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Coefficient Bounds and Fekete-Szego inequalities for A class of Bazilevic type functions associated with new differential operator

Abdul Rahman S. Juma [a],Mushtaq S. Abdulhussain[b] and Saba N. Al-khafaji[c,1]

[a] *Department of Mathematics ,University of Anbar, Ramadi-Iraq,*
dr_juma@hotmail.com
[b] *Department of Mathematics , Mustansiriyah University-Iraq,*
mushtdma8@yahoo.com
[c] *Department of Mathematics, Mustansiriyah University -Iraq,*
sabanf.mc11p@uokufa.edu.iq
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Abstract.
We generalized a class of Bazilevic type functions as defined in the open unit disk associated with a differential operator satisfying some subordination conditions. As well as, we obtained coefficient bounds and Fekete Szego bounds for functions in this subclass.

Keywords: Univalent functions, Subordination, Bazilevic type function.

1. Introduction

Let A refer to the usual class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k , \qquad (1.1)$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized with $f(0) = 0$ and $f'(0) = 1$.

Let S denote to the subclass of A consisting of all univalent functions in the open unit disk $\mathcal{U}$. Also, let P refer to the class of functions $h(z)$ in which the real part is positive in $\mathcal{U}$ as following $h(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$.

We refer $w(z)$ be Schwarz function defined on the boundary ∂D of a domain in $\mathbb{C}$ is an analytic function such that $w(z) = \bar{z}$ on ∂D (see [5]).

It is notable that the following correspondence between the class P and the class of Schwarz functions w exists (see [6])

$$h \in P \Leftrightarrow h = \frac{1+w}{1-w}. \qquad (1.2)$$

Let f and g be two analytic functions in $\mathcal{U}$. Then a function f is called subordinate to a function g, writes as follows

$f \prec g$ or $f(z) \prec g(z),\ (z \in \mathcal{U})$ (1.3)

if there is a Schwarz function w which can be analytic function in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. Thus, if a function $g(z)$ be univalent in $\mathcal{U}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$ (see[5]).

A function $f(z)$ which has the form (1.1) called univalent starlike, if it satisfies the condition

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, (z \in \mathcal{U})$$

we refer to the former class of functions as S^* (see[5]). From (1.1), we can write

$$(f(z))^{\alpha} = \left(z + \sum_{k=2}^{\infty} a_k z^k\right)^{\alpha}, \qquad (1.4)$$

where α is real number greater than zero, then we obtain

$$(f(z))^{\alpha} = (z + a_2z^2 + a_3z^3 + a_4z^4 + \ldots.)^{\alpha}.$$

This is equivalent to

$$(f(z))^{\alpha} = z^{\alpha}(1 + a_2z + a_3z^2 + a_4z^3 + \ldots.)^{\alpha}, \qquad (1.5)$$

Using the binomial expansion for (1.5), we get

$$(f(z))^{\alpha} = z^{\alpha}\left[1+\alpha(a_2z + a_3z^2 + a_4z^3 + \ldots.) + \frac{\alpha(\alpha-1)}{2!}(a_2z + a_3z^2 + a_4z^3 + \ldots.)^2 + \cdots\right].$$

Since the expansion continues, then

$$(f(z))^{\alpha} = z^{\alpha}(1+\alpha(a_2z + a_3z^2 + a_4z^3 + \ldots.)).$$

Therefore,

$$(f(z))^{\alpha} = z^{\alpha} + \alpha a_2 z^{\alpha+1} + \alpha a_3 z^{\alpha+2} + \alpha a_4 z^{\alpha+3} + \cdots .$$

We denote the class of analytic functions of A_{α} as

$$(f(z))^{\alpha} = z^{\alpha} + \sum_{k=2}^{\infty} a_{\mathrm{k}}(\alpha) z^{\alpha+k-1}. \qquad (1.6)$$

For a function $(f(z))^{\alpha}$ given in (1.6), we will define differential operator on space of analytic functions $H(A_{\alpha})$:

$\mathcal{D}^{m,\xi}_{\lambda_1,\lambda_2,\ell}: H(A_{\alpha}) \longrightarrow H(A_{\alpha})$ as follows:

$$\mathcal{D}^{0,\xi}_{\lambda_1,\lambda_2,\ell}\left(\left(f(z)\right)^{\alpha}\right) = (f(z)^{\alpha}),$$

$$D_1 = \frac{\zeta(\lambda_1+\lambda_2)z}{1+\lambda_1(\ell-1)}D + \left(\frac{1-\zeta(\lambda_1+\lambda_2)}{1+\lambda_1(\ell-1)}\right)$$

$$\mathcal{D}^{1,\xi}_{\lambda_1,\lambda_2,\ell}((f(z))^{\alpha}) = D_1(\mathcal{D}^{0,\xi}_{\lambda_1,\lambda_2,\ell}((f(z))^{\alpha}) = D_1\left[z^{\alpha} + \textstyle\sum_{k=2}^{\infty} a_{\mathrm{k}}(\alpha) z^{\alpha+k-1}\right]$$

$$\mathcal{D}^{1,\xi}_{\lambda_1,\lambda_2,\ell}((f(z))^{\alpha}) = \left(\frac{1+\zeta(\lambda_1+\lambda_2)(\alpha-1)}{1+\lambda_1(\ell-1)}\right)^1 z^{\alpha} + \sum_{k=2}^{\infty}\left(\frac{1+\zeta(\lambda_1+\lambda_2)(\alpha+k-2)}{1+\lambda_1(\ell-1)}\right)^1 a_k(\alpha) z^{\alpha+k-1}. \qquad (1.7)$$

$$\mathcal{D}^{2,\xi}_{\lambda_1,\lambda_2,\ell}\left(\left(f(z)\right)^{\alpha}\right) = D_1(\mathcal{D}^{1,\xi}_{\lambda_1,\lambda_2,\ell}\left(\left(f(z)\right)^{\alpha}\right) = \left(\frac{1+\zeta(\lambda_1+\lambda_2)(\alpha-1)}{1+\lambda_1(\ell-1)}\right)^2 z^{\alpha} + \sum_{k=2}^{\infty}\left(\frac{1+\zeta(\lambda_1+\lambda_2)(\alpha+k-2)}{1+\lambda_1(\ell-1)}\right)^2 a_k(\alpha) z^{\alpha+k-1}.$$

In general, we have

$$\mathcal{D}^{m,\xi}_{\lambda_1,\lambda_2,\ell}((f(z))^{\alpha}) = D_1(\mathcal{D}^{m-1,\xi}_{\lambda_1,\lambda_2,\ell}\left(\left(f(z)\right)^{\alpha}\right) = \left(\frac{1+\zeta(\lambda_1+\lambda_2)(\alpha-1)}{1+\lambda_1(\ell-1)}\right)^m z^{\alpha} + \sum_{k=2}^{\infty}\left(\frac{1+\zeta(\lambda_1+\lambda_2)(\alpha+k-2)}{1+\lambda_1(\ell-1)}\right)^m a_k(\alpha) z^{\alpha+k-1}, \qquad (1.8)$$

where $(\lambda_2 \geq \lambda_1, \ell \geq 0, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\ \xi \in \mathbb{N};\ \alpha > 0, z \in \mathcal{U})$.

We remarked that, the linear operator which is defined in (1.8) is generalized many operators by giving specific values of the parameters which is studied by several earlier authors, for as instance:

i. If $\lambda_1 = 0, \alpha = 1$ and $\lambda_2 = \xi = 1$, then the operator $\mathcal{D}^{m,\xi}_{\lambda_1,\lambda_2,\ell}(f(z)^{\alpha})$ reduces to the Salagean derivative operator [14].
ii. If $\lambda_1 = 0, \alpha = 1$ and $\lambda_2 = 0$, then the operator $\mathcal{D}^{m,\xi}_{\lambda_1,\lambda_2,\ell}(f(z)^{\alpha})$ reduces to the generalized Salagean derivative operator which presented by Al-Oboudi [1].

By applying of the operator above we will present a general class of Bazilevic type functions as the following:

Definition 1.1. Assume that $0 \leq \delta \leq 1$ and $0 < \beta \leq 1$. Let $\mathcal{K}^{m,\xi}_{\lambda_1,\lambda_2,\ell}(\delta,\beta,\psi)$ denotes the class of functions $(f(z))^{\alpha}$ belonging to in A_{α} if it satisfying the inequality:

$$\left|\frac{\mathcal{D}^{m,\xi}_{\lambda_1,\lambda_2,\ell} f(z)^{\alpha}}{\left(\frac{1+\zeta(\lambda_1+\lambda_2)(\alpha-1)}{1+\lambda_1(\ell-1)}\right)^m z^{\alpha}} - 1\right| < \beta\left|\frac{\delta(\mathcal{D}^{m,\xi}_{\lambda_1,\lambda_2,\ell} f(z)^{\alpha})}{\left(\frac{1+\zeta(\lambda_1+\lambda_2)(\alpha-1)}{1+\lambda_1(\ell-1)}\right)^m z^{\alpha}} + 1\right|, (z \in \mathcal{U}). \qquad (1.9)$$

The above condition is equivalent to

$$\frac{\mathcal{D}^{m,\xi}_{\lambda_1,\lambda_2,\ell}(f(z))^\alpha}{\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}{1+\lambda_1(\ell-1)}\right)^m z^\alpha} \prec \psi(z), \qquad (1.10)$$

where $\psi(z)=\frac{1+\beta z}{1-\delta\beta z}$ be univalent function with $\psi(0)=1$ and $\psi'(z)>0$.

If $\lambda_1=\delta=0$ and $m=\xi=\lambda_2=\beta=1$, we obtain the subclass of Bazilevic univalent functions [4] defined as following

$$\left|f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1}\right|<1. \qquad (1.11)$$

If $\lambda_1=0$ and $m=\xi=\lambda_2=1$, we obtain a different subclass of Bazilevic function $\breve{\mathcal{K}}^{1,1}_{0,1}(\delta,\beta,\psi)(f(z))^\alpha$

$$f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1} \prec \psi(z) \qquad (1.12)$$

If $\lambda_1=0$, $m=\xi=\lambda_2=\alpha=1$ and putting

$\psi(z)=\frac{1+z}{1-z}$, we have the well-known subclass

$$f'(z)\prec\frac{1+z}{1-z}.$$

The class defined in (1.9) arose from a generalization of Bazilevic [4]. Which he introduced and studied this function as the following

$$f(z)=\left\{\frac{\xi}{1+\eta^2}\int_0^z (h(t)-i\eta)t^{-(1+\frac{i\xi\eta}{1+\eta^2})}g(t)^{\frac{\xi}{1+\eta^2}}dt\right\}^{\frac{1+i\eta}{\xi}}, \qquad (1.13)$$

where $h(t)\in P$ and $g(z)\in S^*$, and $\xi,\eta\in\mathbb{R}$ with $\xi>0$.

To discuss the main results we need the following Lemmas.

Lemma 1.2. [10] Let $h(z)=1+c_1z+c_2z^2+\cdots$, $z\in\mathcal{U}$ belongs to the class P and $\mu\in\mathbb{C}$, then

$$|c_2-\mu c_1^2|\le 2\max\{1,|2\mu-1|\}.$$

The result can be sharp for the following functions $h(z)=\frac{1+z^2}{1-z^2}$ and $h(z)=\frac{1+z}{1-z}, z\in\mathcal{U}$.

Lemma 1.3. [8] Let the function $h(z)$ is analytic in $\mathcal{U}$, with $|h(z)|<1$ and let

$$h(z)=c_0+c_1z+c_2z^2+c_3z^3 \ldots \; .$$

Then

$$|c_0|\le 1 \text{ and } |c_k|\le 1-|c_0|^2 \quad \text{for } k>0.$$

Lemma 1.4. [10] If $h(z)=1+c_1z+c_2z^2+\cdots$, $z\in\mathcal{U}$ belongs to the class P, then

$$|c_2-\gamma c_1^2|\le\begin{cases} -4\gamma+2 & if\ \ \gamma\le 0,\\ 2 & if\ \ 0\le\gamma\le 1,\\ 4\gamma-2 & if\ \ \gamma\ge 1.\end{cases}$$

Where $\gamma<0$ or $\gamma>1$, the equality holds if and only if $h(z)$ is equal to $\frac{1+z}{1-z}$ or one of it is rotations. If $0<\gamma<1$, then equality holds if and only if $h(z)$ is equal to $\frac{1+z^2}{1-z^2}$ or one of it is rotations. Inequality becomes equality when $\gamma=0$ if and only if $h(z)=\left(\frac{1+\lambda}{2}\right)\frac{1+z}{1-z}+\left(\frac{1-\lambda}{2}\right)\frac{1-z}{1+z}$, $0\le\lambda\le 1$, or one of its rotations. While for $\gamma=1$, equality holds if and only if the function h is the reciprocal of one of the functions such that equality holds in the case of $\gamma=0$.

Although the above upper bound is sharp, it can be improved as follows when $0<\gamma<1$:

$$|c_2-\gamma c_1^2|+\gamma|c_1|^2\le 2, \qquad 0<\gamma\le\frac{1}{2}$$

and

$$|c_2-\gamma c_1^2|+(1-\gamma)|c_1|^2\le 2 \qquad \frac{1}{2}\le\gamma<1.$$

The objective of the present paper is to obtain coefficient Bounds and Fekete- Szego inequalities for the subclass $\mathcal{K}^{m,\xi}_{\lambda_1,\lambda_2,\ell}(\delta,\beta,\psi)$ of Bazilev´c type functions.

Bazilevic functions has been investigated by many authors, we refer to ([2] , [3],[7], [9], [11], [12], [13], and [15]).

2. Main Results

Theorem 2.1. Let $(f(z))^\alpha\in A_\alpha$ which is given in (1.6). If $(f(z))^\alpha$ belongs to the class $\mathcal{K}^{m,\xi}_{\lambda_1,\lambda_2,\ell}(\delta,\beta,\psi)$. Then

$$|a_2(\alpha)|\le\frac{\beta(1+\delta)}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m},$$

$$|a_3(\alpha)|\le\frac{\beta(1+\delta)}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}\left[1-\frac{(1-\delta\beta)P_1+\beta(1+\delta)P_3}{2P_1}\right],$$

and

$$|a_4(\alpha)| \le \frac{\beta(1+\delta)}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+3)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}\Bigg[1 - \frac{(\alpha-1)P_3+2(1-\delta\beta)P_2}{2P_2} + \frac{3(\alpha-1)(1-\delta\beta)P_1P_4+3\beta(\alpha-1)(1+\delta)P_3P_4}{12P_1P_2} - \frac{\beta(\alpha-2)(1+\delta)P_3P_4}{12P_1P_2} - \frac{3}{12}(1-2\delta\beta)\Bigg],$$

where

$$P_1 = 2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{2m},$$

$$P_2 = \alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{m}\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{m},$$

$$P_3 = (\alpha-1)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{m},$$

$$P_4 = \beta(1+\delta)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+3)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{m}.$$

Proof: If $(f(z))^\alpha \in \mathcal{K}^{m,\xi}_{\lambda_1,\lambda_2,\ell}(\delta,\beta,\psi)$, then from (1.10) there is Schwarz function $w(z)$ which is analytic in open unit disk $\mathcal{U}$ with $w(0)=0$ and $|w(z)|<1$ such that

$$\frac{\mathcal{D}^{m,\xi}_{\lambda_1,\lambda_2,\ell}(f(z))^\alpha}{\left(\frac{1+\zeta(\lambda_1+\lambda_2)(\alpha-1)}{1+\lambda_1(\ell-1)}\right)^m z^\alpha} \prec \psi(w(z)), \tag{2.1}$$

where

$$\psi(z) = \frac{1+\beta z}{1-\delta\beta z} = 1+\beta(1+\delta)z + \delta\beta^2(1+\delta)z^2 + \delta^2\beta^3(1+\delta)z^3 + \cdots \ . \tag{2.2}$$

Define the function $h(z)$ by

$$h(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1h + c_2h^2 + c_3h^3 + \cdots \ .$$

It follows from (1.2), $h \in P$ and

$$w(z) = \frac{h(z)-1}{h(z)+1} = \frac{1}{2}c_1z + \frac{1}{2}(c_2 - \frac{1}{2}c_1^2)z^2 + \frac{1}{2}(c_3 - c_1c_2 + 1/4\, c_1^3)z^3 + \cdots \tag{2.3}$$

In view from of (2.1), (2.2) and (2.3), we obtain

$$\frac{\mathcal{D}^{m,\xi}_{\lambda_1,\lambda_2,\ell}(f(z))^\alpha}{\left(\frac{1+\zeta(\lambda_1+\lambda_2)(\alpha-1)}{1+\lambda_1(\ell-1)}\right)^m z^\alpha} \prec \psi(w(z))$$

$$= \psi(z)\left(\frac{h(z)-1}{h(z)+1}\right)$$

$$= \psi(\frac{1}{2}c_1z + \frac{1}{2}(c_2 - \frac{1}{2}c_1^2)z^2 + \frac{1}{2}(c_3 - c_1c_2 + 1/4\, c_1^3)z^3 + \cdots),$$

$$= 1 + \tfrac{1}{2}\beta(1+\delta)c_1z + \left[\tfrac{1}{2}\beta(1+\delta)(c_2 - \tfrac{1}{2}c_1^2) + \tfrac{1}{4}\delta\beta^2(1+\delta)c_1^2\right]z^2 + \left[\tfrac{1}{2}\beta(1+\delta)\left(c_3 - c_1c_2 + \tfrac{1}{4}c_1^3\right) + \tfrac{1}{2}\delta\beta^2(1+\delta)(c_2 - \tfrac{1}{2}c_1^2)c_1 + \tfrac{1}{8}\delta^2\beta^3(1+\delta)c_1^3\right]z^3 + \cdots \ . \tag{2.4}$$

Using the series expansion of $1 + \sum_{k=2}^{\infty}\left(\frac{1+\zeta(\lambda_1+\lambda_2)(k+\alpha-1)}{1+\zeta(\lambda_1+\lambda_2)(\alpha-1)}\right)^m a_k(\alpha)z^{k-1}$, we have

$$\frac{\mathcal{D}^{m,\xi}_{\lambda_1,\lambda_2,\ell}(f(z))^\alpha}{\left(\frac{1+\zeta(\lambda_1+\lambda_2)(\alpha-1)}{1+\lambda_1(\ell-1)}\right)^m z^\alpha} = 1 + \alpha\left(\frac{1+\zeta(\lambda_1+\lambda_2)(\alpha+1)}{1+\zeta(\lambda_1+\lambda_2)(\alpha-1)}\right)^m a_2(\alpha)z + \left[\alpha a_3(\alpha) + \frac{\alpha(\alpha-1)}{2!}a_2^2(\alpha)\right]\left(\frac{1+\zeta(\lambda_1+\lambda_2)(\alpha+2)}{1+\zeta(\lambda_1+\lambda_2)(\alpha-1)}\right)^m z^2 + \left[\alpha a_4(\alpha) + \alpha(\alpha-1)a_2(\alpha)a_3(\alpha) + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}a_2^3(\alpha)\right]\left(\frac{1+\zeta(\lambda_1+\lambda_2)(\alpha+3)}{1+\zeta(\lambda_1+\lambda_2)(\alpha-1)}\right)^m z^3 + \cdots. \tag{2.5}$$

Comparing the coefficients of z, z^2 and z^3 in (2.1) based on (2.4) and (2.5), we get

$$a_2(\alpha) = \frac{\beta(1+\delta)c_1}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}, \tag{2.6}$$

$$a_3(\alpha) \le \frac{\beta(1+\delta)}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}\left[c_2 - \frac{(1-\delta\beta)P_1 + \beta(1+\delta)P_3}{2P_1}c_1^2\right], \tag{2.7}$$

and

$$a_4(\alpha) = \frac{\beta(1+\delta)}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+3)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}\Bigg[c_3 - \frac{(\alpha-1)P_3+2(1-\delta\beta)P_2}{2P_2}c_1\,c_2 + \left(\frac{3(\alpha-1)(1-\delta\beta)P_1P_4+3\beta(\alpha-1)(1+\delta)P_3P_4}{12P_1P_2} - \frac{\beta(\alpha-2)(1+\delta)P_3P_4}{12P_1P_2} - \frac{3}{12}(1-2\delta\beta)\right)c_1^3\Bigg]. \qquad (2.8)$$

Since $h(z)$ has an analytic function and is bounded in $\mathcal{U}$, we have

$$|c_0| \le 1 \text{ and } |c_k| \le 1-|c_0|^2 \quad \text{for } k>0. \qquad (2.9)$$

By using (2.9), we obtain

$$|a_2(\alpha)| \le \frac{\beta(1+\delta)}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m},$$

$$|a_3(\alpha)| \le \frac{\beta(1+\delta)}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}\left[1 - \frac{(1-\delta\beta)P_1+\beta(1+\delta)P_3}{2P_1}\right],$$

and

$$|a_4(\alpha)| \le \frac{\beta(1+\delta)}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+3)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}\Bigg[1 - \frac{(\alpha-1)P_3+2(1-\delta\beta)P_2}{2P_2} + \frac{3(\alpha-1)(1-\delta\beta)P_1P_4+3\beta(\alpha-1)(1+\delta)P_3P_4}{12P_1P_2} - \frac{\beta(\alpha-2)(1+\delta)P_3P_4}{12P_1P_2} - \frac{3}{12}(1-2\delta\beta)\Bigg].$$

The proof is complete.

Putting $\psi(z) = \frac{1+z}{1-z}$ in Theorem 2.1, we have the next.

Corollary 2.1. Let $(f(z))^\alpha \in A_\alpha$ which is given in (1.6). If $(f(z))^\alpha$ belonging to the class $\mathcal{K}_{\lambda_1,\lambda_2,\ell}^{m,\xi}(1,1,\psi)$. Then

$$|a_2(\alpha)| \le \frac{1}{\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m},$$

Proof: From (2.6) and (2.7), we have

$$|a_3(\alpha)| \le \frac{1}{\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m} - \frac{P_3}{\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m P_1},$$

and

$$|a_4(\alpha)| \le \frac{1}{\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+3)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}\left[1 - \frac{2(\alpha-1)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}{2P_2} + \frac{4(2\alpha-1)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m P_3}{12P_1P_2}\right].$$

Putting $\alpha = 1$ and $m = 0$ in Corollary 2.1, we get the result.

Corollary 2.2. Let $f(z) \in A$ which is given in (1.6). If $f(z)$ belongs to the class $\mathcal{K}_{\lambda_1,\lambda_2,\ell}^{0,\xi}(1,1,\psi)$. Then

$$|a_2(\alpha)| \le 1$$
$$|a_3(\alpha)| \le 1,$$

and

$$|a_4(\alpha)| \le 1.$$

Theorem 2.2. Let $\psi(z) = 1+\beta(1+\delta)z+\delta\beta^2(1+\delta)z^2+\cdots$, where $\psi(z)\in A$ and $\psi'(z)>0$. If $(f(z))^\alpha$ which is given in (1.6) belongs to the class $\mathcal{K}_{\lambda_1,\lambda_2,\ell}^{m,\xi}(\delta,\beta,\psi)$ and $\mu\in\mathbb{N}$, then

$$|a_3(\alpha)-\mu a_2^2(\alpha)| \le \frac{\beta(1+\delta)}{\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}\max\left\{1, \left|\delta\beta + \beta(1+\delta)\left(\frac{[2\xi+\alpha-1]\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{2m}}\right)\right|\right\}.$$

$$
\begin{aligned}
&a_3(\alpha)-\mu a_2^2(\alpha) \\
&=\frac{\beta(1+\delta)c_2}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m} \\
&-\frac{\left[2\alpha\beta(1+\delta)(1-\delta\beta)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{2m}+(\alpha-1)\beta^2(1+\delta)^2\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m\right]}{8\alpha^2\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{2m}\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}c_1^2 \\
&-\mu\frac{\beta^2(1+\delta)^2c_1^2}{4\alpha^2\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{2m}}.
\end{aligned}
$$

Therefore

$$
|a_3(\alpha)-\mu a_2^2(\alpha)|=\frac{\beta(1+\delta)}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}[c_2-vc_1^2],
$$

where

$$
v=\frac{1}{2}\left[1-\delta\beta+\beta(1+\delta)\left(\frac{(2\mu+\alpha-1)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{2m}}\right)\right].
$$

Applying Lemma 1.2, we get

$$
\begin{aligned}
|a_3(\alpha)-\mu a_2^2(\alpha)|\leq\frac{\beta(1+\delta)}{\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}\max\Bigg\{1,\Bigg|\delta\beta \\
+\beta(1+\delta)\left(\frac{[2\mu+\alpha-1]\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{2m}}\right)\Bigg|\Bigg\}.
\end{aligned}
$$

The result be sharp for $(f(z))^\alpha$ which is given as the follows

$$
\frac{\mathcal{D}_{\lambda_1,\lambda_2,\ell}^{m,\xi}(f(z))^\alpha}{\left(\frac{1+\zeta(\lambda_1+\lambda_2)(\alpha-1)}{1+\lambda_1(\ell-1)}\right)^m z^\alpha}\prec\psi(z^2),
$$

or

$$
\frac{\mathcal{D}_{\lambda_1,\lambda_2,\ell}^{m,\xi}(f(z))^\alpha}{\left(\frac{1+\zeta(\lambda_1+\lambda_2)(\alpha-1)}{1+\lambda_1(\ell-1)}\right)^m z^\alpha}\prec\psi(z).
$$

The proof is complete.

In virtue of Lemma 1.4, we get the following results:

Theorem 2.3. Let $\psi(z)=1+\beta(1+\delta)z+\delta\beta^2(1+\delta)z^2+\cdots$, $0\leq\delta\leq1$ and $0<\beta\leq1$. If $(f(z))^\alpha$ given by (1.6) belongs to the class $\mathcal{K}_{\lambda_1,\lambda_2,\ell}^{m,\xi}(\delta,\beta,\psi)$, then $\sigma\in\mathbb{C}$

$$|a_3(\alpha)-\sigma a_2^2(\alpha)| \le \begin{cases} \dfrac{\beta(1+\delta)}{\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}\left[\delta\beta-\beta(1+\delta)\left(\dfrac{[2\sigma+\alpha-1]\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{2m}}\right)\right], & if\ \sigma\le\sigma_1 \\ \dfrac{\beta(1+\delta)}{\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m} & if\ \sigma_1\le\sigma\le\sigma_2 \\ \dfrac{\beta(1+\delta)}{\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}\left[\beta(1+\delta)\left(\dfrac{[2\sigma+\alpha-1]\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{2m}}\right)-\delta\beta\right], & if\ \sigma\ge\sigma_2, \end{cases}$$

where

$$\sigma_1=\frac{2\alpha(\delta\beta-1)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{2m}-\beta(1+\delta)(\alpha-1)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}{2\beta(1+\delta)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m},$$

$$\sigma_2=\frac{2\alpha(\delta\beta+1)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{2m}-\beta(1+\delta)(\alpha-1)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}{2\beta(1+\delta)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}.$$

Proof. Since $(f(z))^\alpha\in\mathcal{K}_{\lambda_1,\lambda_2,\ell}^{m,\xi}(\delta,\beta,\psi)$ and $\psi(z)$ given by (2.4), then $a_2(\alpha)$ and $a_3(\alpha)$ are given in Theorem 2.1. Furthermore,

$$|a_3(\alpha)-\sigma a_2^2(\alpha)| = \frac{\beta(1+\delta)}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}[c_2-tc_1^2],$$

where

$$t=\frac{1}{2}\left[1-\delta\beta+\beta(1+\delta)\left(\frac{(2\sigma+\alpha-1)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{2m}}\right)\right].$$

Thus, the inequality (1.10) can be established as following by using Lemma 1.4.
If $t\le 0$, then $\sigma\le\sigma_1$ and by applying Lemma 1.4. yields

$$|a_3(\alpha)-\sigma a_2^2(\alpha)|\le\frac{\beta(1+\delta)}{\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}\left[\delta\beta-\beta(1+\delta)\left(\frac{[2\sigma+\alpha-1]\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{2m}}\right)\right].$$

For $0\le t\le 1$, then $\sigma_1\le\sigma\le\sigma_2$ and by applying Lemma 1.4, gives

$$|a_3(\alpha)-\sigma a_2^2(\alpha)|\le\frac{\beta(1+\delta)}{\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}.$$

For $t\ge 1$, then $\sigma_2\le\sigma$. Also by an applying Lemma 1.4, we obtain

$$|a_3(\alpha)-\sigma a_2^2(\alpha)| \le\frac{\beta(1+\delta)}{\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}\left[\beta(1+\delta)\left(\frac{[2\sigma+\alpha-1]\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{2m}}\right)-\delta\beta\right].$$

Putting $\psi(z)=\frac{1+z}{1-z}$ in Theorem 2.3, we get the following result:

Corollary 2.3. Let $\psi(z) = 1 + 2z + 2z^2 + \cdots$. If $(f(z))^\alpha$ which is given in (1.6) belongs to the class $\mathcal{K}_{\lambda_1,\lambda_2,\ell}^{m,\xi}(1,1,\psi)$, then for any real complex number σ

$$|a_3(\alpha) - \sigma a_2^2(\alpha)| \le \begin{cases} \dfrac{2}{\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}\left[1-\left(\dfrac{[2\sigma+\alpha-1]\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}{\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{2m}}\right)\right], & if\ \sigma \le \sigma_1 \\ \dfrac{2}{\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m} & if\ \sigma_1 \le \sigma \le \sigma_2 \\ \dfrac{2}{\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}\left[\left(\dfrac{[2\sigma+\alpha-1]\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}{\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{2m}}\right)-1\right], & if\ \sigma \ge \sigma_2, \end{cases}$$

where

$$\sigma_1 = \frac{-(\alpha-1)}{2(1+\delta)},$$

$$\sigma_2 = \frac{4\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{2m} - 2(\alpha-1)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}{4\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}.$$

.

Note that for $\alpha = 1$ and $m = 0$ in Corollary 2.3, we get the following Corollary:

Corollary 2.4. If $\psi(z) = 1 + 2z + 2z^2 + \cdots$. If $(f(z))^\alpha$ given by (1.6) belongs to the class $\mathcal{K}_{\lambda_1,\lambda_2,\ell}^{0,\xi}(1,1,\psi)$, then for any complex number σ

$$|a_2 - \sigma a_1^2| \le \begin{cases} 2-4\sigma & if\quad \sigma \le \sigma_1, \\ 2 & if\quad \sigma_1 \le \sigma \le \sigma_2, \\ 4\sigma - 2 & if\quad \sigma \ge \sigma_2, \end{cases}$$

where $\sigma_1 = 0$, and $\sigma_2 = 1$ reduces to the result [4, Lemma 1] of W. Ma. D. Minda.

Corollary 2.5. Let $\psi(z) = 1 + \beta(1+\delta)z + \delta\beta^2(1+\delta)z^2 + \cdots$, $0 \le \delta \le 1$ and $0 < \beta \le 1$. If $(f(z))^\alpha$ given by (1.6) belongs to the class $\mathcal{K}_{\lambda_1,\lambda_2,\ell}^{m,\xi}(\delta,\beta,\psi)$, and $\sigma_1 \le \sigma_3 \le \sigma_2$. Let

$$\sigma_3 = \frac{1}{2\beta(1+\delta)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}\left(2\alpha\delta\beta\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{2m} - \beta(1+\delta)(\alpha-1)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m\right).$$

If $\sigma_1 \le \sigma \le \sigma_3$, then

$$\le \frac{\beta(1+\delta)}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}, \tag{2.10}$$

If $\sigma_3 \le \sigma \le \sigma_2$, then

$$|a_3(\alpha) - \sigma a_2^2(\alpha)| + \frac{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{2m}}{\beta(1+\delta)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m} \times\left[1-\delta\beta + \beta(1+\delta)\left(\frac{(2\sigma+\alpha-1)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^m}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{2m}}\right)\right]|a_2(\alpha)|^2$$

$$|a_3(\alpha)-\sigma a_2^2(\alpha)|+\frac{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{2m}}{\beta(1+\delta)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{m}} \times\left[\delta\beta+1-\beta(1+\delta)\left(\frac{(2\sigma+\alpha-1)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{m}}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{2m}}\right)\right]|a_2(\alpha)|^2,$$
$$\leq \frac{\beta(1+\delta)}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{m}} \quad (2.11)$$

Proof. For

$$|a_3(\alpha)-\sigma a_2^2(\alpha)|+(\sigma-\sigma_1)|a_2(\alpha)|^2 = \left(\sigma-\frac{1}{2\beta(1+\delta)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{m}}\left[2\alpha(\delta\beta-1)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{2m}+\beta(1+\delta)(\alpha-1)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{m}\right]\right)$$
$$\times\frac{\beta^2(1+\delta)^2}{4\alpha^2\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{2m}}|c_1|^2+\frac{\beta(1+\delta)}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{m}}[c_2-tc_1^2]$$

$$=\frac{\beta(1+\delta)}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{m}}\{|c_2-tc_1^2(\alpha)|+t|c_1|^2\}.$$

It follows from applying Lemma 1.4, yields

$$|a_3(\alpha)-\sigma a_2^2(\alpha)|+(\sigma-\sigma_1)|a_2(\alpha)|^2 \leq \frac{\beta(1+\delta)}{\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{m}},$$

which is the inequality of (2.10). For

$$|a_3(\alpha)-\sigma a_2^2(\alpha)|+(\sigma-\sigma_1)|a_2(\alpha)|^2 =$$

$$\left(\frac{2\alpha(\delta\beta+1)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{2m}}{2\beta(1+\delta)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{m}}-\frac{\beta(1+\delta)(\alpha-1)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{m}}{2\beta(1+\delta)\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{m}}-\sigma\right)$$

$$\times\frac{\beta^2(1+\delta)^2}{4\alpha^2\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+1)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{2m}}|c_1|^2+\frac{\beta(1+\delta)}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{m}}[c_2-tc_1^2]$$
$$=\frac{\beta(1+\delta)}{2\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{m}}\{|c_2-tc_1^2|+t|c_1|^2\}.$$

It follows from applying Lemma 1.4, we obtain

$$|a_3(\alpha)-\sigma a_2^2(\alpha)|+(\sigma_2-\sigma)|a_2(\alpha)|^2 \leq \frac{\beta(1+\delta)}{\alpha\left(\frac{1+\xi(\lambda_1+\lambda_2)(\alpha+2)}{1+\xi(\lambda_1+\lambda_2)(\alpha-1)}\right)^{m}},$$

which is the inequality of (2.11).

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