On a New Class of Meromorphic Multivalent Functions Defined by Fractional Differ-integral Operator

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***Abstract*— *In this work, we submite and study a new class of meromorphic multivalent functions defined by fractional differ-integral operator . We gain some geometric properties , such as, coefficient inequality, growth and distortion bounds, convolution properties, integral representation, radii of starlikeness and convexity, extreme points, weighted mean and arithmetic mean for functions belonging to the class* $\Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$.

Keywords*— *Meromorphic multivalent function; fractional differ-integral operator; distortion bounds; radius of starlikeness; extreme points.

1. Introduction

Let Σ_p be symbolize the class of functions of the shape:

$$f(z) = \frac{1}{z^p} - \sum_{n=p}^{\infty} a_n z^n, (a_n \geq 0\,;\, p \in N = \{1,2,\ldots\}),\quad (1.1)$$

which are analytic and multivalent in the punctured unit disk

$U^* = \{z: z \in \mathbb{C}\ and\ 0 < |z| < 1\} = U\setminus\{0\}$.

A function $f \in \Sigma_p$ is meromorphic multivalent starlike function of order φ $(0 \leq \varphi < p)$ if

$$Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \varphi, (0 \leq \varphi < p\,;\, z \in U^*). \quad (1.2)$$

A function $f \in \Sigma_p$ is meromorphic multivalent convex function of order φ $(0 \leq \varphi < p)$ if

$$Re\left\{-\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \varphi, (0 \leq \varphi < p\,;\, z \in U^*). \quad (1.3)$$

For functions $f(z) \in \Sigma_p$ presented in (1.1) and $g(z) \in \Sigma_p$ presented by

$$g(z) = \frac{1}{z^p} - \sum_{n=p}^{\infty} b_n z^n, (b_n \geq 0;\ p \in \mathbb{N} = \{1,2,\ldots\}),\quad (1.4)$$

we define the convolution (or Hadamard product) of $f(z)$ and $g(z)$ by

$$(f * g)(z) = \frac{1}{z^p} - \sum_{n=p}^{\infty} a_n b_n z^n = (g * f)(z). \quad (1.5)$$

In this work, we explore and study a new class of meromorphic multivalent functions by making use of the fractional differ-integral operator contained in the following definition:

Definition (1.1)[5]: Let $f(z) \in \Sigma_p$ defined in (1.1). Then

$$\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta} f(z) = \begin{cases} \dfrac{\Gamma(\mu+v+\eta-\lambda)\Gamma(\eta)}{\Gamma(\mu+\eta)\Gamma(v+\eta)} z^{-p-\eta+1} J_{0,z}^{\lambda,\mu,v,\eta}[z^{\mu+p} f(z)] (0 \leq \lambda < 1) \\ \dfrac{\Gamma(\mu+v+\eta-\lambda)\Gamma(\eta)}{\Gamma(\mu+\eta)\Gamma(v+\eta)} z^{-p-\eta+1} I_{0,z}^{-\lambda,\mu,v,\eta}[z^{\mu+p} f(z)] (-\infty < \lambda < 0), \end{cases} \quad (1.6)$$

where $J_{0,z}^{\lambda,\mu,v,\eta} f(z)$ is the generalized fractional derivative operator of order λ defined by

$J_{0,z}^{\lambda,\mu,v,\eta} f(z) =$

$$\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\left\{z^{\lambda-\mu}\int_0^z t^{\eta-1}(z-t)^{-\lambda}\,{}_2F_1\left(\mu-\lambda,1-v;1-\lambda;1-\frac{t}{z}\right)f(t)dt\right\} \quad (1.7)$$

$\left(0\le\lambda<1,\mu,\eta\in R,v\in R^+ \ and\ r>\ (\max\{0,\mu\}-\eta)\right)$,

where $f(z)$ is an analytic function in a simply–connected region of the z-plane containing the origin and the multiplicity of $(z-t)^{-\lambda}$ is removed by requiring $log(z-t)$ to be real when $(z-t)>0$, provided that

$$f(z)=O(|z|^r)\ (z\to 0) \quad (1.8)$$

and $I_{0,z}^{-\lambda,\mu,v,\eta}$ is the generalized fractional integral operator of order $-\lambda$ $(-\infty<\lambda<0)$ defined by

$$I_{0,z}^{\lambda,\mu,v,\eta}f(z)=\frac{z^{-(\lambda+\mu)}}{\Gamma(\lambda)}\int_0^z t^{\eta-1}(z-t)^{\lambda-1}\,{}_2F_1\left(\lambda+\mu,-v;\lambda;1-\frac{t}{z}\right)f(t)dt \quad (1.9)$$

$\left(\lambda>0,\mu,\eta\in R,v\in R^+ \ and\ r>\ (\max\{0,\mu\}-\eta)\right)$,

where $f(z)$ is constrained, the multiplicity of $(z-t)^{\lambda-1}$ is removed as above and r is presented by the order estimate (1.8).

It follows from (1.7) and (1.9) that

$$J_{0,z}^{\lambda,\mu,v,1}f(z)=J_{0,z}^{\lambda,\mu,v}f(z) \quad (1.10)$$

and

$$I_{0,z}^{\lambda,\mu,v,1}f(z)=I_{0,z}^{\lambda,\mu,v}f(z), \quad (1.11)$$

where $J_{0,z}^{\lambda,\mu,v}$ and $I_{0,z}^{\lambda,\mu,v}$ are the Owa-Saigo-Srivastava generalized fractional derivative and integral operators (see [8,2,4]).

Also

$$J_{0,z}^{\lambda,\lambda,v,1}f(z)=D_z^{\lambda}f(z),\ \ (0\le\lambda<1) \quad (1.12)$$

and

$$I_{0,z}^{\lambda,-\lambda,v,1}f(z)=D_z^{-\lambda}f(z)\ \ (\lambda>0), \quad (1.13)$$

where D_z^{λ} and $D_z^{-\lambda}$ are the familiar Owa-Srivastava fractional derivative and integral of order λ, respectively (see [9,3]).

Furthermore, in terms of Gamma function, we have

$$J_{0,z}^{\lambda,\mu,v,\eta}z^n=\frac{\Gamma(n+\eta)\Gamma(n+\eta-\mu+v)}{\Gamma(n+\eta-\mu)\Gamma(n+\eta-\lambda+v)}z^{n+\eta-\mu-1} \quad (1.14)$$

$\left(0\le\lambda<1,\mu,\eta\in R,v\in R^+ and\ n>(\max\{0,\mu\}-\eta)\right)$

and

$$I_{0,z}^{\lambda,\mu,v,\eta}z^n=\frac{\Gamma(n+\eta)\Gamma(n+\eta-\mu+v)}{\Gamma(n+\eta-\mu)\Gamma(n+\eta+\lambda+v)}z^{n+\eta-\mu-1} \quad (1.15)$$

$\left(\lambda>0,\mu,\eta\in R,v\in R^+ and\ n>(\max\{0,\mu\}-\eta)\right)$.

Now, using (1.1) , (1.14) and (1.15) in (1.6), we get

$$\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta}f(z)=\frac{1}{z^p}-\sum_{n=p}^{\infty}\Gamma_{p,n}^{\lambda,\mu,v,\eta}a_n\,z^n \quad (1.16)$$

provided that $-\infty<\lambda<1,\ \mu+v+\eta>\lambda,\mu>-\eta,v>-\eta,\eta>0,p\in N,f\in\Sigma_p$ and

$$\Gamma_{p,n}^{\lambda,\mu,v,\eta}=\frac{(\mu+\eta)_{p+n}(v+\eta)_{p+n}}{(\mu+v+\eta-\lambda)_{p+n}(\eta)_{p+n}}. \quad (1.17)$$

It may be worth noting that, by choosing $\mu=\lambda$ and $\eta=p=1$, the operator $\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta}f(z)$ reduces to the well-known Ruscheweyh derivative $D^{\lambda}f(z)$ for meromorphic univalent functions [7].

Following our new subclass of meromorphic multivalent functions.

Definition (1.2): Let $\Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$ be symbolize the new class of functions $f\in\Sigma_p$ which satisfy the condition:

$$\left|\frac{z(\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta}f(z))''+(p+1)\left(\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta}f(z)\right)'}{\gamma\,z(\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta}f(z))''-\alpha\left(\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta}f(z)\right)'}\right|<\beta, \quad (1.18)$$

where $0\le\gamma\le 1,\ 0<\alpha\le 1,\ 0<\beta\le 1,\ -\infty<\lambda<1,$ $\mu+v+\eta>\lambda,\mu>-\eta,v>-\eta\ and\ \eta>0,\ p\in N\ and\ \ z\in U^*$.

Such type of study was carried out by several different authors such as, Aouf et al. [6], Atshan et al. [11], Panigrahi and Jena [10] and Juma and Dhayea [1] but for another class.

2. Coefficient Inequality

In the following theorem, we gain the necessary and sufficient condition for the function f to be in the class $\Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$.

Theorem (2.1): Let $f\in\Sigma_p$. Then f is in the class $\Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$ if and only if

$$\sum_{n=p}^{\infty}n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}a_n$$

$\leq \beta p(\gamma(p+1)+\alpha),$ (2.1)

where $\Gamma_{p,n}^{\lambda,\mu,v,\eta}$ is presented in (1.17), $0 \leq \gamma \leq 1, 0 < \alpha \leq 1, 0 < \beta \leq 1$ and $z \in U^*$.

The result is sharp for the function

$$f(z) = \frac{1}{z^p} - \frac{\beta p(\gamma(p+1)+\alpha)}{n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}} z^n, n \geq p. \quad \text{... (2.2)}$$

proof: Assume that the inequality (2.1) holds and $|z| = 1$. Then, we have

$$\left| z\left(\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta} f(z)\right)'' + (p+1)\left(\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta} f(z)\right)' \right|$$

$$-\beta \left| \gamma z\left(\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta} f(z)\right)'' - \alpha\left(\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta} f(z)\right)' \right|$$

$$= \left| p\ (p+1)z^{-p-1} - \sum_{n=p}^{\infty} n(n-1)\Gamma_{p,n}^{\lambda,\mu,v,\eta}\ a_n z^{n-1} \right.$$

$$\left. -p(p+1)\ z^{-p-1} - \sum_{n=p}^{\infty} n\ (p+1)\Gamma_{p,n}^{\lambda,\mu,v,\eta} a_n z^{n-1} \right|$$

$$-\beta \left| \gamma p(p+1)\ z^{-p-1} - \sum_{n=p}^{\infty} \gamma n(n-1)\ \Gamma_{p,n}^{\lambda,\mu,v,\eta} a_n z^{n-1} \right.$$

$$\left. +\alpha p z^{-p-1} + \sum_{n=p}^{\infty} \alpha n \Gamma_{p,n}^{\lambda,\mu,v,\eta} a_n z^{n-1} \right|$$

$$= \left| \sum_{n=p}^{\infty} n(n+p)\Gamma_{p,n}^{\lambda,\mu,v,\eta} a_n z^{n-1} \right|$$

$$-\beta \left| p(\gamma(p+1)+\alpha)z^{-p-1} - \sum_{n=p}^{\infty} n(\gamma(n-1)-\alpha)\Gamma_{p,n}^{\lambda,\mu,v,\eta} a_n z^{n-1} \right|$$

$$\leq \sum_{n=p}^{\infty} n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta} a_n$$

$$-\beta p(\gamma(p+1)+\alpha) \leq 0,$$

by hypothesis.

Hence, by maximum modulus principle, $f \in \Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$.

Conversely, suppose that $f \in \Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$. Then from (1.18) we get

$$\left| \frac{z\left(\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta} f(z)\right)'' + (p+1)\left(\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta} f(z)\right)'}{\gamma z\left(\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta} f(z)\right)'' - \alpha\left(\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta} f(z)\right)'} \right|$$

$$= \left| \frac{\sum_{n=p}^{\infty} n(n+p)\Gamma_{p,n}^{\lambda,\mu,v,\eta} a_n z^{n-1}}{p(\gamma(p+1)+\alpha)z^{-p-1} - \sum_{n=p}^{\infty} n(\gamma(n-1)-\alpha)\Gamma_{p,n}^{\lambda,\mu,v,\eta} a_n z^{n-1}} \right| < \beta.$$

Since $Re(z) \leq |z|$ $for\ all\ z$ $(z \in U^*)$, we get

$$Re\left\{ \frac{\sum_{n=p}^{\infty} n(n+p)\Gamma_{p,n}^{\lambda,\mu,v,\eta} a_n z^{n-1}}{p(\gamma(p+1)+\alpha)z^{-p-1} - \sum_{n=p}^{\infty} n(\gamma(n-1)-\alpha)\Gamma_{p,n}^{\lambda,\mu,v,\eta} a_n z^{n-1}} \right\} < \beta.$$

We choose the value of z on the real axis so

$\dfrac{z\left(\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta} f(z)\right)''}{\left(\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta} f(z)\right)'}$ is real.

Letting $z \to 1^-$, through real values, we gain the inequality (2.1).

Finally, sharpness follows if we take

$$f(z) = \frac{1}{z^p} - \frac{\beta p(\gamma(p+1)+\alpha)}{n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}} z^n, \quad n \geq p.$$

Corollary: Let $f(z) \in \Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$. Then

$$a_n \leq \frac{\beta p(\gamma(p+1)+\alpha)}{n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}, n \geq p. \quad (2.3)$$

3. Growth and Distortion Bounds

Next, we gain the growth and distortion bounds for the linear operator $\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta}$.

Theorem (3.1): If $f \in \Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$, then

$$\frac{1}{r^p} - \frac{\beta(\gamma(p+\ 1)+\alpha)}{[2p+\beta(\gamma(p-\ 1)-\alpha)]} r^p \leq \left|\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta} f(z)\right|$$

$$\leq \frac{1}{r^p} + \frac{\beta(\gamma(p+1)+\alpha)}{[2p+\beta(\gamma(p-1)-\alpha)]} r^p,$$

$(0 < |z| = r < 1).$ (3.1)

The outcome is sharp for the function

$$f(z) = \frac{1}{z^p} - \frac{\beta(\gamma(p+1)+\alpha)}{[2p+\beta(\gamma(p-1)-\alpha)]\Gamma_{p,p}^{\lambda,\mu,v,\eta}} z^p. \quad (3.2)$$

Proof: Let $f \in \Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$. Then by Theorem (2.1), we get

$$p[2p+\beta(\gamma(p-1)-\alpha)]\Gamma_{p,p}^{\lambda,\mu,v,\eta}\sum_{n=p}^{\infty}a_n$$

$$\leq \sum_{n=p}^{\infty} n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}a_n$$

$$\leq \beta p(\gamma(p+1)+\alpha)$$

or

$$\sum_{n=p}^{\infty}a_n \leq \frac{\beta(\gamma(p+1)\ +\alpha)}{[2p+\beta(\gamma(p-1)-\alpha)]\Gamma_{p,p}^{\lambda,\mu,v,\eta}} \qquad (3.3)$$

Hence,

$$\left|\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta}f(z)\right| \leq \frac{1}{|z|^p}+\sum_{n=p}^{\infty}\Gamma_{p,n}^{\lambda,\mu,v,\eta}a_n\,|z|^n$$

$$\leq \frac{1}{|z|^p}+\Gamma_{p,p}^{\lambda,\mu,v,\eta}|z|^p\sum_{n=p}^{\infty}a_n$$

$$=\frac{1}{r^p}+\Gamma_{p,p}^{\lambda,\mu,v,\eta}r^p\sum_{n=p}^{\infty}a_n$$

$$\leq \frac{1}{r^p}+\frac{\beta(\gamma(p+1)\ +\alpha)}{[2p+\beta(\gamma(p-1)\ -\alpha)]}r^p. \qquad (3.4)$$

Similarly,

$$\left|\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta}f(z)\right| \geq \frac{1}{|z|^p}-\sum_{n=p}^{\infty}\Gamma_{p,n}^{\lambda,\mu,v,\eta}\,a_n|z|^n$$

$$\geq \frac{1}{|z|^p}-\Gamma_{p,p}^{\lambda,\mu,v,\eta}|z|^p\sum_{n=p}^{\infty}a_n$$

$$=\frac{1}{r^p}-\Gamma_{p,p}^{\lambda,\mu,v,\eta}r^p\sum_{n=p}^{\infty}a_n$$

$$\geq \frac{1}{r^p}-\frac{\beta(\gamma(p+1)\ +\alpha)}{[2p+\beta(\gamma(p-1)\ -\alpha)]}r^p. \qquad (3.5)$$

From (3.4) and (3.5), we obtain (3.1) and the proof is complete.

Theorem (3.2): If $f \in \Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$, then

$$\frac{p}{r^{p+1}}-\frac{\beta p(\gamma(p+1)+\alpha)}{[2p+\beta(\gamma(p-1)-\alpha)]}r^{p-1} \leq \left|(\Gamma_{p,n}^{\lambda,\mu,v,\eta}f(z))'\right|$$

$$\leq \frac{p}{r^{p+1}}+\frac{\beta p(\gamma(p+1)\ +\alpha)}{[2p+\beta(\gamma(p-1)\ -\alpha)]}r^{p-1},$$

$$(0<|z|=r<1). \qquad (3.6)$$

The outcome is sharp for the function f is presented in (3.2).

Proof: The proof is similar to that of Theorem (3.1).

4. Convolution Properties

Theorem (4.1): Let the function f_j $(j=1,2)$ defind by

$$f_j(z)=\frac{1}{z^p}-\sum_{n=p}^{\infty}a_{n,j}\,z^n, \qquad (a_{n,j}\geq 0, j=1,2), \qquad (4.1)$$

which in the class $\Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$.

Then $f_1 * f_2 \in \Sigma_p(\gamma,\alpha,\delta,\lambda,\mu,v,\eta)$, where

$$\delta \leq \frac{n[n+p+\beta(\gamma(n-1)-\alpha)]^2\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\beta^2 p(n+p)[\gamma(p+1)+\alpha]}$$

$$-\frac{\beta^2 p[\gamma(p+1)+\alpha][\gamma(n-1)-\alpha]}{\beta^2 p(n+p)[\gamma(p+1)+\alpha]}$$

Proof: We must find the largest δ such that

$$\sum_{n=p}^{\infty}\frac{n[n+p+\delta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\delta p(\gamma(p+1)+\alpha)}a_{n,1}a_{n,2}\leq 1\,.$$

Since $f_j \in \Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$, $(j=1,2)$, then

$$\sum_{n=p}^{\infty}\frac{n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\beta p(\gamma(p+1)+\alpha)}a_{n,j}\leq 1,$$

$$(j=1,2). \qquad (4.2)$$

By Cauchy- Schwarz inequality, we have

$$\sum_{n=p}^{\infty}\frac{n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\beta p(\gamma(p+1)+\alpha)}\sqrt{a_{n,1}a_{n,2}}\leq 1. \qquad (4.3)$$

So, we only show that

$$\frac{n[n+p+\delta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\delta p(\gamma(p+1)+\alpha)}a_{n,1}a_{n,2}$$

$$\leq \frac{n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\beta p(\gamma(p+1)+\alpha)}\sqrt{a_{n,1}a_{n,2}}\,.$$

This equivalently to

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{\delta[n+p+\beta(\gamma(n-1)-\alpha)]}{\beta[n+p+\delta(\gamma(n-1)-\alpha)]}.$$

From (4.3), we have

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{\beta p(\gamma(p+1)+\alpha)]}{n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}.$$

Which is sufficient to prove that

$$\frac{\beta p(\gamma(p+1)+\alpha)]}{n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}$$

$$\leq \frac{\delta[n+p+\beta(\gamma(n-1)-\alpha)]}{\beta[n+p+\lambda(\gamma(n-1)-\alpha)]},$$

that implies to

$$\delta \leq \frac{n[n+p+\beta(\gamma(n-1)-\alpha)]^2\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\beta^2 p(n+p)[\gamma\ (p+1)+\alpha]}$$

$$-\frac{\beta^2 p[\gamma(p+1)\ +\alpha)][\gamma(n-1)-\alpha)]}{\beta^2 p(n+p)[\gamma(p+1)+\alpha]}$$

Theorem (4.2): Let the function f_j $(j=1,2)$ that defined in (4.1) be in the class $\Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$. Then the function k defined by

$$k(z)=\frac{1}{z^p}-\sum_{n=p}^{\infty}\left(a_{n,1}^2+a_{n,2}^2\right)z^n, \qquad (4.4)$$

belong to the class $\Sigma_p(\gamma,\alpha,\epsilon,\lambda,\mu,v,\eta)$, where

$$\epsilon \leq \frac{n[n+p+\beta(\gamma(n-1)-\alpha)]^2\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{2p\beta^2(n+p)(\gamma(p+1)+\alpha)}$$

$$-\frac{2p\beta^2(\gamma(n-1)-\alpha)(\gamma(p+1)+\alpha)}{2p\beta^2(n+p)(\gamma(p+1)+\alpha)}$$

Proof: We must find the largest ϵ such that

$$\sum_{n=p}^{\infty}\frac{n[n+p+\epsilon(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\epsilon p(\gamma(p+1)+\alpha}\left(a_{n,1}^2+a_{n,2}^2\right)\leq 1.$$

Since $f_j \in \Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)(j=1,2)$, we get

$$\sum_{n=p}^{\infty}\left(\frac{n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\beta p(\gamma(p+1)+\alpha)}\right)^2 a_{n,1}^2$$

$$\leq \left(\sum_{n=p}^{\infty}\frac{n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\beta p(\gamma(p+1)+\alpha)}a_{n,1}\right)^2\leq 1 \qquad (4.5)$$

and

$$\sum_{n=p}^{\infty}\left(\frac{n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\beta p(\gamma(p+1)+\alpha)}\right)^2 a_{n,2}^2$$

$$\leq \left(\sum_{n=p}^{\infty}\frac{n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\beta p(\gamma(p+1)+\alpha)}a_{n,2}\right)^2\leq 1. \quad (4.6)$$

Combining the inequalities (4.5) and (4.6), gives

$$\sum_{n=p}^{\infty}\frac{1}{2}\left(\frac{n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\beta p(\gamma(p+1)+\alpha)}\right)^2\left(a_{n,1}^2+a_{n,2}^2\right)$$

$$\leq 1. \qquad (4.7)$$

But, $k \in \Sigma_p(\gamma,\alpha,\epsilon,\lambda,\mu,v,\eta)$ if and only if

$$\sum_{n=p}^{\infty}\frac{n[n+p+\epsilon(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\epsilon p(\gamma(p+1)+\alpha)}\left(a_{n,1}^2+a_{n,2}^2\right)$$

$$\leq 1. \qquad (4.8)$$

The inequality (4.8) will be satisfied if

$$\frac{n[n+p+\epsilon(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\epsilon p(\gamma\ (p+1)+\alpha)}$$

$$\leq \frac{n^2[n+p+\beta(\gamma\ (n-1)-\alpha]^2[\Gamma_{p,n}^{\lambda,\mu,v,\eta}]^2}{2\beta^2p^2(\gamma(p+1)+\alpha)^2}, n\geq p.$$

So that

$$\epsilon \leq \frac{n[n+p+\beta(\gamma(n-1)-\alpha)]^2\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{2p\beta^2(n+p)(\gamma(p+1)+\alpha)}$$

$$-\frac{2p\beta^2(\gamma(n-1)-\alpha)(\gamma(p+1)+\alpha)}{2p\beta^2(n+p)(\gamma(p+1)+\alpha)}$$

Theorem (4.3): If $f(z)=\frac{1}{z^p}-\sum_{n=p}^{\infty}a_n z^n$ and $g(z)=\frac{1}{z^p}-\sum_{n=p}^{\infty}b_n z^n$ with $|b_n|\leq 1$ are in the class $\Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$, then $f(z)*g(z)\in \Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$.

Proof: From Theorem (2.1) we get

$$\sum_{n=p}^{\infty} n\,[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}a_n$$

$$\leq \beta p(\gamma(p+1)+\alpha).$$

Since

$$\sum_{n=p}^{\infty}\frac{n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\beta p(\gamma\ (p+1)+\alpha)}|a_n b_n|$$

$$=\sum_{n=p}^{\infty}\frac{n[n+p+\beta(\gamma\ (n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\beta p(\gamma(p+1)+\alpha)}a_n|b_n|$$

$$\leq\sum_{n=p}^{\infty}\frac{n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\beta p(\gamma(p+1)+\alpha)}a_n\leq 1.$$

Thus $f(z)*g(z)\in\Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$.

Hence the proof is complete.

Corollary (4.1): If $f(z)=\frac{1}{z^p}-\sum_{n=p}^{\infty}a_n z^n$ and $g(z)=\frac{1}{z^p}-\sum_{n=p}^{\infty}b_n z^n$ with $0\leq b_n\leq 1$ are in the class $\Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$, then $f(z)*g(z)\in\Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$.

5. Integral Representation

Theorem (5.1): Let $f\in\Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$. Then

$$\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta}f(z)=\int_0^z\exp[\int_0^z\frac{\alpha\beta\psi(t)+(p+1)}{t(\gamma\beta\psi(t)-1)}dt]dt,$$

where $|\psi(t)|<1,\ z\in U^*$.

Proof: By letting $\frac{z\left(\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta}f(z)\right)''}{\left(\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta}f(z)\right)'}=Q(z)$ in (1.18), we get

$$\left|\frac{Q(z)+(p+1)}{\gamma\,Q(z)-\alpha}\right|<\beta,$$

or equivalently

$$\frac{Q(z)+(p+1)}{\gamma\,Q(z)-\alpha}=\beta\psi(z),\qquad (|\psi(z)|<1,z\in U^*).$$

So

$$\frac{\left(\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta}f(z)\right)''}{\left(\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta}f(z)\right)'}=\frac{\alpha\beta\psi(z)+(p+1)}{z(\gamma\beta\psi(z)-1)},$$

after integration, we obtain

$$\log\left(\left(\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta}f(z)\right)'\right)=\int_0^z\frac{\alpha\beta\psi(t)+(p+1)}{t(\gamma\beta\psi(t)-1)}dt.$$

Therefore

$$\left(\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta}f(z)\right)'=\exp[\int_0^z\frac{\alpha\beta\psi(t)+(p+1)}{t(\gamma\beta\psi(t)-1)}dt].$$

Again by integration, we have

$$\mathcal{F}_{0,z}^{\lambda,\mu,v,\eta}f(z)=\int_0^z\exp[\int_0^z\frac{\alpha\beta\psi(t)+(p+1)}{t(\gamma\beta\psi(t)-1)}dt]dt$$

and this is the required result.

6. Radii of Starlikeness and Convexity

In the following theorems, we introduce the radii of starlikeness and convexity.

Theorem (6.1): If $f\in\Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$, then f is multivalent meromorphic starlike of order $\varphi(0\leq\varphi<p)$ in the disk $|z|<r_1$, where

$$r_1=\inf_n\left\{\frac{n(p-\varphi)[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\beta p(n-\varphi+2p)(\gamma(p+1)+\alpha)}\right\}^{\frac{1}{n+p}},\quad n\geq p.$$

The outcome is sharp for the function f that presented in (2.2).

Proof: It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}+p\right|\leq p-\varphi\ for\ |z|<r_1. \tag{6.1}$$

But

$$\left|\frac{zf'(z)+pf(z)}{f(z)}\right|\leq\frac{\sum_{n=p}^{\infty}(n+p)a_n|z|^{n+p}}{1-\sum_{n=p}^{\infty}a_n|z|^{n+p}}.$$

Thus, (6.1) will be satisfied if

$$\frac{\sum_{n=p}^{\infty}(n+p)\ a_n|z|^{n+p}}{1-\sum_{n=p}^{\infty}\ a_n|z|^{n+p}}\leq p-\varphi,$$

or if

$$\sum_{n=p}^{\infty}\frac{(n-\varphi+2p)}{p-\varphi}a_n|z|^{n+p}\leq 1. \tag{6.2}$$

Since $f\in\Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$, we have

$$\sum_{n=p}^{\infty}\frac{n[n+p+\beta(\gamma\ (n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\beta p(\gamma\ (p+1)+\alpha)}a_n\leq 1.$$

Hence, (6.2) will be true if

$$\frac{n-\varphi+2p}{p-\varphi}\ |z|^{n+p}\leq\ \frac{n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\beta p(\gamma(p+1)+\alpha)},$$

or equivalently

$$|z|\leq\left\{\frac{n(p-\varphi)[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\beta p(n-\varphi+2p)(\gamma(p+1)+\alpha)}\right\}^{\frac{1}{n+p}},\quad n\geq p$$

which follows the result.

Theorem (6.2): If $f \in \Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$, then f is multivalent meromorphic convex of order $\varphi(0 \le \varphi < p)$ in the disk $|z| < r_2$, where

$$r_2 = \inf_n \left\{ \frac{(p-\varphi)[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\beta(n-\varphi+2p)(\gamma(p+1)+\alpha)} \right\}^{\frac{1}{n+p}}, \quad n \ge p.$$

The outcome is sharp for the function f presented by (2.2).

Proof: It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)} + 1 + p\right| \le p - \varphi \ \ for \ \ |z| < r_2. \quad (6.3)$$

But

$$\left|\frac{zf''(z)}{f'(z)} + 1 + p\right| = \left|\frac{zf''(z) + (p+1)f'(z)}{f'(z)}\right|$$

$$\le \frac{\sum_{n=p}^{\infty} n\,(n+p)a_n|z|^{n+p}}{p - \sum_{n=p}^{\infty} n\,a_n|z|^{n+p}}.$$

Thus, (6.3) will be satisfied if

$$\frac{\sum_{n=p}^{\infty} n\,(n+p)a_n|z|^{n+p}}{p - \sum_{n=p}^{\infty} na_n|z|^{n+p}} \le p - \varphi,$$

or if

$$\sum_{n=p}^{\infty} \frac{n\,(n-\varphi+2p)}{p\,(p-\varphi)} a_n|z|^{n+p} \le 1. \quad (6.4)$$

Since $f \in \Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$, we have

$$\sum_{n=p}^{\infty} \frac{n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\beta p(\gamma(p+1)+\alpha)} a_n \le 1.$$

Hence, (6.4) will be true if

$$\frac{n(n-\varphi+2p)}{p(p-\varphi)}|z|^{n+p} \le \frac{n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\beta p(\gamma(p+1)+\alpha)},$$

or equivalent

$$|z| \le \left\{ \frac{(p-\varphi)[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\beta(n-\varphi+2p)(\gamma(p+1)+\alpha)} \right\}^{\frac{1}{n+p}}, \quad n \ge p$$

which follows the result.

7. Extreme Points

We gain here extreme points of the class $\Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$.

Theorem (7.1): Let $f_{p-1}(z) = z^{-p}$ and

$$f_n(z) = z^{-p} - \frac{\beta p(\gamma(p+1)+\alpha)}{n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}} z^n, \quad (7.1)$$

where all parameters are constrained as in Theorem (2.1).

Then the function f is in the class $\Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$ if and only if

$$f(z) = \theta_{p-1}z^{-p} + \sum_{n=p}^{\infty} \theta_n f_n(z), \quad (7.2)$$

where $(\theta_{p-1} \ge 0,\ \theta_n \ge 0,\ n \ge p)$ and $\theta_{p-1} + \sum_{n=p}^{\infty} \theta_n = 1$.

Proof: Suppose that f is expressed in (7.2). Then

$$f(z) = \theta_{p-1}z^{-p} + \sum_{n=p}^{\infty} \theta_n \left[z^{-p} - \frac{\beta p(\gamma(p+1)+\alpha)}{n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}} z^n \right]$$

$$= z^{-p} - \sum_{n=p}^{\infty} \frac{\beta p(\gamma(p+1)+\alpha)}{n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}} \theta_n z^n.$$

Hence,

$$\sum_{n=p}^{\infty} \frac{n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\beta p(\gamma(p+1)+\alpha)} \times \frac{\beta p(\gamma(p+1)+\alpha)\theta_n}{n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}$$

$$= \sum_{n=p}^{\infty} \theta_n = 1 - \theta_{p-1} \le 1.$$

Then $f \in \Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$.

Conversely, suppose that $f \in \Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$. We may set

$$\theta_n = \frac{n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}}{\beta p(\gamma(p+1)+\alpha)} a_n,$$

where a_n is presented in (2.3). Then

$$f(z) = z^{-p} - \sum_{n=p}^{\infty} a_n z^n$$

$$= z^{-p} - \sum_{n=p}^{\infty} \frac{\beta p(\gamma(p+1)+\alpha)}{n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}} \theta_n z^n$$

$$= z^{-p} - \sum_{n=p}^{\infty} [z^{-p} - f_n(z)]\, \theta_n$$

$$= (1 - \sum_{n=p}^{\infty} \theta_n)\, z^{-p} + \sum_{n=p}^{\infty} \theta_n f_n(z)$$

$$= \theta_{p-1} z^{-p} + \sum_{n=p}^{\infty} \theta_n f_n(z).$$

This completes the proof .

8. Weighted Mean and Arithmetic Mean

Definition (8.1): Let f and g belong to Σ_p. Then the weighted mean $E_q(z)$ of $f(z)$ and $g(z)$ is presented by

$$E_q(z) = \frac{1}{2}[(1-q)f(z) + (1+q)g(z)], (0 < q < 1).$$

The following theorem shows the weighted mean for this class.

Theorem (8.1): Let f and g be in the class $\Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$. Then the weighted mean of f and g is also in the class $\Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$.

Proof: By Definition (8.1), we have

$$E_q(z) = \frac{1}{2}\ [(1-q)f(z)\ + (1+q)g(z)]$$

$$= \frac{1}{2}\left[(1-q)\ (z^{-p} - \sum_{n=p}^{\infty} a_n z^n\ ) + (1+q)(z^{-p} - \sum_{n=p}^{\infty} b_n z^n\,)\right]$$

$$= z^{-p} - \sum_{n=p}^{\infty} \frac{1}{2}[(1-q)a_n + (1+q)b_n] z^n\,.$$

Since f and g are in the class $\Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$ so by Theorem (2.1), we get

$$\sum_{n=p}^{\infty} n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta} a_n$$

$$\leq \beta p(\gamma(p+1)+\alpha)$$

and

$$\sum_{n=p}^{\infty} n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta} b_n$$

$$\leq \beta p(\gamma(p\ +1)+\alpha).$$

Hence,

$$\sum_{n=p}^{\infty} n[n+p+\beta(\gamma\ (n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta} \left(\frac{1}{2}(1-q)a_n + \frac{1}{2}(1+q)b_n\right)$$

$$= \frac{1}{2}(1-q) \sum_{n=p}^{\infty} n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta} a_n$$

$$+ \frac{1}{2}(1+q)\ \sum_{n=p}^{\infty} n[n+p+\beta(\gamma(n-1)\ -\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta} b_n$$

$$\leq \frac{1}{2}(1-q)\beta p(\gamma(p\ +1)+\alpha) + \frac{1}{2}(1+q)\beta p(\gamma(p+\ 1)+\alpha)$$

$$= \beta p(\gamma(p+1)+\alpha).$$

This shows that $E_q \in \Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$.

In the following theorem, we shall demonstrate that the class $\Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$ is closed under arithmetic mean.

Theorem (8.2): Let $f_1(z), f_2\,(z), \dots, f_s(z)$ that defined by

$$f_k(z) = z^{-p} - \sum_{n=p}^{\infty} a_{n,k} z^n,$$

$$\left(a_{n,k} \geq 0, k = 1,2,\dots,s, n \geq p\right) \tag{8.1}$$

are in the class $\Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$. Then the arithmetic mean of $f_k(z)$ $(k\ = 1,2,\dots,s)$ that defined by

$$h(z) = \frac{1}{s}\sum_{k=1}^{s} f_k(z), \tag{8.2}$$

is also in the class $\Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$.

Proof: By (8.1) and (8.2), we can write

$$h(z) = \frac{1}{s}\sum_{k=1}^{s}(\,z^{-p} - \sum_{n=p}^{\infty} a_{n,k} z^n)$$

$$= z^{-p} - \sum_{n=p}^{\infty}(\frac{1}{s}\sum_{k=1}^{s} a_{n,k})\, z^n.$$

Since $f_k(z) \in \Sigma_p(\gamma,\alpha,\beta,\lambda,\mu,v,\eta)$ for every $k = 1,2,\dots,s,$ so by Theorem (2.1), we have

$$\sum_{n=p}^{\infty} n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}\,(\frac{1}{s}\sum_{k=1}^{s} a_{n,k})$$

$$=\frac{1}{s}\sum_{k=1}^{s}(\sum_{n=p}^{\infty} n[n+p+\beta(\gamma(n-1)-\alpha)]\Gamma_{p,n}^{\lambda,\mu,v,\eta}\,a_{n,k})$$

$$\leq \frac{1}{s}\sum_{k=1}^{s} \beta p(\gamma(p+1)+\alpha)=\beta p(\gamma(p+1)+\alpha).$$

This ends the proof .

References

[1] A. R. S. Juma and H. I. Dhayea, On a subclass of meromoephic function with fixed second coefficient involving Fox-Wright′s generalized Hypergeometic function, General Mathematical Notes, 28(2), 2015, pp: 30-41.

[2] H. M. Srivastava, M. Saigo and S. Owa, "A class of distortion theorems involving certain operators of fractional calculus", Journal of Mathematical Analysis and Applications, 131, 1988, pp: 412-420.

[3] H. M. Srivastava and S. Owa, Univalent functions, fractional calculus and their applications, Halsted Press (Ellis Horwood Limited, Chichester) (John Wiley and Sons, New York, Chichester, Brisbane and Toronto), 1989.

[4] H. M. Srivastava and S. Owa, Current topics in analytic function theory, World Scientific Publising Company, Singapore, New Jersey, London and Hong Kong, 1992.

[5] M. Bhagtani and P. Vijaywargiya, "On a subclass of meromorphic multivalent functions defined by fractional calculus operators", Tamsui Oxford Journal of Mathematical Sciences, 25(1), 2009, pp:15-25.

[6] M. K. Aouf, T. M. Seoudy and G. M. El-Hawsh, Subclass of meromorphic univalent function defined by Ruscheweyh derivative with fixed second coefficients, Southeast Asian Bulletin of Mathematics, 41, 2017, pp: 473-480.

[7] S. Ruscheweyh, New criteria for univalent functions, Proceedings of the American Mathematical Society, 49(1), 1975, pp:109-115.

[8] S. Owa, M. Saigo and H. M. Srivastava, Some characterization theorems for starlike and convex functions involving a certain fractional integral operator, Journal of Mathematical Analysis and Applications, 140, 1989, pp: 419-426.

[9] S. Owa, On the distortion theorems I, Kyungpook Mathematical Journal, 18, 1978, pp: 53-59.

[10] T. Panigrahi and L. Jena, A new class of meromorphic multivalent functions defined by linear operator, Journal of Analysis and Number Theory, 4(2), 2016, pp: 91- 99.

[11] W. G. Atshan, L. A. Alzopee and M. M. Alcheikh, "On fractional calculus operators of a class of meromorphic multivalent functions", General Mathematical Notes, 18(2), 2013, pp: 92-103.