gamma]}\right]^{\frac{1}{k-p}}. \quad (10)$$

The theorem follows from (10) ■

Theorem 2.3.3. Let the function $f(z)$ defined by (2) be in the class $A(p,\gamma,\omega)$. Then $f(z)$ is multivalent close-to-convex of order δ $(0 \leq \delta < p)$ in the disc $|z| < r_3$, where

$$r_3(p,\gamma,\omega,\delta) = \inf_k \left[\frac{(k-1)(p-\delta)[k-p-\omega(k+1)+\gamma]}{p(p-1)[\omega(p+1)-\gamma]}\right]^{\frac{1}{k-p}}.$$

$(k \geq p + 1)$.

The result is sharp for the external function $f(z)$ given by (6).

Proof: We must show that:

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p - \delta\,, \quad (0 \leq \delta < p),$$

for $|z| < r_3(p,\gamma,\omega,\delta)$.

We have: $\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq \sum_{k=p+1}^{\infty} a_k|z|^{k-p}$.

Thus

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p - \delta$$

If $\sum_{k=p+1}^{\infty} \frac{ka_k|z|^{k-p}}{(p-\delta)} \leq 1$.

Hence by Corollary (2.1.1), the last statement will be true if:

$$\frac{k|z|^{k-p}}{(p-\delta)} \leq \frac{k(k-1)[k-p-\omega(k+1)+\gamma]}{p(p-1)[\omega(p+1)-\gamma]},$$

equivalently if

$$|z| \leq \left[\frac{(k-1)(p-\delta)[k-p-\omega(k+1)+\gamma]}{p(p-1)[\omega(p+1)-\gamma]}\right]^{\frac{1}{k-p}}. \quad (11)$$

The theorem follows easily from (11) ■

2.4. Extreme Points.

The following theorem discuss the extreme points of the class $A(p,\gamma,\omega)$.

Theorem 2.4.1. Let $f_p(z) = z^p$ and

$$f_k(z) = z^p + \frac{p(p-1)[\omega(p+1)-\gamma]}{k(k-1)[k-p-\omega(k+1)+\gamma]} z^k ,$$

$$where \left(k \ge p+1 , p \ge 1, \frac{1}{2} \le \omega < 1, 0 < \gamma \le \frac{1}{2}\right).$$

Then the function f belongs to the class $A(p,\gamma,\omega)$ if and only if it can be written as:

$$f(z) = \mathcal{L}_p z^p + \sum_{k=p+1}^{\infty} \mathcal{L}_k f_k(z), \tag{12}$$

such that

$$\left(\mathcal{L}_p \ge 0, \mathcal{L}_k \ge 0, k \ge p+1\right) \ and \ \mathcal{L}_p + \textstyle\sum_{k=p+1}^{\infty} \mathcal{L}_k = 1$$

Proof: Suppose that $f(z)$ that defined in (12). Then

$$f(z) = \mathcal{L}_p z^p + \sum_{k=p+1}^{\infty} \mathcal{L}_k \left[z^p + \frac{p(p-1)[\omega(p+1)-\gamma]}{k(k-1)[k-p-\omega(k+1)+\gamma]} z^k\right]$$

$$= z^p + \sum_{k=p+1}^{\infty} \frac{p(p-1)[\omega(p+1)-\gamma]}{k(k-1)[k-p-\omega(k+1)+\gamma]} \mathcal{L}_k z^k.$$

Hence

$$\sum_{k=p+1}^{\infty} \frac{k(k-1)[k-p-\omega(k+1)+\gamma]}{p(p-1)[\omega(p+1)-\gamma]} \times \frac{p(p-1)[\omega(p+1)-\gamma]}{k(k-1)[k-p-\omega(k+1)+\gamma]} \mathcal{L}_k$$

$$= \sum_{k=p+1}^{\infty} \mathcal{L}_k = 1 - \mathcal{L}_p \le 1 .$$

Thus $f \in A(p,\gamma,\omega)$.

Conversely, suppose that $f \in A(p,\gamma,\omega)$, we may set

$$\mathcal{L}_k = \frac{k(k-1)[k-p-\omega(k+1)+\gamma]}{p(p-1)[\omega(p+1)-\gamma]} a_k ,$$

where a_k is defined in (5). Then

$$f(z) = z^p + \textstyle\sum_{k=p+1}^{\infty} a_k z^k$$

$$= z^p + \sum_{k=p+1}^{\infty} \frac{p(p-1)[\omega(p+1)-\gamma]}{k(k-1)[k-p-\omega(k+1)+\gamma]} \mathcal{L}_k z^k$$

$$= z^p + \sum_{k=p+1}^{\infty} [f_k(z) - z^p]$$

$$= \sum_{k=p+1}^{\infty} \mathcal{L}_k f_k(z) + (1 - \sum_{k=p+1}^{\infty} \mathcal{L}_k) z^p =$$

$$= f(z) = \mathcal{L}_p z^p + \sum_{k=p+1}^{\infty} \mathcal{L}_k f_k(z)$$

This complete the proof of Theorem (7) ■

3. Convolution Properties.

The following theorems shows the convolution properties for the functions in the class $A(p,\gamma,\omega)$.

Theorem 3.1 Let the functions $f_r(z) \in A(p,\gamma,\omega)$ such that

$$f_r(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,r} z^k, \ \left(a_{k,r} \ge 0, \quad r = 1,2\right). \tag{13}$$

Then $(f_1 * f_2) \in A(p,\gamma,d)$, where

$$d \ge \frac{p(p-1)[(\omega p+1)-\gamma]^2(k-p+\gamma) + \gamma k(k-1)[k-p-(\omega k+1)+\gamma]^2}{k(p+1)(k-1)[k-p-(\omega k+1)+\gamma]^2 + p(k+1)(p-1)[(\omega p+1)-\gamma]^2}.$$

The result is sharp for the functions f_r $(r = 1,2)$ given by (6).

Proof: We will find the smallest d such that

$$\sum_{k=p+1}^{\infty} \frac{k(k-1)[k-p-d(k+1)+\gamma]}{p(p-1)[d(p+1)-\gamma]} a_{k,1} a_{k,2} \le 1$$

Since $f_r \in A(p,\gamma,\omega), (r = 1,2)$, then

$$\sum_{k=p+1}^{\infty} \frac{k(k-1)[k-p-(\omega k+1)+\gamma]}{p(p-1)[(\omega p+1)-\gamma]} a_{k,r} \le 1 , \quad (r = 1,2)$$

By Cauchy-Schwarz inequality, we get

$$\sum_{k=p+1}^{\infty} \frac{k(k-1)[k-p-(\omega k+1)+\gamma]}{p(p-1)[(\omega p+1)-\gamma]} \sqrt{a_{k,1} a_{k,2}} \le 1. \tag{14}$$

Now, we need only to show that:

$$\frac{k(k-1)[k-p-d(k+1)+\gamma]}{p(p-1)[d(p+1)-\gamma]}a_{k,1}a_{k,2} \le \frac{k(k-1)[k-p-(\omega k+1)+\gamma]}{p(p-1)[(\omega p+1)-\gamma]}\sqrt{a_{k,1}a_{k,2}}\ ,$$

and this equivalently to:

$$\sqrt{a_{k,1}a_{k,2}} \le \frac{[d(p+1)-\gamma][k-p-(\omega k+1)+\gamma]}{[k-p-d(k+1)+\gamma][(\omega p+1)-\gamma]}.$$

From (14), we have

$$\sqrt{a_{k,1}a_{k,2}} \le \frac{p(p-1)[(\omega p+1)-\gamma]}{k(k-1)[k-p-(\omega k+1)+\gamma]}.$$

Thus, it is sufficient to show that

$$\frac{p(p-1)[(\omega p+1)-\gamma]}{k(k-1)[k-p-(\omega k+1)+\gamma]} \le \frac{[d(p+1)-\gamma][k-p-(\omega k+1)+\gamma]}{[k-p-d(k+1)+\gamma][(\omega p+1)-\gamma]}\ ,$$

which implies to

Thus, the theorem is established ■

Theorem 3.2. Let the functions $f_r(z)$ in Theorem 3.1 belongs to the class $A(p,\gamma,\omega)$. Then the function $h(z)=z^p+\sum_{k=p+1}^{\infty}\left(a_{k,1}^2+a_{k,2}^2\right)z^k$, belongs also to the class $A(p,\gamma,\omega)$

where

$$p(p+1)[1-(\omega(p+1)+1)+\gamma]-2p(p-1)[(\omega p+1)-\gamma] \ \ \ge 0.$$

Proof: Since $f_1(z)\in A(p,\gamma,\omega)$, we get

$$\sum_{k=p+1}^{\infty}\left[\frac{k(k-1)[k-p-(\omega k+1)+\gamma]}{p(p-1)[(\omega p+1)-\gamma]}\right]^2 a_{k,1}^2 \le \left(\sum_{k=p+1}^{\infty}\left[\frac{k(k-1)[k-p-(\omega k+1)+\gamma]}{p(p-1)[(\omega p+1)-\gamma]}\right]a_{k,1}\right)^2 \le 1\ , \tag{15}$$

and

$$\sum_{k=p+1}^{\infty}\left[\frac{k(k-1)[k-p-(\omega k+1)+\gamma]}{p(p-1)[(\omega p+1)-\gamma]}\right]^2 a_{k,2}^2 \le \left(\sum_{k=p+1}^{\infty}\left[\frac{k(k-1)[k-p-(\omega k+1)+\gamma]}{p(p-1)[(\omega p+1)-\gamma]}\right]a_{k,2}\right)^2 \le 1\ . \tag{16}$$

Combining the inequalities (15) and (16), gives

$$\sum_{k=p+1}^{\infty}\frac{1}{2}\left[\frac{k(k-1)[k-p-(\omega k+1)+\gamma]}{p(p-1)[(\omega p+1)-\gamma]}\right]^2\left(a_{k,1}^2+a_{k,2}^2\right) \le 1\ . \tag{17}$$

According to Theorem (2.1), it is sufficient to show that:

$$\sum_{k=p+1}^{\infty}\left[\frac{k(k-1)[k-p-(\omega k+1)+\gamma]}{p(p-1)[(\omega p+1)-\gamma]}\right]\left(a_{k,1}^2+a_{k,2}^2\right) \le 1.$$

Thus the last inequality, will be satisfies if, for $k=p+1,p+2,p+3,\ldots$

$$\left[\frac{k(k-1)[k-p-(\omega k+1)+\gamma]}{p(p-1)[(\omega p+1)-\gamma]}\right] \le \frac{1}{2}\left[\frac{k(k-1)[k-p-(\omega k+1)+\gamma]}{p(p-1)[(\omega p+1)-\gamma]}\right]^2.$$

Or if

$$k(k-1)[k-p-(\omega k+1)+\gamma]-2p(p-1)[(\omega p+1)-\gamma] \ge 0. \tag{18}$$

For $(k=p+1,p+2,p+3,\ldots)$ the left hand side of (18) is increasing function of k, hence it is satisfied for all k if:

$$p(p+1)[1-(\omega(p+1)+1)+\gamma]-2p(p-1)[(\omega p+1)-\gamma] \ge 0.$$

which is true by our assumption. Therefor the prove is complete ■

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