Study of the Existence and Uniqueness of Global Solution for the Fractional Second Order Nonlinear Integro –Differential Equations with Boundary Conditions

Ammar Isam Edress

Geology of Petroleum and Sediments Department, Technical Collage of Petroleum and Minerals, Duhok Polytechnic University, Kurdistan Region-Iraq

Ammar.isam@dpu.edu.krd

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Abstract— *In this paper, the existence of the global solution and its uniqueness is studied for the Second Order Nonlinear Integro–Differential Fractional Equations with boundary conditions by utilizing the Picard approximation method which is given by Sturble, 1962. Furthermore, several given results by Butris, 2010 have been extended.*

Keywords— Differential Equations, Fractional Second Order Nonlinear Integro –Differential Equations

I. INTRODUCTION

Enormous of studies have discussed the solution of the differential and integral equation for fractional order such as [1], [2], [3], [4], [5], [6] and [7]. In this study, the focus will be on the 2nd order nonlinear integro–differential fractional equations [8], and [9].

At the beginning, it's worth to set some definitions and lemmas to be employed then for the main theorems proofs.

Definition 1: [1], [10]
Let f be a function which is defined a.e. (almost everywhere) on [a, b].
For, we define:

$$\overset{b\alpha}{\underset{a}{I}} f = \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1} f(s)ds$$

provided this integral (Lebesgue) exists.

Definition 2: [3], [10]
If $\alpha > 0$, then Gamma's function is denoted by Γ and defined by the form:

$$\Gamma(\alpha) = \int_0^\infty e^{-s} s^{\alpha-1} ds$$

Lemma 1: [1], [11]

If $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions, is defined over the set E⊆R such that $|f_n| \le M_n$, where M_n is a positive number, then $\sum_{n=1}^{\infty} f_n$ is uniformly converted on E if $\sum_{n=1}^{\infty} M_n$ is convergent.

Lemma 2: [3]

Let

$$E_\alpha(m;x)=\sum_{m=1}^{\infty}\frac{m^{n-1}x^{n\alpha-1}}{\Gamma(n\alpha)}$$

where m = R, then:

i. the series converges for $x \neq o$ and $\alpha > 0$

ii. the series converges everywhere when $\alpha \geq 1$

iii. if $\alpha = 1$, then $E1(m,x)=\exp(mx)$

For the definitions and lemmas see [2], [8].

Consider the following fractional second order nonlinear integro–differential equations, which have the form:

$$x^{(2\alpha)} = f\left(t, x(t)), \bar{x}(t), \int_{-\infty}^{t} w(t,s)g(s,x(s), \bar{x}(s)ds\right), 0<\alpha\leq 1 \qquad \ldots (1)$$

Subject to the following boundary conditions:

$$x(a)=A`,\ x(b)=B \qquad \ldots (2)$$

where the function $f(t,x,\dot{x},y)$ is a continuous in $t,x,\dot{x},y$ and defined over the domain:

$$(t,x,\dot{x},y)\in R'\times G_\alpha \times G_{1\alpha}\times G_{2\alpha} = [0,T]\times G_\alpha\times G_{1\alpha}\times G_{2\alpha}, T=\infty \qquad \ldots (3)$$

where G_α, $G_{1\alpha}$ and $G_{2\alpha}$ are close and bounded domains subset of the Euclidian space R' where A, B are positive constant.

Suppose that the function $f(t,x,\dot{x},y)$ is satisfying following inequities:

$$\| f(t,x,\dot{x},y)\| \leq M, \qquad \ldots (4)$$

$$\| f(t,x_1,\dot{x}_1,y_1) - f(t,x_2,\dot{x}_2,y_2)\| \leq K_1\|x_1-x_2\| + K_2\|\dot{x}_1-\dot{x}_2\| + K_3\|y_1-y_2\| \qquad \ldots (5)$$

$$\| g(t,x_1,\dot{x}_1,y_1) - g(t,x_2,\dot{x}_2,y_2)\| \leq L_1\|x_1-x_2\| + L_2\|\dot{x}_1-\dot{x}_2\| \qquad \ldots (6)$$

for all $t\in[R_1]$ and $x,x_1,x_2\in G_R, \dot{x},\dot{x}_1,\dot{x}_2\in G_{1\alpha}$ and $y,y_1,y_2\in G_{2\alpha}$ where M,K_1,K_2,K_3,L_1,L_2 are positive constants and $y(t,x_0)=\int_{-\infty}^{t} w(t,s)g(s,x(s),\dot{x}(s))ds$ where the function $w(t,s)$ is defined and continuous on the domain $-\infty\leq 0 < a \leq s \leq t \leq b \leq T \leq \infty$ provided that where γ, s are a positive constants.

We define the non–empty sets as follows:

$$G_a f = G_\alpha - \frac{(\Gamma-a)^{\alpha+1}}{\Gamma(\alpha+1)}M$$

$$G_{1a} f = G_{1\alpha} - \frac{(\Gamma-a)^{\alpha}}{\Gamma(\alpha+1)}M$$

$$G_{2a} f = G_{2\alpha} - \frac{\delta}{\gamma}\left[L_1\frac{(\Gamma-a)^{\alpha+1}}{\Gamma(\alpha+1)} + L_2\frac{(\Gamma-a)^{\alpha+1}}{\Gamma(\alpha+1)}M\right] \qquad \ldots (7)$$

where $\|.\| = \max|.|$

Furthermore, we assumed that the greatest eigen–value of the following matrix:

$$H_o = \begin{pmatrix} \frac{(\Gamma-a)^{\alpha+1}}{\Gamma(\alpha+1)}q_1 & \frac{(\Gamma-a)t^{\alpha+1}}{\Gamma(\alpha)} + \frac{(\Gamma-a)^{\alpha+1}}{\Gamma(\alpha+1)}q_2 \\ \frac{(\Gamma-a)^{\alpha+1}}{\Gamma(\alpha+1)}q_1 & \frac{(\Gamma-a)^{\alpha}}{\Gamma(\alpha+1)}q_2 \end{pmatrix}$$

is less than unity i.e.

$$h_{\max}(H_o) < 1$$

where

$$q_1 = K_1 + K_3 \frac{\delta}{\gamma} L_1 \text{ and } q_2 = K_2 + K_3 \frac{\delta}{\gamma} L_2$$

.. (8)

II. EXISTENCE OF THE SOLUTION:

The study for existence of solution for the problem (1), (2) will be introduced by the following:

1. Theorem 1: (Global Existence Theorem):

Let the function $f(t,x,\dot{x},y)$ be defined in the domain (3), continuous in $t,x,\dot{x}$ and satisfy the inequalities (4), (5) and (6), then the sequence of functions:

$$x_{m+1}(t,x_o) = \frac{[A+(t-a)\dot{x}_m(a)]t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_a^t [(w(t,s) f((s,m(s),\dot{x}_m(s),y_m(s)))](t-s)^{\alpha-1} ds$$

… (9)

with

$$x_o = A + \frac{(t-a)(B-A)}{(b-a)}, \text{ and } \dot{x}_o(t) = \frac{(B-A)}{(b-a)}$$

Converges uniformly on the domain

$$(t,x_o,\dot{x}_o,y_o) \in [0,T] \times G_{1\alpha f} \times G_{2\alpha f}$$

… (10)

to the limit function $x_\infty(t,x_o)$ which is satisfying the integral equation:

$$x(t,x_o) = \frac{[A+(t-a)\dot{x}(a)]t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_a^t [w(t,s) f(s,m_,(s),\dot{x}_m(s),y_m(s))](t-s)^{\alpha-1} ds$$

… (11)

provided that

$$\|x_\infty(t,x_o) - x_o\| < \frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)} M ,$$

… (12)

$$\|\dot{x}_\infty(t,x_o)\| < \frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)} M$$

… (13)

and

$$\begin{pmatrix} \|x_\infty(t,x_o) - x_m(t,x_o)\| \\ \|\dot{x}_\infty(t,x_o) - \dot{x}_m(t,x_o)\| \end{pmatrix} \leq H_o^m (E - H_o)^{-1} \psi_o$$

… (14)

for $t \in [o,T], x \in G_{\alpha f}$

where $\psi_o = \begin{pmatrix} \frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)} \\ \frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)} \end{pmatrix}$ and E is identity matrix

Proof:

The proof of this theorem has been given in details by [6].

Moreover, to prove that the solution $x(t_1,x_o) \in G_f$ for $x_o \in G_f$ start by taking the following:

$$\left|\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)f(s,x_m(s),\dot{x}_m(s))(t-s)^{\alpha-1}ds - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)f(s,x(s),\dot{x}(s))(t-s)^{\alpha-1}ds\right|$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)\left[q_1\|x_m(s)-x(s)\| + q_2\|\dot{x}_m(s)-\dot{x}(s)\|\right](t-s)^{\alpha-1}ds$$

$$\lim_{m\to\infty}\left|\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)f(s,x_m(s),\dot{x}_m(s))(t-s)^{\alpha-1}ds - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)f(s,x(s),\dot{x}(s))(t-s)^{\alpha-1}ds\right|$$

$$\lim_{m\to\infty}\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)\left[q_1\|x_m(s)-x(s)\| + q_2\|\dot{x}_m(s)-\dot{x}(s)\|\right](t-s)^{\alpha-1}ds$$

… (15)

Since that the sequence $\{x_m(t,x_o)\}_{m=o}^{\infty}$ is uniformly convergent on $[o,T]$ from the function $x_m(t,x_0)$ on the same interval, then

$$\lim_{m\to\infty}\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)f(s,x_m(s),\dot{x}_m(s),y_m(s))(t-s)^{\alpha-1}ds$$

$$=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)f(s,x_m(s),\dot{x}_m(s),y_m(s))(t-s)^{\alpha-1}ds$$

… (16)

Then the solution $x(t,x_0)\in G_f$

III. The Uniqueness of the Solution:

The solution uniqueness study of the problem (1) and (2) to be gain by the following theorem.

Theorem 2: Global Uniqueness Theorem [1]

Let all the assumptions and conditions of theorem 1 be given. Then the problem (1), (2) has a unique solution on the domain (3).

By assuming $u(t,x_o)$ is another solution for (1), (2), i.e

$$u(t,x_o)=\frac{[A+(t-a)\dot{x}(a)]t^{\alpha-1}}{\Gamma(\alpha)}+$$

$$\frac{1}{\Gamma(\alpha)}\int_a^t [(t-s)f(s,x(s),\dot{x}(s),y(s))](t-s)^{\alpha-1}ds$$

… (17)

Now, to prove $x(t,x_o,\dot{x}_o,y_o)=u(t,x_o,\dot{x}_o,y_o)$ and $\dot{x}(t,x_o`)=\dot{u}(t,x_o)$ the following inequality required to be proved using the principle of induction:

$$\begin{pmatrix}\|u(t,x_o)-x_m(t,x_o)\|\\ \|\dot{u}(t,x_o)-\dot{x}_m(t,x_o)\|\end{pmatrix} < H_o^m(E-H_o)^{-1}\Psi_o$$

… (18)

Same above technique could be utilized to prove the theorem 1, the uniqueness solution of (1) and (2) to be got, due to $H_o^m \to o$ as $m\to\infty$ so that proceeding in the last inequality to the limit

The equalities to be obtained

$x_m(t,x_o)=u(t,x_o)$ and $\dot{x}_m(t,x_o)=\dot{u}(t,x_o)$

… (19)

Remark: By setting α=1, the results of Butris will be produced as it given by [2].

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