$\mathrm{ZT_2}$ space and semi-$\mathrm{ZT_2}$ space

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DOI : http://dx.doi.org/10.31642/JoKMC/2018/050305

Abstract: The aim of this paper is to introduce new types of separation axioms which are ZT_2 space and semi-ZT_2 space, with examples and theorems. The relationships between them and with each of T_2 and semi-T_2 spaces have been given. The product of two ZT_2 (semi-ZT_2) spaces is ZT_2 (semi-ZT_2) space.

Keywords: T_2 space, semi-T_2 space, ZT_2 space, semi ZT_2 space.

1 Introduction

The concept of the semi-open set defined by Levine in 1963 [7], is as a follows: in a space X, A is a semi-open set if there exists an open set U such that $U \subseteq A \subseteq \overline{U}$. In 1971 S. Gene Crossley and S. K. Hildebrand [9], defined the semi-closed set, semi-interior, and semi-closure as the complement of semi-open set, the union of all semi-open sets of X contained in A and the intersection of all semi-closed sets containing A, respectively. In 1973 P.Das [9], defined semi-boundary of A as $\overline{A}^s - A^{\circ s}$. We shall use A^c, $A^{\circ s}$, $\overline{A}^s$, A^{bs}, A°, $\overline{A}$, A^e, A^b, $(X \times Y, \tau_{pro})$, $\mathbb{R}$, $\mathbb{Z}$ to the complement of A, semi-interior, semi-closure, semi-boundary, interior of A, closure of A, exterior of A, boundary of A, the product space, real number, integer number, respectively.

2 Fundamental Concepts

Definition 2.1[5] A space (X,T) is called T_2 space iff for any $x, y \in X, x \neq y$ there exist open sets U, V such that $U \cap V = \phi$, $x \in U$ and $y \in V$.

Definition 2.2[6] A space (X,T) is called a semi-T_2 space iff for any $x, y \in X, x \neq y$ there exist semi-open sets U, V such that $U \cap V = \phi$, $x \in U$ and $y \in V$.

Theorem 2.3[4] Let (X, τ) and (Y, τ^*) be spaces, $A \subset X$ and $B \subset Y$, then $(A \times B)^b = (A^b \times \overline{B}) \cup (\overline{A} \times B^b)$.

Theorem 2.4[2] If $f : X \to Y$ is a homomorphism, then $f(A^b) = (f(A))^b$, for all $A \subset X$

Theorem 2.5[3] Let (X, τ) and (Y, τ^*) be spaces, $A \subset X$ and $B \subset Y$, then $\overline{A \times B}^s \subset \overline{A}^s \times \overline{B}^s$.

Remark 2.6[1] In a space X, $A^{bs} = \overline{A}^s \cap \overline{A^c}^s$.

Remark 2.7[9] In a space X, $A^{\circ s} = A - A^{bs}$.

Theorem 2.8 [10] The product of semi-open sets is a semi-open set.

Theorem 2.9[8] If $f : X \longrightarrow Y$ is a semi-homomorphism then:

i. $\overline{f^{-1}(B)}^s = f^{-1}(\overline{B}^s)$, for all $B \subseteq Y$

ii. $(f^{-1}(B))^{\circ s} = f^{-1}(B^{\circ s})$, for all $B \subseteq Y$

iii. $\overline{f(A)}^s = f(\overline{A}^s)$, for all $A \subset X$

iv. $(f(B))^{\circ s} = f(B^{\circ s})$, for all $A \subset X$

Remark 2.10 [5] In any space, $A^b = \phi$ iff A is a clopen set.

Theorem 2.11[8] Let X be a space and A be a subset of X, then:

i. A is semi-open set iff $A = A^{\circ s}$

ii. A is semi-closed set iff $A = \overline{A}^s$

Theorem 2.12 If $A \subset X,\ B \subset Y$ and $f : X \longrightarrow Y$ be a semi-homomorphism, then:

i. $f(A^{bs}) = [f(A)]^{bs}$

ii. $f^{-1}(B^{bs}) = [f^{-1}(B)]^{bs}$

Proof: i. Since f is a semi-homomorphism, then $f(\overline{A}^s) = \overline{(f(A))}^s$ and $f(A^{\circ s}) = (f(A))^{\circ s}$, for all $A \subset X$, [Theorem 2.9 (iii, iv)].
$A^{bs} = \overline{A}^s - A^{\circ s}$ implies $f(A^{bs}) = f(\overline{A}^s - A^{\circ s}) = f(\overline{A}^s) - f(A^{\circ s})$=$\overline{(f(A))}^s - (f(A))^{\circ s} = (f(A))^{bs}$.

ii. Since f is a semi-homomorphism, then $f^{-1}(\overline{B}^s) = \overline{(f^{-1}(B))}^s$ and $f^{-1}(B^{\circ s}) = (f^{-1}(B))^{\circ s}$, for all $B \subset Y$, [Theorem 2.9 (i, ii)].
$B^{bs} = \overline{B}^s - B^{\circ s}$ implies $f^{-1}(B^{bs}) = f^{-1}(\overline{B}^s - B^{\circ s}) = f^{-1}(\overline{B}^s) - f^{-1}(B^{\circ s})$=$\overline{(f^{-1}(B))}^s - (f^{-1}(B))^{\circ s} = (f^{-1}(B))^{bs}$.

□

Theorem 2.13 In a space X, A is a semi-clopen (semi-open and semi-closed) set iff $A^{bs} = \phi$.

proof: $\Rightarrow$) Since A is semi-clopen, then $\overline{A}^s = A$ and $A^{\circ s} = A$, [Theorem 2.11 (i, ii)], so $A^{bs} = \overline{A}^s - A^{\circ s} = A - A = \phi$
$\Leftarrow$) Since $A^{\circ s} = A - A^{bs}$, [Remark 2.7], and $A^{bs} = \phi$, then $A^{\circ s} = A$, so A is a semi-open set, [Theorem 2.11 (i)].
$A^{bs} = \overline{A}^s - A^{\circ s} = \phi$ implies $\overline{A}^s \subseteq A^{\circ s}$. Since $A^{\circ s} \subseteq A \subseteq \overline{A}^s$, then $A^{\circ s} = A = \overline{A}^s$ and A is semi-closed, [Theorem 2.11 (ii)].

Remark 2.14 In a space X, $A^{bs} \subset A^b$

3 Main Results

Definition 3.1 A space (X, τ) is called ZT_2 space iff for any $x, y \in X,\ x \neq y$ there exist two open sets U, V such that $U \cap V = \phi$, $x \in U, y \in V$, and $U^b \cap V^b = \phi$.

Example 3.2 The space $(\mathbb{R}, \tau_l)$ where τ_l is the lower limit topology is a ZT_2 space :
let $x, y \in \mathbb{R},\ x < y$ and let $d = d(x, y) = |x - y|$.
put $U = [x - 1, x + \frac{d}{4}), V = [y - \frac{d}{2}, y + 1)$. Note that
$U, V \in \tau_l,\ x \in U, y \in V$ and $U \cap V = \phi$
$[x-1, x+\frac{d}{4})^\circ = [x-1, x+\frac{d}{4})$, $[x-1, x+\frac{d}{4})^c = \mathbb{R} \setminus [x - 1, x + \frac{d}{4}) = (-\infty, x - 1) \cup [x + \frac{d}{4}, \infty)$,$([x - 1, x + \frac{d}{4})^c)^\circ = \mathbb{R} \setminus [x - 1, x + \frac{d}{4}) = [x - 1, x + \frac{d}{4})^e$ so $[x - 1, x + \frac{d}{4})^b = \phi$, similarity $V^b = \phi$ then $U^b \cap V^b = \phi$ and so $(\mathbb{R}, \tau_l)$ is ZT_2 space.

Example 3.3 The discrete space of more than one point, (X, τ_d), is a ZT_2 space:

For any $a, b \in X, a \neq b$, $U = \{a\}$, $V = \{b\}$ are open sets such that $a \in U,\ b \in V$ and $U \cap V = \phi$. Since $U^b = \phi$ and $V^b = \phi$ then $U^b \cap V^b = \phi$. Hence (X, τ_d) is ZT_2 space.

Example 3.4 The Cofinite space (X, τ_c), X is an infinite set, is not a ZT_2 space.

Example 3.5 The indiscrete space (X, τ_{ind}) is not a ZT_2 space.

Example 3.6 Let $X = \{1, 2, 3, 4\}, \tau = \{X, \phi, \{1\}, \{2\}, \{1, 2\}\}$
(X, τ) is not a ZT_2 space for $1, 4 \in X, 1 \neq 4$ but can not find disjoint open sets containing 1 and 4.

Theorem 3.7 ZT_2 property is a topological property.

proof: Let $(X,\tau),(Y,\tau^*)$ be spaces such that X is homomorphic to Y and X is ZT_2 space. Since $X \cong Y$, then there exist a homomorphism $f : X \to Y$. For any $y_1, y_2 \in Y, y_1 \neq y_2$ we have $f^{-1}(y_1), f^{-1}(y_2) \in X$ and $f^{-1}(y_1) \neq f^{-1}(y_2)$ for f is one to one.
Since X is a ZT_2 space then there exist $U, V \in \tau$ such that $f^{-1}(y_1) \in U, f^{-1}(y_2) \in V$, $U \cap V = \phi$, and $U^b \cap V^b = \phi$. Now,
$y_1 \in f(U)$ and $y_2 \in f(V)$. Since f is open function then $f(U)$, $f(V)$ are open sets in Y and $f(U) \cap f(V) = f(U \cap V) = f(\phi) = \phi$.
Now, $(f(U))^b \cap (f(V))^b = f(U^b) \cap f(V^b)$, [Theorem 2.4], $= f(U^b \cap V^b) = f(\phi) = \phi$. Hence (Y,τ^*) is a ZT_2 space.

□

Theorem 3.8 If each of (X,τ) and (Y,τ^*) is a ZT_2 space, then the product space $X \times Y$ is ZT_2 space.

proof: Let (X,τ) and (Y,τ^*) be ZT_2 spaces. To prove $(X \times Y, \tau_{pro})$ is also ZT_2 space, let (x_1, y_1), $(x_2, y_2) \in X \times Y$, $(x_1, y_1) \neq (x_2, y_2)$. Suppose that $x_1 \neq x_2$. Since X is a ZT_2 space, $x_1 \neq x_2$ then there exist $U_1, U_2 \in \tau$ such that $x_1 \in U_1, x_2 \in U_2$, $U_1 \cap U_2 = \phi$ and ${U_1}^b \cap {U_2}^b = \phi$.
$U_1 \times Y, U_2 \times Y$ are open sets in $X \times Y$ and $(x_1, y_1) \in U_1 \times Y = G$, $(x_2, y_2) \in U_2 \times Y = H$ and $G \cap H = (U_1 \times Y) \cap (U_2 \times Y) = (U_1 \cap U_2) \times (Y \cap Y) = \phi \times Y = \phi$.

Now, $(U_1 \times Y)^b \cap (U_2 \times Y)^b = \phi$
$(U_1 \times Y)^b \cap (U_2 \times Y)^b = (({U_1}^b \times \overline{Y}) \cup (\overline{U_1} \times Y^b))$
$\cap (({U_2}^b \times \overline{Y}) \cup (\overline{U_2} \times Y^b))$, [Theorem 2.3]
$= (({U_1}^b \times \overline{Y}) \cup (\overline{U_1} \times \phi)) \cap (({U_2}^b \times \overline{Y}) \cup (\overline{U_2} \times \phi))$, [Remark 2.10]
$= (({U_1}^b \times Y) \cup \phi) \cap (({U_2}^b \times Y) \cup \phi)$
$= ({U_1}^b \times Y) \cap ({U_2}^b \times Y)$
$= ({U_1}^b \cap ({U_2}^b) \times (Y \cap Y) = \phi \times Y = \phi$
Hence $X \times Y$ is a ZT_1 space.

□

Remark 3.9 The continuous image of ZT_2 needs not be a ZT_2 space :
$f : (\mathbb{Z}, \tau_d) \to (\mathbb{R}, \tau_{ind})$, $f(x) = x$ is continuous function and, $(\mathbb{Z}, \tau_d)$ is ZT_2 space but $f(\mathbb{Z}) = \mathbb{Z}$ with the relative indiscrete topology is not a ZT_2 space.

Definition 3.10 A space (X,τ) is called semi-ZT_2 space iff for any $x, y \in X$, $x \neq y$ there exist disjoint semi-open sets U, V such that $x \in U, y \in V$, and $U^{bs} \cap V^{bs} = \phi$.

Example 3.11 The real number with the usual topology $(\mathbb{R}, \tau_u)$, is semi-ZT_2 space:

let $x, y \in \mathbb{R}$ with $x < y$ and let $d = d(x, y) = |x-y|$. Put $U = (x-1, x+\frac{d}{4}), V = (y-\frac{d}{2}, y+1)$ which are disjoint semi-open sets such that $x \in U$, $y \in V$.
Since U is semi-clopen, then $U^{bs} = \phi$, [Theorem 2.13].
similarity, $V^{bs} = \phi$, so $U^{bs} \cap V^{bs} = \phi$. Then $(\mathbb{R}, \tau_u)$ is a semi-ZT_1 space.

Example 3.12 The discrete space of more than one point, (X, τ_d), is a semi-ZT_2 space.

Example 3.13 The Cofinite space (X, τ_c), X is an infinite set, is not a semi-ZT_2 space.

Example 3.14 The indiscrete space (X, τ_{ind}), is not a semi-ZT_2 space.

Example 3.15 let $X = \{a, b\}, \tau = \{X, \phi, \{a\}\}$
(X,τ) is not semi-ZT_2 space for $a, d \in X, a \neq d$ but can not find disjoint semi-open sets containing a and d.

Example 3.16 Let $X = \{a, b, c, d\}, \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. (X,τ) is semi-ZT_2 space:
$a, b \in X, a \neq b$, $\{a\}$ and $\{b\}$ are disjoint semi-open sets containing a and b respectively with $\{a\}^{bs} \cap \{b\}^{bs} = \phi$.

$a, c \in X, a \neq c$, $\{a\}$ and $\{b, c\}$ are disjoint semi-open sets containing a and c respectively with $\{a\}^{bs} \cap \{b, c\}^{bs} = \phi$.
$a, d \in X, a \neq d$, $\{a\}$ and $\{b, d\}$ are disjoint semi-open sets containing a and d respectively with $\{a\}^{bs} \cap \{b, d\}^{bs} = \phi$.
$b, c \in X, b \neq c$, $\{b\}$ and $\{a, c\}$ are disjoint semi-open sets containing b and c respectively with $\{b\}^{bs} \cap \{a, c\}^{bs} = \phi$.
$b, d \in X, b \neq d$, $\{b\}$ and $\{a, d\}$ are disjoint semi-open sets containing b and a, d respectively with $\{b\}^{bs} \cap \{a, d\}^{bs} = \phi$.
$c, d \in X, c \neq d$, $\{c, b\}$ and $\{a, d\}$ are disjoint semi-open sets containing c and d respectively with $\{c, b\}^{bs} \cap \{a, d\}^{bs} = \phi$.

Theorem 3.17 Semi-ZT_2 property is a semi-topological property, and then it is a topological property.

proof: Let $(X, \tau), (Y, \tau^*)$ be spaces such that X is semi-homomorphic to Y and X is semi-ZT_2 space. Then there exists a semi-homomorphism $f : X \to Y$. For any $y_1, y_2 \in Y, y_1 \neq y_2$ we have $f^{-1}(y_1), f^{-1}(y_1) \in X$ and $f^{-1}(y_1) \neq f^{-1}(y_2)$ for f is one to one. Since X is semi-ZT_2 space then there exist U, V semi-open sets such that
$f^{-1}(y_1) \in U, f^{-1}(y_2) \in V, U \cap V = \phi$ and $U^{bs} \cap V^{bs} = \phi$. Now,
$y_1 \in f(U)$ $y_2 \in f(V)$ and $f(U) \cap f(V) = f(U \cap V) = f(\phi) = \phi$. Since f pre-semi-open then $f(U)$ and $f(V)$ are semi-open in Y. Now, $(f(U))^{bs} \cap (f(V))^{bs}$, [Theorem 2.12 (i)], $= f(U^{bs}) \cap f(V^{bs}) = f(U^{bs} \cap V^{bs}) = f(\phi) = \phi$. Hence (Y, τ^*) is semi-ZT_2 space.

□

Theorem 3.18 Every ZT_2 space is a semi-ZT_2 space.

Proof: Let (X, τ) be a ZT_2 space and $x, y \in X, x \neq y$, then there exist disjoint open sets U and V such that $x \in U, y \in V$, and $U^b \cap V^b = \phi$. Since every open set is semi-open set and $U^{bs} \subset U^b$, $V^{bs} \subset V^b$, [Remark 2.14], and $U^b \cap V^b = \phi$, then $U^{bs} \cap V^{bs} = \phi$. Hence (X, τ) is a semi-ZT_2 space.

Remark 3.19 Every ZT_2 space is a T_2 space and every semi-ZT_2 space is a semi-T_2 space.

Remark 3.20 The converse of **Theorem 3.18** is not true for the space in the **Example 3.6** is semi-ZT_2 space but not ZT_2 space.

Theorem 3.21 The product space of two semi-ZT_2 spaces is a semi-ZT_2 space.

proof: Let (X, τ) and (Y, τ^*) be semi-ZT_2 spaces. To prove $(X \times Y, \tau_{pro})$ is also semi-ZT_2 space, let (x_1, y_1), $(x_2, y_2) \in X \times Y$, $(x_1, y_1) \neq (x_2, y_2)$. Suppose $x_1 \neq x_2$. Since X is semi-ZT_2 space, $x_1 \neq x_2$ then there exist U_1, U_2 semi-open sets such that $x_1 \in U_1, x_2 \in U_2$, $U_1 \cap U_2 = \phi$ and ${U_1}^{bs} \cap {U_2}^{bs} = \phi$.
$U_1 \times Y, U_2 \times Y$ are semi-open sets in $X \times Y$, [Theorem 2.8], such that $(x_1, y_1) \in U_1 \times Y = G$, $(x_2, y_2) \in U_2 \times Y = H$ and,
$G \cap H = (U_1 \times Y) \cap (U_2 \times Y) = (U_1 \cap U_2) \times (Y \cap Y) = \phi \times Y = \phi$. Now,
$(U_1 \times Y)^{bs} = (\overline{U_1 \times Y}^s) \cap (\overline{(U_1 \times Y)^c}^s)$, [Remark 2.6]

$$= (\overline{U_1 \times Y}^s) \cap (\overline{({U_1}^c \times Y) \cup (X \times Y^c)}^s)$$
$$= (\overline{U_1 \times Y}^s) \cap (\overline{({U_1}^c \times Y) \cup (X \times \phi)}^s)$$
$$= \overline{(U_1 \times Y)}^s \cap \overline{({U_1}^c \times Y)}^s$$
$$\subset (\overline{U_1}^s \times \overline{Y}^s) \cap (\overline{{U_1}^c}^s \times \overline{Y}^s), \text{[Theorem 2.5]}$$
$$= (\overline{U_1}^s \times Y) \cap (\overline{{U_1}^c}^s \times Y)$$
$$= (\overline{U_1}^s \cap \overline{{U_1}^c}^s) \times (Y \cap Y)$$
$$= (\overline{U_1}^s \cap \overline{{U_1}^c}^s) \times Y = {U_1}^{bs} \times Y.$$

Similarity, $(U_2 \times Y)^{bs} \subset {U_2}^{bs} \times Y$.
So $(U_1 \times Y)^{bs} \cap (U_2 \times Y)^{bs} \subset ({U_1}^{bs} \times Y) \cap ({U_2}^{bs} \times Y) = ({U_1}^{bs} \cap {U_2}^{bs}) \times Y = \phi \times Y = \phi$.
Hence the product space of two semi-ZT_2 spaces is a semi-ZT_2 space.

□

Remark 3.22 The semi continuous image of the semi-ZT_2 needs not be a semi-ZT_2 space. $f : (\mathbb{Z}, \tau_d) \to (\mathbb{R}, \tau_{ind})$, $f(x) = x$ is semi continuous function and, $(\mathbb{Z}, \tau_d)$ is semi-ZT_2 space but $f(\mathbb{Z}) = \mathbb{Z}$ with the relative indiscrete topology is not a semi-ZT_2 space.

4 Conclusion

The two separation axioms have the following relationships in diagram.

$$\begin{array}{ccc} \mathbf{ZT_2}\ \textbf{space} & \Rightarrow & \textbf{semi-}\mathbf{ZT_2}\ \textbf{space} \\ \Downarrow & & \Downarrow \\ \mathbf{T_2}\ \textbf{space} & \Rightarrow & \textbf{semi-}\mathbf{T_2}\ \textbf{space} \end{array}$$

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