On a Differential Subordination of a Certain Subclass of Univalent Functions

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Abstract **:** *In this paper, we introduce and discuss a certain subclass* $\boldsymbol{A}(\boldsymbol{\alpha},\boldsymbol{\beta})$ *of univalent functions in the open unit disc, we obtain some properties like coefficient estimates and results of integral means by using differential subordination.*

Keywords: ***Univalent Function; Coefficient Estimates; Integral Means; Differential Subordination; Convolution.***

I. INTRODUCTION

Let W be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U, \mathbb{N} = \{1,2,3,\dots\}) \tag{1}$$

which are analytic and univalent in open unit disc $U = \{z: |z| < 1\}$.

The Hadamard product or (convolution) of function $f(z)$ given by (1) and function $g(z)$ is defined by:

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad (z \in U, \mathbb{N} = \{1,2,3,\dots\}) \tag{2}$$

in the class W is

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad (z \in U, n \in \mathbb{N} = \{1,2,3,\dots\}). \tag{3}$$

Let A denotes the subclass of W of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0, n \in \mathbb{N} = \{1,2,3,\dots\}). \tag{4}$$

A function f in the class W is said to be univalent convex function [2] of order δ if: $Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \delta$

$$(0 \leq \delta < 1, z \in U, f'(z) \neq 0). \tag{5}$$

In the following definition, we give the condition for the function f which is defined in (4) and belongs to the class $A(\alpha,\beta)$.

Definition 1.1: A function $f \in A$ is in the class $A(\alpha,\beta)$ if it satisfies the following condition:

$$\left|\frac{\left(1 + \frac{zf''(z)}{f'(z)}\right) + 1}{2\left(1 + \frac{zf''(z)}{f'(z)}\right) + 2\alpha}\right| < \beta$$

$$where \left(\frac{1}{2} < \beta \leq 1, \quad \alpha > 1\right). \tag{6}$$

Many different authors studied classes of univalent functions for other classes like, Darus [1], Goodman [3], Gupta and Jain [4], Owa [8], Schild and Silvermen [9], Swag [10] and others.

In this paper, we obtain coefficient estimates and proof of several theorems by using the definition of subordination of function which was introduced by (Miller and Mocanu [7]) and Littlewood theorem of subordination [5], (see also Duren [2]).

Definition 1.2_[6]. If f *and* g be two analytic functions in the open unit disc U. Then g is said to be subordinate to f, written $g \prec f$ or $g(z) \prec f(z)$, if there exists a Schwarz function w, which is analytic in U, with $w(0) = 0$ *and* $|w(z)| = 1$, $(z \in U)$, such that $g(z) = f(w(z))$, $(z \in U)$. Indeed it is known that $g(z) \prec f(z), \ (z \in U) \Rightarrow g(0) = f(0)$ *and* $g(U) \subset f(U)$. In particular, if f is univalent in U, we have the following equivalence:

$$g(z) \prec f(z), \quad (z \in U) \iff g(0) = f(0) \text{ and } g(U) \subset f(U).$$

Theorem 1.1 [5] (Littlewood Theorem)

If the functions f *and* g are analytic in U such that $g \prec f$, then, for $\tau > 0$ *and* $z = re^{i\theta}(0 < r < 1)$

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\tau} d\theta \le \int_0^{2\pi} \left|g(re^{i\theta})\right|^{\tau} d\theta. \qquad (7)$$

Theorem 1.2 [2] (Maximum Modulus Theorem)

Suppose that a function f is continuous on a boundary of D (D any disk or region). Then, the maximum value of $|f(z)|$, which is always reached, occurs somewhere on the boundary of D and never in the interior.

2. Cofficient Estimates.

Here, we give necessary and sufficient condition for the function f to be in the class $A(\alpha, \beta)$, as follows :

Theorem 2.1 The function f be in the class $A(\alpha, \beta)$ of the form (4) if and only if

$$\sum_{n=2}^{\infty} n[(n+1) - 2\beta(\alpha+n)]a_n \le 2[\beta(\alpha+1) - 1], \qquad (8)$$

where $\left(\frac{1}{2} < \beta \le 1, \ \alpha > 1, \quad n \in \mathbb{N} = \{1,2,3,\dots\}\right)$,

and the result is a sharp for the function

$$f(z) = z + \frac{2[\beta(\alpha+1) - 1]}{n[(n+1) - 2\beta(\alpha+n)]} z^n, \qquad (n \ge 2). \quad (9)$$

Proof: Let f *in the class* $A(\alpha, \beta)$, *then* f satisfies the inequality (6) which is equivalent to :

$$\left|\frac{2f'(z) + zf''(z)}{2(1+\alpha)f'(z) + 2zf''(z)}\right| < \beta$$

$$= \left|\frac{2[1 + \sum_{n=2}^{\infty} na_n z^{n-1}] + z[\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}]}{2(1+\alpha)[1 + \sum_{n=2}^{\infty} na_n z^{n-1}] + 2z[\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}]}\right| < \beta$$

$$= \left|\frac{2 + \sum_{n=2}^{\infty} 2na_n z^{n-1} + \sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{2(1+\alpha) + \sum_{n=2}^{\infty} 2n(1+\alpha)a_n z^{n-1}] + \sum_{n=2}^{\infty} 2n(n-1)a_n z^{n-1}}\right| < \beta$$

$$= \left|\frac{2 + \sum_{n=2}^{\infty} n(n+1)a_n z^{n-1}}{2(1+\alpha) + \sum_{n=2}^{\infty} 2n(n+\alpha)a_n z^{n-1}]}\right| < \beta.$$

Since $|Re(z)| \le |z|$ for all z, we have

$$Re\left\{\frac{2 + \sum_{n=2}^{\infty} n(n+1)a_n z^{n-1}}{2(1+\alpha) + \sum_{n=2}^{\infty} 2n(n+\alpha)a_n z^{n-1}]}\right\} < \beta.$$

Then by choosing the value of z on the real axis and letting $z \to 1^-$ through values, we get:

$$2 + \sum_{n=2}^{\infty} n(n+1)a_n < 2\beta(1+\alpha) + \sum_{n=2}^{\infty} 2n\beta(n+\alpha)a_n$$

Hence

$$\sum_{n=2}^{\infty} n[(n+1) - 2\beta(n+\alpha)]a_n < 2[\beta(1+\alpha) - 1].$$

Conversely, we assume that (8) satisfies and $|z| = 1$, then :

$$|2f'(z) + zf''(z)| - \beta|2(1+\alpha)f'(z) + 2zf''(z)|$$

$$= \left|2[1 + \sum_{n=2}^{\infty} na_n z^{n-1}] + z[\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}]\right| - \beta\left|2(1+\alpha)[1 + \sum_{n=2}^{\infty} na_n z^{n-1}] + 2z[\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}]\right|$$

$$= \left|2 + \sum_{n=2}^{\infty} n(n+1)a_n z^{n-1}\right| - \beta\left|2(1+\alpha) + \sum_{n=2}^{\infty} 2n(n+\alpha)a_n z^{n-1}]\right|$$

$$\leq 2+\sum_{n=2}^{\infty} n(n+1)a_n-\beta\left(2(1+\alpha)+\sum_{n=2}^{\infty} 2n(n+\alpha)a_n\right)$$

$$=\sum_{n=2}^{\infty} n[(n+1)-2\beta(\alpha+n)]a_n-2[\beta(\alpha+1)-1]\leq 0,$$

by hypothesis.

Then by Maximum Modulus Theorem, we have $f \in A(\alpha,\beta)$ ■

Corollary 2.1. Let the function $f(z)$ *of the class* $A(\alpha,\beta)$. *Then*

$$a_n \leq \frac{2[\beta(\alpha+1)-1]}{n[(n+1)-2\beta(\alpha+n)]}, \qquad (n\geq 2). \qquad (10)$$

Theorem 2.2. Let $f \in A(\alpha,\beta)$ and f_i is defined by

$$f_i(z)=z+\frac{2[\beta(\alpha+1)-1]}{i[(i+1)-2\beta(\alpha+i)]}z^i.$$

If there exists an analytic function w defined by :

$$[w(z)]^{i-1}=\frac{i[(i+1)-2\beta(\alpha+i)]}{2[\beta(\alpha+1)-1]}\sum_{n=2}^{\infty} na_n z^{n-1} \qquad (11)$$

Then, for $z=re^{i\theta}$ and $(0<r<1)$

$$\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^{\tau}d\theta \leq \int_0^{2\pi}\left|f_i\left(re^{i\theta}\right)\right|^{\tau}d\theta\,, \quad where\ (\tau>0) \qquad (12)$$

Proof. Since $f \in A(\alpha,\beta)$, then

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n\ ,(n\in\mathbb{N})$$

and

$$f_i(z)=z+\frac{2[\beta(\alpha+1)-1]}{i[(i+1)-2\beta(\alpha+i)]}z^i. \qquad (13)$$

Then, we must show that:

$$\int_0^{2\pi}\left|1+\sum_{n=2}^{\infty} a_n z^{n-1}\right|^{\tau} \leq \int_0^{2\pi}\left|1+\frac{2[\beta(\alpha+1)-1]}{i[(i+1)-2\beta(\alpha+i)]}z^{i-1}\right|^{\tau}d\theta.$$

By Theorem 1.1, it is sufficient to show that:

$$1+\sum_{n=2}^{\infty} a_n z^{n-1} \prec 1+\frac{2[\beta(\alpha+1)-1]}{i[(i+1)-2\beta(\alpha+i)]}z^{i-1}\,.$$

Set

$$1+\sum_{n=2}^{\infty} a_n z^{n-1}=1+\frac{2[\beta(\alpha+1)-1]}{i[(i+1)-2\beta(\alpha+i)]}[w(z)]^{i-1} \quad (14)$$

From (14) and (8), we obtain

$$|w(z)|^{i-1}=\left|\frac{i[(i+1)-2\beta(\alpha+i)]}{2[\beta(\alpha+1)-1]}\right|\left|\sum_{n=2}^{\infty} a_n z^{n-1}\right|$$
$$\leq |z|\frac{i[(i+1)-2\beta(\alpha+i)]}{2[\beta(\alpha+1)-1]}\sum_{n=2}^{\infty} a_n$$
$$\leq |z| \quad ■$$

Theorem 2.3. If $f \in A(\alpha,\beta)$ and

$$f_i(z)=z+\frac{2[\beta(\alpha+1)-1]}{i[(i+1)-2\beta(\alpha+i)]}z^i.$$

Then

$$\int_0^{2\pi}\left|f'\left(re^{i\theta}\right)\right|^{\tau}d\theta \leq \int_0^{2\pi}\left|f_i{}'\left(re^{i\theta}\right)\right|^{\tau}d\theta$$

$$where\ \left(z=re^{i\theta},\tau>0\ and\ 0<r<1\ \right).$$

Proof: Since

$$f'(z)=1+\textstyle\sum_{n=2}^{\infty} na_n z^{n-1}\ \ and$$

$$f_i{}'(z)=1+\frac{2[\beta(\alpha+1)-1]}{[(i+1)-2\beta(\alpha+i)]}z^{i-1}.$$

It is sufficient to show that:

$$1+\sum_{n=2}^{\infty} n a_n z^{n-1} \prec 1+\frac{2[\beta(\alpha+1)-1]}{[(i+1)-2\beta(\alpha+i)]} z^{i-1}.$$

Set

$$1+\sum_{n=2}^{\infty} n a_n z^{n-1} = 1 + \frac{2[\beta(\alpha+1)-1]}{[(i+1)-2\beta(\alpha+i)]}[w(z)]^{i-1}. \qquad (15)$$

From (14) and (8), we have:

$$|w(z)|^{i-1} = \left|\frac{[(i+1)-2\beta(\alpha+i)]}{2[\beta(\alpha+1)-1]}\right| \left|\sum_{n=2}^{\infty} n a_n z^{n-1}\right| \le |z| \frac{[(i+1)-2\beta(\alpha+i)]}{2[\beta(\alpha+1)-1]} \sum_{n=2}^{\infty} n a_n \le |z| \ \blacksquare$$

Theorem 2.4. Let $f \in A(\alpha,\beta)$ be of the form (4) and g is given by

$$g(z) = z+\sum_{n=2}^{\infty} b_n z^n \quad (b_n \ge 0, \quad z \in U, \quad n \in \mathbb{N} = \{1,2,3,\dots\}),$$

and let:

$$\frac{R_i}{b_i} = \min_{n\ge 2} \frac{R_n}{b_n} \quad (for\ some\ i \in \mathbb{N}) \ where\ R_n = \frac{n[(n+1)-2\beta(\alpha+n)]}{2[\beta(\alpha+1)-1]}.$$

Also for some $i \in \mathbb{N}$, the functions f_i *and* g_i be defined respectively by:

$$f_i(z) = z+\frac{2[\beta(\alpha+1)-1]}{i[(i+1)-2\beta(\alpha+i)]} z^i,$$
$$g_i(z) = z + b_i z^i. \qquad (16)$$

Then:

$$\int_0^{2\pi} |(f*g)(z)|^{\tau} d\theta \le \int_0^{2\pi} |(f_i * g_i)(z)|^{\tau} d\theta$$

$$where\ \left(z = re^{i\theta}, \tau > 0, 0 < r < 1\right).$$

Proof. The Hadamard product of f *and* g is given by :

$$(f*g)(z) = z+\sum_{n=2}^{\infty} a_n b_n z^n,$$

and from (16), we have :

$$(f_i * g_i)(z) = z+\frac{2[\beta(\alpha+1)-1]b_i}{i[(i+1)-2\beta(\alpha+i)]} z^i.$$

Now, we must show that for $z = re^{i\theta}, \tau > 0$ *and* $0 < r < 1$

$$\int_0^{2\pi} \left|1+\sum_{n=2}^{\infty} a_n b_n z^{n-1}\right|^{\tau} d\theta \le \int_0^{2\pi} \left|1 + \frac{2[\beta(\alpha+1)-1]}{i[(i+1)-2\beta(\alpha+i)]} z^{i-1}\right|^{\tau} d\theta.$$

By applying Theorem (1.1), it would be sufficient to show that

$$1+\sum_{n=2}^{\infty} a_n b_n z^{n-1} \prec 1+\frac{2[\beta(\alpha+1)-1]}{i[(i+1)-2\beta(\alpha+i)]} z^{i-1}. \qquad (17)$$

If the subordination (17) holds true, then there exists an analytic function w with $w(0) = 0$ *and* $|w(1)| < 1$ such that:

$$1+\sum_{n=2}^{\infty} a_n b_n z^{n-1} = 1+\frac{2[\beta(\alpha+1)-1]b_i}{i[(i+1)-2\beta(\alpha+i)]}[w(z)]^{i-1}.$$

From the hypothesis of the Theorem (2.2), there exists an analytic function w given by:

$$[w(z)]^{i-1} = \frac{i[(i+1)-2\beta(\alpha+i)]}{2[\beta(\alpha+1)-1]} \sum_{n=2}^{\infty} a_n b_n z^{n-1},$$

which readily yields $w(0) = 0$. Thus for such function w, using the hypothesis in the coefficient inequality for the class $A(\alpha,\beta)$, we have:

$$|w(z)|^{i-1} = \left|\frac{[(i+1)-2\beta(\alpha+i)]}{2[\beta(\alpha+1)-1]}\right| \left|\sum_{n=2}^{\infty} a_n b_n z^{n-1}\right| \le |z| \frac{[(i+1)-2\beta(\alpha+i)]}{2[\beta(\alpha+1)-1]} \sum_{n=2}^{\infty} a_n b_n \le |z|.$$

Therefore the subordination (17) holds true ■

In the next theorem, we discuss the integral means inequalities for $f \in A(\alpha,\beta)$ and h defined by :

$$h(z) = z + b_i z^i + b_{2i-1} z^{2i-1} \ (b_i \ge 0, i \ge n) \qquad (18)$$

Theorem 2.5. Let $h(z)$ given by (18) and let $\in A(\alpha,\beta)$, if f satisfies :

$$\sum_{n=2}^{\infty} a_n \leq \frac{b_{2i-1}}{2(i-1)} - \frac{b_i}{(i-1)} \quad , \qquad where \ \frac{b_i}{(i-1)} < \frac{b_{2i-1}}{2(i-1)}, \qquad (19)$$

and there exists an analytic function w such that :

$$b_{2i-1}(w(z))^{2(i-1)} + b_i(w(z))^{(i-1)} - \sum_{n=2}^{\infty} a_n z^{n-1} = 0 .$$

Then , for $z = re^{i\theta}, \tau > 0 \ and \ \ 0 < r < 1$

$$\int_0^{2\pi} |f(z)|^\tau d\theta \leq \int_0^{2\pi} |h(z)|^\tau d\theta \ .$$

Proof . By $z = re^{i\theta} \ and \ \ 0 < r < 1$, we see that

$$\int_0^{2\pi} |f(z)|^\tau d\theta = \int_0^{2\pi} \left| z + \sum_{n=2}^{\infty} a_n z^n \right|^\tau d\theta$$
$$= (r)^\tau \int_0^{2\pi} \left| 1 + \sum_{n=2}^{\infty} a_n z^{n-1} \right|^\tau d\theta$$

and

$$\int_0^{2\pi} |h(z)|^\tau d\theta = \int_0^{2\pi} \left| z + b_i z^i + b_{2i-1} z^{2i-1} \right|^\tau d\theta$$
$$= \ (r)^\tau \left| 1 + b_i z^{i-1} + b_{2i-1} z^{2(i-1)} \right|^r d\theta.$$

By using Theorem (1.1), we have to show that:

$$1 + \sum_{n=2}^{\infty} a_n z^{n-1} \prec \ 1 + b_i z^{i-1} + b_{2i-1} z^{2(i-1)}.$$

We define the function w by:

$$1 + \sum_{n=2}^{\infty} a_n z^{n-1} = \ 1 + b_i(w(z))^{i-1} + b_{2i-1}(w(z))^{2(i-1)}, (20)$$

or equivalent to:

$$b_{2i-1}(w(z))^{2(i-1)} + b_i(w(z))^{i-1} - \sum_{n=2}^{\infty} a_n z^{n-1} = 0.$$

Now if $z = 0$, then $(w(0))^{i-1}\{b_{2i-1}(w(0))^{(i-1)} + b_i\} = 0.$

So there exists an analytic function $w \ in \ U \ such \ that \ w(0) = 0$.

Next , we prove the function w which is analytic in U and satisfies $|w(z)| = 1, (z \in U)$, for the condition (19). From (20), we have that:

$$\left| b_{2i-1}\big(w(z)\big)^{2(i-1)} + b_i\big(w(z)\big)^{i-1} \right| = \left| \sum_{n=2}^{\infty} a_n z^{n-1} \right| < \sum_{n=2}^{\infty} a_n.$$

For $z \in U$, hence

$$b_{2i-1}\left|\big(w(z)\big)\right|^{2(i-1)} + b_i\left|\big(w(z)\big)\right|^{(i-1)} - \textstyle\sum_{n=2}^{\infty} a_n < 0 \qquad (21)$$

Letting $c = \left|\big(w(z)\big)\right|^{(i-1)} \ \ (c \geq 0) \ \ in \ (21)$ and we define $Q(c)$ by:

$$Q(c) = b_{2i-1}c^2 + b_i c - \sum_{n=2}^{\infty} a_n.$$

If $Q(c) \geq 0, where \ c < 1 \ \ for \ Q(c) < 0, we \ obtain$

$$Q(1) = b_{2i-1} - b_i - \sum_{n=2}^{\infty} a_n \geq 0$$

That is

$$\sum_{n=2}^{\infty} a_n \leq b_{2i-1} - b_i \blacksquare$$

Theorem 2.6. Let $f_i \in A(\alpha,\beta) \ \ where \ (i = 1,2, \dots, m)$ and

$$h(z) = z + \sum_{n=2}^{\infty} \big(\sum_{i=1}^{m} a_{n,i}^2\big) \, z^n \qquad (22)$$

Then, $h(z) \in A(\alpha, l)$, where

$$l \leq \frac{mn(n+1)\left[2(\beta_0(\alpha+1)-1)^2 - 2n\big((n+1) - 2\beta_0(\alpha+n)\big)^2\right]}{2(\alpha+1)[n((n+1)-2\beta_o(\alpha+n)]^2 - 2nm(\alpha+n)[2(\beta_0(\alpha+1)-1)]^2},$$

where, $[\beta_0 = \min(\beta_1, \dots, \beta_m)]$.

The result is a sharp for the function f_i which is given by

$$f_i\,(z) = z + \frac{[\beta_i(\alpha+1)-1]}{[3-2\beta_i(\alpha+2)]} z^2 \,, (i = 1, \dots, m). \qquad (23)$$

Proof. From Theorem (2.1), we have:

$$\sum_{n=2}^{\infty}\left\{\frac{n[(n+1)-2\beta_i(\alpha+n)]}{2[\beta_i(\alpha+1)-1]}\right\}^2 a_{n,i}^2 \leq \sum_{n=2}^{\infty}\left\{\frac{n[(n+1)-2\beta_i(\alpha+n)]}{2[\beta_i(\alpha+1)-1]}a_{n,i}\right\}^2 \leq 1.$$

For $(i=1,\dots,m)$, we have

$$\sum_{n=2}^{\infty}\frac{1}{m}\left\{\frac{n[(n+1)-2\beta_i(\alpha+n)]}{2[\beta_i(\alpha+1)-1]}\right\}^2\left(\sum_{i=1}^{m}a_{n,i}^2\right)\leq 1.$$

Now, we try to find the largest l such that:

$$\sum_{n=2}^{\infty}\left\{\frac{n[(n+1)-2l(\alpha+n)}{2[l(\alpha+1)-1]}\right\}\left(\sum_{i=1}^{m}a_{n,i}^2\right)\leq 1,$$

and the last inequality is true if

$$\left\{\frac{n[(n+1)-2l(\alpha+n)}{2[l(\alpha+1)-1]}\right\}\leq\frac{1}{m}\left\{\frac{n[(n+1)-2\beta_i(\alpha+n)]}{2[\beta_i(\alpha+1)-1]}\right\}^2.$$

From the last expression, we get:

$$l\leq\frac{mn(n+1)[2(\beta_i(\alpha+1)-1)^2-2n\big((n+1)-2\beta_i(\alpha+n)\big)^2]}{2(\alpha+1)[n((n+1)-2\beta_i(\alpha+n)]^2-2nm(\alpha+n)[2(\beta_i(\alpha+1)-1)]^2}$$

That is: $l\leq\frac{mn(n+1)\left[2(\beta_0(\alpha+1)-1)^2-2n\big((n+1)-2\beta_0(\alpha+n)\big)^2\right]}{2(\alpha+1)[n((n+1)-2\beta_o(\alpha+n)]^2-2nm(\alpha+n)[2(\beta_0(\alpha+1)-1)]^2}$,

where, $[\beta_0=\min(\beta_1,\dots,\beta_m)]$ ■

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