# Near-Legendre Differential Equations

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***Abstract***—*A differential equation of the form* $\left((1-x^{2m})y^{(k)}\right)^{(2m-k)}+\lambda y=0, -1\le x\le 1, 0\le k\le 2m; k, m$ *integers is called a near-Legendre equation. We show that such an equation has infinitely many polynomial solutions corresponding to infinitely many λ. We list all of these equations for* $1\le m\le 2$*. We show, for* $m=1$*, that these solutions are 'partially' orthogonal with respect to some weight functions and show how to expand functions using these polynomials. We give few applications to partial differential equations.*

***Keywords—Near-Legendre equation; Euler form; eigen polynomial.***

## I. INTRODUCTION

A Legendre polynomial $P_n, n = 0,1,2,\ldots$ is a polynomial solution of the differential equation

$$\left((1-x^2)y'\right)' + \lambda y = y'' - x^2y'' - 2xy' + \lambda y$$
$$= 0, \lambda = n(n+1), -1 \le x \le 1.$$

Usually a Legendre polynomial $P_n$ is defined as a polynomial solution of the differential equation

$$((1-x^2)y')' + n(n+1)y = 0,$$

$n$ nonnegative integer. In the way of generalization some research was done by replacing $n(n+1)$ by $A(A+1)$ where A is an analytic function. Another generalization was done by replacing $n(n+1)$by $A(t)(A(t)+1)$, where $A(t)$ is a probabilistic function as done by [3]. In this article we generalize $(1-x^2)$ to $(1-x^{2m})$ in the original Legendre differential equation. Some authors studied the Legendre differential equation from the perspective of operator theory and eigen values as done in [5]. In this article we use some aspects of this approach but in a slight manner. Some authors applied Legendre equation in solving partial differential equations as in [4]. In this article we have given some examples of partial differential equations that can be solved by near-Legendre differential equations.

Let us call a finite sum $a_n y^{(n)}x^n + a_{n-1}y^{(n-1)}x^{n-1} + \ldots + a^0 y, y \in C^{\{\infty\}}$, an *Euler form*. Let $k$ be an integer. A $k$*-Euler form* $E_k(y)$ is a finite sum $E_k(y) = \sum_{i\ge 0} a_i x^i y^{(i+k)}$. Thus a 0-Euler form is an Euler form. When a $k$-Euler form is multiplied by $x^k$ we get a 0-Euler form. When an Euler form is equated to 0 we get an Euler homogeneous equation. We notice that the right hand side of a Legendre equation is a sum of a derivative, which is a 2-Euler form, and an Euler form involving some parameter λ. So let us call a differential equation
$E_k(y) + E_0(y) = 0, E_0$ is involving $\lambda, -a \le x \le a, a > 0$, a *near- Legendre equation*. It is proved below, Proposition (4), that
$((1-x^{2n})y^{(k)})^{(2n-k)} + \lambda y = 0, -1 \le x \le 1,$
is a near-Legendre differential equation that has polynomial solutions. In such equations sometimes we call the λ eigen values and the corresponding polynomial solutions the *eigen polynomials* although this is an abuse of language.

## II. EXAMPLES

We give below several examples of near-Legendre equations with polynomial solutions.

**Example (1)**: Consider the differential equation

$$((a^2-x^2)y')' + \lambda y = 0, -a \le x \le a.$$

This equation can be written as

$$((a^2-x^2)y')' + \lambda y = (a^2-x^2)y'' - 2xy' + \lambda y$$
$$= a^2y'' - x^2y'' - 2xy' + \lambda y = 0, -a \le x \le a.$$

So it is a near -Legendre equation. We notice that $g(x) = a^2 - x^2$ has no zeros in the interval $(-a, a)$ and $g(x) = 0\ at\ -a, a$.

This is similar to $1 - x^2$ in the original Legendre equation on the interval $[-1,1]$. If we let $u = \frac{x}{a}$ then

$$\frac{dy}{dx} = \left(\frac{1}{a}\right).\frac{dy}{du}, \frac{d^2y}{dx^2} = \left(\frac{1}{a}\right).\frac{d\left(\frac{\left(\frac{1}{a}\right)dy}{du}\right)}{du} = \frac{\left(\frac{1}{a^2}\right)d^2y}{du^2}.$$

Thus the equation reduces to

$$(a^2 - a^2u^2)\left(\frac{\left(\frac{1}{a^2}\right)d^2y}{du^2}\right) - 2au\left(\frac{1}{a}\right)\frac{dy}{du} + \lambda y$$
$$= \frac{(1-u^2)d^2y}{du^2} - \frac{2udy}{du} + \lambda y = 0, -1 \le u \le 1.$$

This is the same as the usual equation

$(1-x^2)y'' - 2xy' + \lambda y = 0, -1 \le x \le 1.$

To get polynomial solutions we must set

$$\lambda = k(k+1), k = 0,1,2,\ldots.$$

Thus the solution to this equation with

$\lambda = k(k+1)$is $q_k(u) = p_k\left(\frac{x}{a}\right), k = 0,1,2,\ldots$

We notice that the polynomial solutions are in powers of $(x-0)$, 0 is the center of the interval $[-a, a]$ and $(a^2 - x^2) = 0$ at $a, -a$. Using a change of variable and orthogonality of $p_k(x)$ , it is seen that the polynomials $p_k\left(\frac{x}{a}\right)$ are orthogonal on the interval $[-a, a]$.

**Example (2)**: Consider the differential equation

$$((c + dx - x^2)\, y')' + \lambda y$$
$$= (dx - 2x)\, y' + (c + dx - x^2)\, y'' + \lambda y$$
$$= 0, d \ne 0, c > 0.$$

This is not a near- Legendre equation but it can be reduced to such equation. Let $h(x) = c + dx - x^2$. The discriminant of $h(x) = 0$ is $d^2 + 4c > 0$ since $c, d^2 > 0$. Thus $h(x) = 0$ has two distinct roots $a, b$ and

$c = \left(\frac{b-a}{2}\right)^2 + \left(\frac{b+a}{2}\right)^2, d = \frac{b+a}{2}.$

Then $h(x) = \left(\frac{b-a}{2}\right)^2 - \left(x - \frac{a+b}{2}\right)^2$ $, a \le x \le b,$
which is a generalization of the function
$f(x) = 1 - x^2, -1 \le x \le 1$ since $h(a) = h(b) = 0.$
The center of the interval $[a, b]$ is $\frac{a+b}{2}$.
Thus using the chain rule, this equation reduces to the differential equation

$$\frac{d}{d\left(x-\left(\frac{a+b}{2}\right)\right)}\left[\left\{\left(\frac{b-a}{2}\right)^2 - \left(x-\left(\frac{a+b}{2}\right)\right)^2\right\}\frac{dy}{d\left(x-\left(\frac{a+b}{2}\right)\right)}.\frac{d\left(x-\left(\frac{a+b}{2}\right)\right)}{dx}\right]$$

$$.\frac{d\left(x-\left(\frac{a+b}{2}\right)\right)}{dx} + \lambda y = 0,$$

which is similar to $((a^2 - x^2)y')' + \lambda y = 0$, a near-Legendre equation. We suggest a solution of the form

$$y = \sum\nolimits_0 a_n\left(x - \left(\frac{a+b}{2}\right)\right)^{\mathrm{n}}.$$

If we let

$$u = x - \left(\frac{a+b}{2}\right), A = \left(\frac{b-a}{2}\right)$$

then the last differential equation reduces to

$\frac{d}{du}[(A^2 - u^2)\frac{dy}{du}] + \lambda y = 0, -A \le u \le A.$
Thus we see that for a polynomial solution to exist, λ has to be of the form $k(k+1)$ and the polynomial solutions sought are

$$Q_k = p_k\left(\frac{x-\left(\frac{a+b}{2}\right)}{\left(\frac{b-a}{2}\right)}\right), a \le x \le b, k = 0,1,2,\ldots$$

We prove the following remark.

**Remark (1):** Let $c > 0$ and let $a < b$ be the distinct roots of the equation $(c + dx - x^2) = 0$. Let $p_k(x)$ be the Legendre polynomials. Any two distinct solutions of differential equation $((c + dx - x^2)y')' + \lambda y = 0$ are orthogonal over the interval $[a, b]$, Furthermore, $\int_a^b Q_k^2(x) = \left(\frac{b-a}{2}\right)\int_{-1}^1 p_k^2(x)dx$. For, let $m \ne n$ be two nonnegative integers and consider
$I = \int_a^b Q_m Q_n dx.$
Then

$$I = \int_a^b p_m\left(\frac{x-\left(\frac{a+b}{2}\right)}{\left(\frac{b-a}{2}\right)}\right)\ p_n\left(\frac{x-\left(\frac{a+b}{2}\right)}{\left(\frac{b-a}{2}\right)}\right)dx.$$

$$Let\ t = \frac{x-\left(\frac{a+b}{2}\right)}{\left(\frac{b-a}{2}\right)}, a \le x \le b.$$

Then

$$I = \left(\frac{b-a}{2}\right)\int_{-1}^1 p_m(t)p_n(t)dt. -1 \le t \le 1.$$

The result follows from orthogonality of the Legendre polynomials $p_k$.

This way we have found a orthogonal polynomial basis for the space $P[a, b]$ of all polynomials over the interval $[a, b]$.

**Example (3):** Consider the interval $[0, b]$ and the equation

$$\left((bx - x^2)y'\right)' + \lambda y = 0\ , 0 \le x \le b\ , b > 0.$$

This equation can be written as

$$\left((bx - x^2)y'\right)' + \lambda y$$
$$= bxy'' - x^2y'' + by' - 2xy' + \lambda y = 0{,}0 \le x \le b, b > 0.$$

This is a near-Legendre equation and we can expect a polynomial solutions. We can use the preceding example to solve it but we prefer to do it directly. We look for a solution of the form $y = \sum_0 a_n x^{\mathrm{n}}$. Expanding we get

$$\sum\nolimits_0 b.n(n-1)a_n x^{\mathrm{n}-1} - \sum\nolimits_0 n(n-1)a_n x^{\mathrm{n}}$$
$$+\sum\nolimits_0 b.na_n x^{\mathrm{n}-1} - \sum\nolimits_0 2na_n x^{\mathrm{n}}$$
$$+\sum\nolimits_0 \lambda a_n x^{\mathrm{n}} = 0.$$

Changing the summation variables we get

$$\sum\nolimits_{-1} bn(n+1)a_{n+1}x^{\mathrm{n}} - \sum\nolimits_0 (n-1)na_n x^{\mathrm{n}} -$$
$$\sum\nolimits_0 2na_n x^{\mathrm{n}} + \sum\nolimits_{-1} b(n+1)a_{n+1}x^{\mathrm{n}} + \sum\nolimits_0 \lambda a_n x^{\mathrm{n}} = 0.$$

Then we get

$$[-b(0)1a_0 + b(0)a_0]x^{-1} = 0,$$
$$\sum\nolimits_0 [-n(n-1)a_n - 2na_n + \lambda a_n + bn(n+1)a_{n+1}$$
$$+b(n+1)a_{n+1}]x^{\mathrm{n}} = 0$$

Thus $a_0$ is free and

$$a_{n+1} = \frac{[n(n+1) - \lambda]a_n}{b(n+1)^2}\ , n = 0,1,\ldots.$$

Now to ensure solutions to be polynomials we must have $\lambda = m(m+1)$. Then

$$a_{n+1} = \frac{[n(n+1) - m(m+1)]a_n}{b(n+1)^2}, n = 0,1,\ldots.$$

Let us take $m = 0$.Then $a_0$ is free and $a_1 = 0, a_2 = 0,$ $a_3 = 0, \ldots$ . The solution is $y_0 = a_0$. Now let $m = 1$. Then $\lambda = 2$. Then $a_0$ is free and

$$a_1 = \left(-\frac{2}{b}\right)a_0, a_2 = \left(\frac{2-2}{4b}\right)a_2 = 0, a_3 = 0, a_4 = 0, \ldots$$

The solution is $y_2 = a_0 - \left(\frac{2}{b}\right)a_0 x$. Let $m = 2$.

Then $\lambda = 6$. Let $a_0$ be free then

$$a_1 = \left(-\frac{6}{b}\right) a_0, a_2 = \left(\frac{2-6}{4b}\right) a_1 = \left(\frac{6}{b^2}\right) a_0, a_3 = 0,$$

$a_4 = 0, \ldots$ .

The solution is $y_6 = a_0 - \left(\frac{6}{b}\right) a_0 x + \left(\frac{6}{b^2}\right) a_0 x^2$.

We notice that all solutions are constant multiple of

$p_k\left(\frac{x-\left(\frac{0+b}{2}\right)}{\frac{0+b}{2}}\right), 0 \le x \le b.$

**Remark (2):** The solutions of the equation $((a^2 - x^2)y')' + \lambda y = 0$ are orthogonal on the interval $[0, b]$. As an example

$$y_2.y_6 = \int_0^b a_0^2 \left(1 - \frac{2}{b}x\right)\left(1 - \frac{6}{b}x + \frac{6}{b^2}x^2\right) dx$$

$$= \int_0^b \left(1 - \frac{8}{b}x + \frac{18}{b^2}x^2 - \frac{12}{b^3}x^3\right) dx$$

$$= b - 4b + 6b - 3b = 0.$$

We prove the orthogonality of the solutions directly. Let $y, z$ be a polynomial solutions related to $m$ and $k$ respectively. Thus

$$\left((x^2 - bx)y'\right)' - m(m+1)y = 0,$$

$$((x^2 - bx)z')' - k(k+1)z = 0.$$

Then

$$z[((x^2 - bx)y')' - m(m+1)y] - y[\left((x^2 - bx)z'\right)'$$
$$-k(k+1)z] = 0.$$

This reduces to

$$z[(x^2 - bx)y'' + (x^2 - bx)'y'] - y[(x^2 - bx)z''$$
$$+(x^2 - bx)'z'] + [(k(k+1) - m(m+1)]yz = 0$$

This simplifies to

$$[(x^2 - bx)(zy' - yz')]' + [(k(k+1) - m(m+1)]yz = 0.$$

Integrating over the interval $[0, b]$ we get

$$\int_0^b [(k(k+1) - m(m+1)]yz dx = 0.$$

But

$$k(k+1) - m(m+1) = k^2 - m^2 + k - m$$

$$= (k - m)(k + m + 1) \neq 0 \ if \ m \neq k.$$

The result follows.

## III. GENERAL RESULTS

**Notation (1):** let $n > k > 0$ be positive integers. Let $nPk = n(n-1)\ldots(n-k+1)$. Let $n, k$ be positive integers. Let $nLk = n(n+1)\ldots(n+k-1)$.

**Proposition (1):** The differential equation $((1 - x^{2m})y^{(n)})' + \lambda y = 0$, where $m, n$ are positive integers is a near-Legendre equation which has a polynomial solution for some choice for λ if $2m = n + 1$.

***Proof:*** We have

$$(y^{(n)} - x^{2m}y^{(n)})' + \lambda y$$
$$= y^{(n+1)} - x^{2m}y^{(n+1)} - 2m.x^{2m-1}y^{(n)} + \lambda y = 0.$$

If $y = \sum a_k x^k$ then we would have

$$\sum_k a_k x^{k-n-1} - \sum_k a_k x^{k+2m-n-1}$$
$$-\sum_k 2m.a_k x^{k+2m-1-n} + \sum_k \lambda a_k x^k = 0 \qquad (1)$$

To ensure polynomial solutions we must have three of the exponents in (1) the same. In fact they are the last three exponents. Thus

$k + 2m - n - 1 = k, 2m - n - 1 = 0, 2m = n + 1.$

If $m = 1$ then we would have the differential equation $((1 - x^2)y')' + \lambda f = 0$ which is the classical Legendre equation. Notice that we take the power of $x$ to be $2m$ to use it for orthogonality purposes because we need $(1 - x^{2m})$ to be zero when $x = \pm 1$ at least when $m = 1$. As a special case

**Example (4):** Let us discuss the differential equation

$$\left((1 - x^{2m})y^{(2m-1)}\right)' + \lambda y = 0, \qquad -1 \le x \le 1 \qquad (2)$$

This equation is the same as

$$y^{(2m)} - x^{2m}y^{(2m)} - 2mx^{2m-1}y^{(2m-1)} + \lambda y = 0.$$

Equation (2) is a near-Legendre equation.

We propose a solution of the form $y = \sum_0^\infty a_n x^n$.

After expansion we get

$$\sum_{n=0} a_n(n)(n-1)\ldots(n-2m+1)x^{n-2m}$$
$$-\sum_{n=0} a_n(n)(n-1)\ldots(n-2m+1)x^n$$
$$-\sum_{n=0} 2ma_n(n)(n-1)\ldots(n-2m+2)x^n$$
$$+\sum_{n=0} \lambda a_n x^n = 0.$$

Changing summation index we get

$$\sum_{n=-2m} a_{n+2m}(n+2m)(n+2m-1)\ldots(n+1)x^n$$
$$-\sum_{n=0} a_n(n)(n-1)\ldots(n-2m+1)x^n$$
$$-\sum_{n=0} 2ma_n(n)(n-1)\ldots(n-2m+2)x^n$$
$$+\sum_{n=0} \lambda a_n x^n = 0.$$

Thus

$$a_{n+2m} = a_n.\frac{n(n-1)\ldots(n-2m+1) + 2mn(n-1)\ldots(n-2m+2) - \lambda}{(n+2m)(n+2m-1)\ldots(n+1)}$$
$$= -a_n.\frac{-(n+1)(n)(n-1)\ldots(n-2m+2) + \lambda}{(n+2m)(n+2m-1)\ldots(n+1)}$$

$$= -\frac{\lambda - (n+1)P(2m)}{(n+1)L(2m)} a_n. \qquad (3)$$

To get a polynomial solution, recall Notation (1)
$\lambda$ must be of the form
$\lambda = (k+1)P(2m) = (k+1)k(k-1)\ldots(k-2m+2)$,

$$k+1 \geq 2m.$$

Now we write $k+1 = s\,.\,2m + r, 0 \leq r < 2m$ using Euclidean algorithm. Since $a_0, a_1 \ldots, a_{2m-1}$ are arbitrary we set $a_i = 0, i \neq r$ and $a_r = 1$. To get the polynomial solution in this case we have

$$a_{k+2m} = -\frac{\lambda - (n+1)P(2m)}{(n+1)L(2m)} a_k$$
$$= \frac{(k+1)P(2m) - (k+1)P(2m)}{(n+1)L(2m)} a_k = 0$$

and so all $a_{k+2tm} = 0$ , $t = 0,1,2,\ldots$. Thus we get a polynomial solution
$y = x^r + a_{r+2m}x^{r+2m} + a_{r+4m}x^{r+4m} + \cdots$
$+a_{k+1-2m}x^{k+1-2m}$ .
Here the coefficients are gotten using (3). If we try to set some $a_i \neq 0$ for $i \neq 0$ we get a divergent series solution using the ratio test. So essentially we have a unique polynomial solution for every

$$\lambda = (k+1)P(2m) = (k+1)k(k-1)\ldots(k-2m+2)$$

$$, k+1 \geq 2m$$

**Remark (3):** Let $u, v$ be $C^\infty -$ functions and $n$ a positive integer. Leibniz Formula states that

$$(uv)^{(\mathrm{n})} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} v^{(n-k)}$$

**Proposition (2):** Let $y$ be a $C^\infty$ function and let
$0 \leq m < n = s + t$ , $m, n, s, t$ be nonnegative integers.
Let $L(y) = \left((x^m - x^{\mathrm{n}})y^s\right)^{(t)}$. Then
$L(y) = E_{(s+t-m)} - E_{(s+t-n)}$.

***Proof:*** We have, using Remark (3)

$$\left((x^m - x^{\mathrm{n}})y^{(s)}\right)^{(t)} = \sum_{k=0}^{t} \binom{t}{k} (x^m)^{(k)} \left(y^{(s)}\right)^{(t-k)}$$
$$-\sum_{k=0}^{t} \binom{t}{k} (x^{\mathrm{n}})^{(k)} \left(y^{(s)}\right)^{(t-k)}$$
$$= \sum_{k=0}^{t} \binom{t}{k} (x^m)^{(k)} (y^{(s+t-k)}) - \sum_{k=0}^{t} \binom{t}{k} (x^{\mathrm{n}})^{(k)} \left(y^{(s+t-k)}\right).$$
$$= E_{s+t-k-(m-k)} - E_{s+t-k-(n-k)} = E_{s+t-m} - E_{s+t-n},$$

as required.

**Corollary (1):** We have

$$\left((1 - x^{\mathrm{n}})y^{(s)}\right)^{(t)} = E_{s+t} - E_{s+t-n}$$
$$,\left((x - x^{\mathrm{n}})y^{(s)}\right)^{(t)} = E_{s+t-1} - E_{s+t-n}.$$

***Proof:*** Straightforward.

**Remark (4):** Let

$$L(y) = E_0(y) = a_0 y + a_1 x y' + \cdots + a_m x^m y^{(m)}$$
$$= \sum_{k=0}^{m} a_k x^k y^{(k)}$$

be a 0-Euler form. We are interested in finding all real $\lambda$ that ensure the existence of a polynomial solution for the equation$E_0(y) - \lambda y = 0$. We notice that $E_0(y) - \lambda y = 0$ is an $E_0$ form and so we try a solution $y = x^{\mathrm{n}}$.Then we have

$$a_0 x^{\mathrm{n}} + a_1.n.x^{\mathrm{n}} + \ldots + a_m.n.(n-1)\ldots(n-m+1)x^m$$
$$-\lambda x^m = 0,$$
$$(a^0 + a^1.n. + \ldots + a_m.n.(n-1)\ldots(n-m+1) - \lambda)x^m = 0,$$
$$\lambda = a_0 + a_1.n. + \ldots + a_m.n.(n-1)\ldots(n-m+1).$$

Thus for each $n$ and for each
$\lambda = a_0 + a_1.n. + \ldots + a_m.n.(n-1)\ldots(n-m+1)$ we have a monomial solution $y = x^{\mathrm{n}}$. In general let $y = \sum_{i=0}^{n} b_i x^i$ be a polynomial eigen function for $E_0(y) - \lambda y = 0$ for some $\lambda$. Then we have

$$\sum_{k=0}^{m} a_k x^k \left(\sum_{i=0}^{n} b_i x^i\right)^{(k)} - \sum_{i=0}^{n} \lambda b_i x^i = 0,$$
$$\sum_{k=0}^{m} a_k x^k \left(\sum_{i=0}^{n} iLk.\, b_i x^{i-k}\right) - \sum_{i=0}^{n} \lambda b_i x^i = 0,$$
$$\sum_{i=0}^{n} b_i \left(\left(\sum_{k=0}^{m} a_k\, iLk\right) - \lambda\right) x^i = 0$$

Thus we have

$$\sum_{k=0}^{m} (a_k iLk - \lambda) b_i = 0$$
$$, i = 0, \ldots, m.$$

Thus we look for all such solutions $b_i, \lambda$ that satisfy the system and then if there are such solutions the corresponding polynomial solution would be

$$y = \sum_{i=0}^{n} b_i x^i.$$

**Proposition (3):** Let $k, m$ be a positive integers and

$$L(y) = E_k(y)$$
$$= a_0 y^{(k)} + a_1 x y^{(k+1)} + \ldots + a_m x^m y^{(k+m)}$$

be a $k-Euler$ form. Then the only polynomial solution of the equation

$$E_k(y)-\lambda y=0$$

are polynomials of degree less than k corresponding to $\lambda=0$.

***Proof:*** For if there is such a polynomial of degree $s$ greater than or equal to $k$ then from

$a_0y^{(k)}+a_1xy^{(k+1)}+\ldots+a_mx^my^{(k+m)}=\lambda y,$

we get, upon comparing coefficients of highest terms of both sides, that the degree of left hand side is $s-k$ while the degree of the right hand side is $s$ which is absurd.

**Proposition (4):** Let $k,m,n,t$ be nonnegative integers $,k>0,t<k$. Then

1. $\left((1-x^k)y^{(m)}\right)^{(n)}+\lambda y=0$ is a near-Legendre equation and has polynomial solutions if and only if $m+n=k$.

2. $\left((x^t-x^k)y^{(m)}\right)^{(n)}+\lambda y=0, t<k$ is a near-Legendre equation and has a polynomial solution if and only if

$m+n=k$.

***Proof:*** We have

$$((1-x^k)y^{(m)})^{(\mathrm{n})}-\lambda y\ =E_{(m+n)}-E_{(m+n-k)}-E_0.$$

Since $m,n,k$ are nonnegative we see that $m+n-k=0$. The proof of the other part is similar.

**Proposition (5):** When $m=1$ we have only three near-Legendre equations

1. $((1-x^2)y)''+\lambda y=0.$
2. $((1-x^2)y')'+\lambda y=0$, the usual Legendre equation.
3. $(1-x^2)y''+\lambda y=0.$

While when m=2 we have only five near-Legendre equations.

1. $((1-x^4)y)^{(4)}+\lambda y=0.$
2. $((1-x^4)y')^{(3)}+\lambda y=0.$
3. $((1-x^4)y'')''+\lambda y=0.$
4. $((1-x^4)y^{(3)})'+\lambda y=0.$
5. $((1-x^4)y^{(4)}+\lambda y=0.$

In general when $m=k$ we have $2k+1$ cases

$$\left((1-x^{2k})y^{(i)}\right)^{(2k-i)}+\lambda y=0,0\le i\le 2k.$$

***Proof:*** We just apply Proposition (4) with different $m$ and $k$.

## IV. THE EQUATION $(1-x^2)y''+\lambda y=0$.

Consider the differential equation $(1-x^2)y''+\lambda y=0$. This is a near-Legendre differential equation and has polynomial solutions. Let $y$ be a solution corresponding to $\lambda$ and $u$ a solution corresponding to $\mu$. Thus $(1-x^2)u''+\mu u=0$.
Let $y=\sum a_nx^{\mathrm{n}}$.

Then

$$\sum n(n-1)a_nx^{\mathrm{n}-2}-\sum n(n-1)a_nx^{\mathrm{n}} + \sum \lambda n(n-1)a_nx^{\mathrm{n}}=0.$$

This is equivalent to

$$\sum_{n=-2}(n+2)(n+1)a_{n+2}x^{\mathrm{n}} -\sum_{n=0} n(n-1)a_nx^{\mathrm{n}}+\sum_{n=0}\lambda n(n-1)a_nx^{\mathrm{n}}=0.$$

It follows that

$a_{n+2}=-\frac{\lambda-n(n-1)}{(n+2)(n+1)}a_n, n\ge 0,\ a_0,a_1$ free.

It follows that for a polynomial solution to exist

$\lambda=k(k-1),\ k$ is a nonnegative integer.

Thus when $k=1$ or $0,\lambda=0$ and
$a^2=0,a_3=-\left(\frac{0-0}{6}\right)a_1=0,$

and so all $a_n=0,n\ge 2$. The solution is $y_0=a_0+a_1x$. Thus we have two solutions: $1,x$.
If $k=2,\ \lambda=2$ and
$a_2=-\left(\frac{2-0}{2}\right)a_0=-a\_0,a_3=-\left(\frac{2-0}{6}\right)\ a_1$
$=-\frac{a_1}{3},a_4=-\left(\frac{2-2}{12}\right)\ a_3=0,a_5=-ca_1$

for some constant c. If we set $a_1=0$ the polynomial solution is $y_2=a_0-a_0x^2$.

If $k=3,\lambda=6,a_5=0,a_7=0,a_9=0,\ldots$ and the polynomial solution is
$y_6=a_1x-a_3x^3,a_3=-\left(\frac{6-0}{6}\right)a_1=-a_1.$

Thus $y_6=a_1x-a_1x^3$.

If $k=4,\lambda=12,a_6=0=a_8=a_{10}=\ldots$ and so the eigen polynomial is odd and of degree 5 ,

$$y_{12}=a_0+a_2x^2+a_4x^4.$$

Here
$a_2=-\left(\frac{12-0}{2}\right)a_0=-6a_0,a_4=-\left(\frac{12-2}{12}\right)a_2=5a_0.$

Thus the polynomial solution is
$y_{12}=a_0-6a_0x^2+5a_0x^4.$
We notice any polynomial solution $y_\lambda$ is either even and the numerical coefficients are multiple of $a_0$ or it is odd and the numerical coefficients are multiple of $a_1$. When we substitute in these solutions $a_0=1$ and $a_1=1$ we call the resulting solutions normalized polynomial solutions.

**Remark (5):** Consider the differential equation $(1-x^2)y''+\lambda y=0$. We notice that when $\lambda\neq 0$ and from $((1-x^2)y''+\lambda y)(1)=0$ we get $y(1)=y(-1)=0$.

**Proposition (6):** The polynomial solutions $y_\lambda$ of the equation

$$(1-x^2)y^{(2)}+\lambda y=0$$

are orthogonal over the interval $[-1,1]$ with weight $(1-x^2)^{-1}$.

***Proof:*** Assume that we have

$$(1-x^2)y_\lambda''+\lambda y_\lambda=0,(1-x^2)u_\mu''+\mu u_\mu=0,\lambda\neq\mu.$$

Multiply the first equation by $\frac{u_\mu}{1-x^2}$, the second by $\frac{y_\lambda}{1-x^2}$ and subtract to get

$$\int_{-1}^{1}(y_\lambda''u_\mu-u_\mu''y_\lambda)dx+(\lambda-\mu)\int_{-1}^{1}\frac{y_\lambda u_\mu}{(1-x^2)}dx=0\ ...(4)$$

Now

$$\int_{-1}^{1}y_\lambda''u_\mu dx=y_\lambda'u_\mu]_{-1}^{1}-\int_{-1}^{1}y_\lambda'u_\mu'$$

$$=0-\int_{-1}^{1}y_\lambda'u_\mu'dx=-\int_{-1}^{1}y_\lambda'u_\mu'dx\,,\int_{-1}^{1}u_\mu''y_\lambda dx$$

$$=-\int_{-1}^{1}y_\lambda'u_\mu'dx.$$

Thus (4) reduces to

$(\lambda-\mu)\int_{-1}^{1}\left(\frac{y_\lambda u_\mu}{(1-x^2)}\right)dx=0$. Since $\lambda\neq\mu$, we get $\int_{-1}^{1}\left(\frac{y_\lambda u_\mu}{1-x^2}\right)dx=0$. Thus $y_\lambda,u_\mu$ are orthogonal with weight $\frac{1}{1-x^2}$.

We notice that $\int_{-1}^{1}\left(\frac{y_\lambda u_\mu}{1-x^2}\right)dx$ exists since $y_\lambda,u_\mu$ are polynomials that vanish on $1,-1$ and hence $\left(\frac{y_\lambda u_\mu}{1-x^2}\right)$ is a polynomial.

For example if we take the polynomial solutions

$$f=y=a_0-a_0x^2,g=b_0-6b_0x^2+5b_0x^4$$

for the differential equation $(1-x^2)y''+\lambda y=0$ we see that

$$\int_{-1}^{1}\left(\frac{fg}{1-x^2}\right)dx=\int_{-1}^{1}a_0b_0(1-6x^2+5x^4)dx$$
$$=2a_0b_0[x-2x^3+x^5]_0^1=0$$

as expected.

**Conjecture:** It is an open question that the solutions of $(1-x^{2m})y''+\lambda y=0$ are orthogonal with weight $\frac{1}{1-x^{2m}}$ over the interval $[-1,1]$.

**Proposition (7):** Any continuous function $f$ on the interval $[-1,1]$ can be expanded in terms of the normalized eigen polynomials of the differential equation

$$(1-x^2)y''+\lambda y=0,-1\leq x\leq 1.$$

***Proof:*** Write $f=\sum a_\lambda y_\lambda, y_\lambda$ the normalized eigen polynomials of the given differential equation. To find $a_\lambda$ we multiply both sides with $\left(\frac{y_\lambda}{1-x^2}\right)$ and integrate from $-1$ to 1 and use orthogonality to get

$$a_\lambda=\frac{\int_{-1}^{1}f\left(\frac{y_\lambda}{1-x^2}\right)dx}{\int_{-1}^{1}\left(\frac{y_\lambda^2}{1-x^2}\right)dx}.$$

## V. THE EQUATION $\left((1-x^2)y\right)''+\lambda y=0,-1\leq x\leq 1$

The differential equation $((1-x^2)y)''+\lambda y=0$ can be written as

$$\left((1-x^2)y\right)''+\lambda y=(-2xy+y'-x^2y')'+\lambda y$$
$$=-2xy'-2y+y''-x^2y''-2xy'$$
$$=(1-x^2)y''-4xy'+(\lambda-2)y$$
$$=y''-x^2y''-4xy'+(\lambda-2)y$$
$$=0.$$

It follows that the equation is near-Legendre one and so we expect polynomial eigen solutions. Let $y=\sum a_nx^n$ be a power series solution. Then we have

$$0=\sum n(n-1)a_nx^{n-2}-\sum n(n-1)a_nx^n$$
$$-\sum 4n.a_nx^n+(\lambda-2)\sum a_nx^n$$
$$=\sum(n+2)(n+1)a_{n+2}x^n$$
$$-\sum[n(n-1)+4n-\lambda+2]a_nx^n.$$

It follows that $a_{n+2}=-\left(\frac{\lambda-(n+1)(n+2)}{(n+1)(n+2)}\right)a_n, n\geq 0, a_0, a_1$ free. To have a polynomial solution $\lambda$ has to have value $\lambda=(k+1)(k+2),k\geq 0$.

For $k=0,\lambda=2,a_2=0=a_4=a_6=..$. Thus a polynomial solution is $y_2=a_0$.

For $k=1,\lambda=6,a_3=0=a_5=a_7=..$. Thus the polynomial solution is $y_6=a_1x$.

For $k=2,\lambda=12,a_4=0=a_6=a_8=..$. Thus a polynomial solution is

$y_{12}=a_0+a_2x^2, a_2=-\left(\frac{[12-2]}{2}\right)a_0, y_{12}=a_0(1-5x^2)$.

For $k=3,\lambda=20,a_5=0=a_7=a_9=..$. Thus a polynomial solution is

$$y_{20}=a_1x+a_3x^3,a_3=-\left(\frac{[20-6]}{6}\right)a_1,$$

$$y_{20}=a_1(x-\left(\frac{7}{3}\right)x^3).$$

For $k = 4, \lambda = 30, a^6 = 0 = a^8 = a^{10} = \cdots$ Thus a polynomial solution is

$y_{30} = a_0 + a_2x^2 + a_4x^4, a_2 = -\left(\frac{[30-2]}{2}\right) a_0 = -14a_0,$

$$a_4 = -\left(\frac{[30-12]}{12}\right) a_2 = -\left(\frac{3}{2}\right) a^2 = 21a_0,$$

$y_{30} = a_0(1 - 14x^2 + 21x^4)$. We can continue to find infinitely many polynomial eigen solutions.

**Remark(6):** Let $y$ be even differentiable function on $\boldsymbol{R}$. Then

$$\left((1-t^2)y(t)'\right)]_{t=0} = 0, ((1-t^2)y(t)')]_{t=x} = ((1-x^2)y(x))'.$$

For , $((1-t^2)y(t))' = y'(t) - 2ty(t) - t^2y'(t)$. If $y$ is even then $y'$ is odd and we have

$$((1-t^2)y(t))']_{t=0} = (y'(t) - 2ty(t) - t^2y'(t))]_{t=0} = 0 - 0 - 0 = 0$$

and

$$((1-t^2)y(t))']_{t=x} = (y'(t) - 2ty(t) - t^2y'(t))]_{t=x} = y'(x) - 2xy(x) - x^2y'(x) = ((1-x^2)y(x))'$$

as expected.

**Proposition (8):** Let $y_\lambda, u_\mu$ , $\lambda \neq \mu$ be two even polynomial solutions of the differential equation

$$((1-x^2)y)'' + \lambda y = 0, -1 \leq x \leq 1.$$

The antiderivatives $\int_0^x y_\lambda dt$ , $\int_0^x u_\mu dt$ are orthogonal over the intervals $[0,1], [-1,1]$.

***Proof:*** Let $y_\lambda, u_\mu$ be two even polynomial solutions of the given differential equations corresponding to $\lambda \neq \mu$ eigen values. Thus

$$((1-x^2)y_\lambda)'' + \lambda y_\lambda = 0, ((1-x^2)u_\mu)'' + \mu y_\lambda = 0.$$

Let

$$U_\mu(x) = \int_0^x u_\mu(t)dt , Y_\lambda(x) = \int_0^x y_\lambda(t)dt.$$

Then we have

$$\int_0^x ((1-t^2)y_\lambda(t))''dt + \int_0^x \lambda y_\lambda(t)dt = 0.$$

From Remark (6) we have

$$((1-x^2)y_\lambda(x))' + \int_0^x \lambda y_\lambda(t)dt = 0.$$

Now multiply both sides by $U_\mu(x)$ and integrate from 0 to 1 to get

$$\int_0^1 U_\mu(x).\left((1-x^2)y_\lambda(x)\right)'dx + \lambda \int_0^1 U_\mu(x)Y_\lambda(x)dx = 0$$

By integration by parts we have

$$U_\mu(x).\left((1-x^2)y_\lambda(x)\right)]_0^1 - \int_0^1 \left((1-x^2)y_\lambda(x)\right)U_\mu(x)'dx + \lambda \int_0^1 U_\mu(x)Y_\lambda(x)dx = 0$$

This is the same as

$$-\int_0^1 (1-x^2)y_\lambda(x)u_\mu(x)dx + \lambda \int_0^1 U_\mu(x)Y_\lambda(x)dx = 0. \ldots (5)$$

In a similar manner we get

$$-\int_0^1 \left((1-x^2)\, u_\mu(x)\right) Y_\lambda(x)'dx + \mu \int_0^1 U_\mu(x)Y_\lambda(x)dx = 0$$

which is the same as

$$-\int_0^1 (1-x^2)y_\lambda(x)u_\mu(x)dx + \mu \int_0^1 U_\mu(x)Y_\lambda(x)dx = 0. \ldots (6)$$

By subtracting Equations (5), (6) we get

$$(\lambda - \mu) \int_0^1 U_\mu(x)Y_\lambda(x)dx = 0,$$

$$\int_0^1 U_\mu(x)Y_\lambda(x)dx = 0.$$

Since $u, y$ are even $U, Y$ are odd and $U_\mu Y_\lambda$ is even. Thus

$$\int_{-1}^1 U_\mu(x)Y_\lambda(x)dx = 2\int_0^1 U_\mu(x)Y_\lambda(x)dx = 0,$$

as required.

For example consider the polynomial solutions

$$f = a_0(1-5x^2), g = b_0(1-14x^2+21x^4).$$

Then $f, g$ have antiderivatives

$F = a_0(x - \frac{5x^3}{3}), G = b_0(x - \frac{14x^3}{3} + \frac{21x^5}{5})$.
Now

$$45\int_0^1 \left(x-\frac{5x^3}{3}\right)\left(x-\frac{14x^3}{3}+\frac{21x^5}{5}\right)dx$$

$$= \int_0^1 (3x-5x^3)(15x-70x^3+63x^5)dx$$

$$= \int_0^1 (45x^2-285x^4+539x^6-315x^8)dx$$

$$= 15-57+77-35 = 0.$$

Thus $\int_{-1}^1 FGdx = 0$ as expected.

As before we normalize the solutions by putting $a_0 = 1$,
$a_1 = 1$.

**Corollary (2):** Any continuous function $f$ on $[-1,1]$ can be expanded in polynomials derived from the normalized eigen polynomials of the differential equation

$$((1-x^2)y)'' + \lambda y = 0, -1 \le x \le 1.$$

***Proof:*** If $f$ is even then $f = \sum a_\lambda y_\lambda, y_\lambda$ are even.
Let $F(x) = \int_0^x f(t)dt$. Then $F = \sum a_\lambda Y_\lambda$. To get $a_\lambda$ we multiply both sides by $Y_\lambda$, integrate and use orthogonality of the $Y_\lambda$ to get

$$a_\lambda = \frac{\int_0^1 FY_\lambda}{\int_0^1 Y_\lambda^2}.$$

If $f$ is odd we expand $f\,'$and then integrate the result to get back $f$. In general we write $f = f_e + f_o$ as a sum of even and odd functions and expand each and sum up the result is

$$f_e(x) = \left(\frac{f(x)+f(-x)}{2}\right), f_o(x) = \left(\frac{f(x)-f(-x)}{2}\right).$$

One may ask if there are similar-to Rodriguez's Formulas and Recursive Rules for these newly developed eigen polynomials. This is still an open problem.

**Remark(7):**The eigen polynomials of the equation
$((1-x^2)y)'' + \lambda y = 0$ are the same as those of the the equation $y'' - x^2y'' - 4xy' + (\lambda-2)y = 0$ which are the same as those of the equation $y'' - x^2y'' - 4xy' + \lambda y = 0.$

## VI. Applications

1. Suppose we have a steady state partial differential equation

$$(1-x^2)u_{xx} + u_{yy} = 0, -1 \le x \le 1, y \ge 0, u(x,0) = f(x), u(-1,y) = 0 = u(1,y),$$

$f(x)$ continuous on $[-1,1]$.

We use separation of variables technique and assume $u = X(x)Y(y)$. Then we get $(1-x^2)X''Y + XY'' = 0$. Thus upon dividing by $XY$ we get

$$\frac{(1-x^2)X''}{X} + \frac{Y''}{Y} = 0.$$

It follows that

$$\frac{(1-x^2)X''}{X} = -\lambda, \frac{Y''}{Y} = \lambda, \lambda > 0.$$

Thus

$$(1-x^2)X'' + \lambda X = 0, Y'' - \lambda Y = 0, \lambda > 0.$$

The first equation is a near-Legendre equation and the second has general solution $Y = a\, e^{-\sqrt{\lambda y}} + be^{+\sqrt{\lambda y}}$. To get a bounded solution we choose $Y = e^{-\sqrt{\lambda y}}$. Now we take the values of λ that give polynomial solutions to the equation

$$(1-x^2)X'' + \lambda X = 0$$

and take the corresponding polynomial solutions $y_\lambda$. The candidate solution for our partial differential equation is

$$u = \sum a_\lambda y_\lambda(x) e^{-\sqrt{\lambda y}}.$$

Then we have

$$u(x,0) = f(x) = \sum a_\lambda y_\lambda(x).$$

We find $a_n$ using orthogonality in Proposition (6). If the series

$$\sum a_\lambda y_\lambda(x) e^{-\sqrt{\lambda y}}$$

Converges then

$$u = \sum a_\lambda y_\lambda(x) e^{-\sqrt{\lambda y}}$$

is the required solution.

2. Suppose we have a steady state partial differential equation

$$(1-x^2)u_{xx} - 4xu_x + u_{yy} = 0, -1 \le x \le 1,$$

$$u(x,0) = g(x), u(-1,y) = u(1,y),$$

$g(x)$ continuous on $[-1,1]$.

We use separation of variables technique and assume

$u = X(x)Y(y)$.

Then we get

$$(1-x^2)X''Y - 4xX'Y + XY'' = 0.$$

Thus upon dividing by $XY$ we get

$$\frac{(1-x^2)X''}{X}-\frac{4xX'}{X}+\frac{Y''}{Y}=0.$$

It follows that

$$\frac{(1-x^2)X''}{X}-\frac{4xX'}{X}=-\lambda,\frac{Y''}{Y}=\lambda,\lambda>0.$$

This reduces to

$$(1-x^2)X''-4xX+\lambda X=0, Y''-\lambda Y=0,\lambda>0.$$

To have a bounded solution we choose $Y(y)=e^{-\sqrt{\lambda y}}$. The equation

$$(1-x^2)X''-4xX+\lambda X=0,-1\leq x\leq 1$$

is a near-Legendre equation which has an infinitely many eigen polynomials $y_\lambda$ corresponding to infinitely many "eigen values $\lambda$"as referred to in Remark (7). Since we have $X(-1)=X(1)$ we choose the even $y_\lambda$. Thus we try a solution

$$u(x,y)=\sum_{y_\lambda\ even} a_\lambda y_\lambda(x)e^{-\sqrt{\lambda y}}$$

Then $u(x,0)=\sum\ a_\lambda y_\lambda(x)=g(x)$. We find $a_\lambda$ using orthogonality and Corollary (2). If the series

$$\sum_{y_\lambda\ even} a_\lambda y_\lambda(x)e^{-\sqrt{\lambda y}}$$

converges then

$$u=\sum_{y_\lambda\ even} a_\lambda y_\lambda(x)e^{-\sqrt{\lambda y}}$$

is a solution to our problem.

**Remark(8):** We can solve

$$(1-x^2)u_{xx}-u_{yy}=0,-1\leq x\leq 1,y\geq 0,$$
$$u(x,0)=f(x),$$
$$u(-1,y)=0=u(1,y)$$

and we can solve

$$(1-x^2)u_{xx}-4xu_x-u_{yy}=0,-1\leq x\leq 1,$$
$$u(x,0)=g(x),$$
$$u(-1,y)=u(1,y)$$

using the same techniques but with proper replacement of

$$e^{-\sqrt{\lambda y}}.$$

In a future work, we may discuss the other five cases, when $m=2$, of near-Legendre equations mentioned in Proposition (5)

## VII. CONCLUSION

We generalized the Legendre equation

$((1-x^2)y')'+n(n+1)y=0,-1\leq x\leq 1, n$ is a nonnegative integer, to near-Legendre equations

$$((1-x^k)y^{(m)})^{(n)}+\lambda y = 0,$$
$$((x^t-x^k)y^{(m)})^{(n)}+\lambda y=0,$$
$$-1\leq x\leq 1,t<k,m+n=k,$$

$n,m,k,t$ nonnegative integers. We discussed several cases to ensure the existence of polynomial solutions. We then gave applications to solve some partial differential equations.

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