On optimal periodic solution of differential equations

Methaq Hamza Geem
Department of mathematic
College of Education/University of Qadisiyah
Al-Qadisiyah, Iraq
Methaq.Geem@qu.edu.iq

Received September 18, 2017. Accepted for publication January 16, 2018

DOI: http://dx.doi.org/10.31642/JoKMC/2018/050103

Abstract*-this paper dedicated to the construction of solution of a three time scale periodic singular perturbed non-linear quadratic optimal control problem by using the direct method. The algorithm of the method is the direct substitution of the postulated asymptotic expansion of the solution of the problem and then by the conditions of the problem we constructed of a series of problem and find terms of the asymptotic. We find the solution by using the Hamilton's function and maximum principle.*

Keywords- Hamilton's function, periodic differential equations, optimal solution , direct method, differential equations.

I. Introduction:

Period optimal problem have a large applications and theoretical interest. One of common methods is determining periodic which associated with use a small parameter. Therefore, it is interest to study periodic problems of optimal control by asymptotic methods. The direct method was applied by V.A. Plotnikov and his student [4]. In this paper we study a problem periodic singularly perturbed of three scales-time and use maximum principle for finding the solution of this problem.

A singular perturbation problem is one for which the perturbed problem is qualitatively different from the unperturbed problem. One typically obtains an asymptotic, but possibly divergent, expansion of the solution, which depends singularly on the parameter ε. A singular perturbation problem can be described by system of two differential equations such that one equation has small parameter in high derivative and other equation has not small parameter.

A singular perturbation problem of three scales contains three equations such has one of them has small parameter in high its derivative and other has quadratic the same small parameter in its derivative and the third equation has not small parameter.

Each of these equations work in interval such as [0,T] and has condition, if the conditions periodic (x(0)=x(T), y(0)=y(T), z(0)=z(T)) we say the problem is periodic problem.

II.View a problem:

Consider the following problem:

P_ϵ:

$$J_\epsilon(u) = \int_0^T F(w,u,t)dt \to min$$

Such that

$$\dot{x} = f(x,y,z,u,t)\,, \qquad x(0,\epsilon) = x(T,\epsilon)\,, x \in R^n \;\ldots.\ldots.(1)$$

$$\epsilon\dot{y} = g(x,y,z,u,t)\,, \qquad y(0,\epsilon) = y(T,\epsilon)\,, y \in R^m \;\ldots\ldots\;(2)$$

$$\epsilon^2\dot{z} = h(x,y,z,u,t)\,, \qquad z(0,\epsilon) = z(T,\epsilon)\,, z \in R^k \;\ldots\ldots\ldots(3)$$

$w = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, u \in M$ -set of all T-periodic vector functions. T-is fixed positive number.

III. Construct asymptotic expansion of problem P_ϵ

We can be problem of smaller dimension which we call the degenerate problem P_0. We mean by the dimension of optimal control problem is dimension of the phase space. We write formula of optimal control by using principle of maximum:

$$\left.\begin{aligned}\dot{\emptyset}_1 &= -H_x = -f_x'\emptyset_1 - \frac{g_x'\widetilde{\emptyset}_2}{\epsilon} - \frac{h_x'\widetilde{\emptyset}_3}{\epsilon^2} + F_x \\ &\emptyset_1(T,\epsilon) = \emptyset_1(0,\epsilon) \\ \dot{\emptyset}_2 &= -H_y = -f_y'\emptyset_1 - \frac{g_y'\widetilde{\emptyset}_2}{\epsilon} - \frac{h_y'\widetilde{\emptyset}_3}{\epsilon^2} + F_y \\ &\emptyset_2(T,\epsilon) = \emptyset_2(0,\epsilon) \\ \dot{\emptyset}_3 &= -H_z = -f_z'\emptyset_1 - \frac{g_z'\widetilde{\emptyset}_2}{\epsilon} - \frac{h_z'\widetilde{\emptyset}_3}{\epsilon^2} + F_z \\ &\emptyset_3(T,\epsilon) = \emptyset_3(0,\epsilon)\end{aligned}\right\} \dots\dots\dots\dots\dots (4)$$

Such that

$$H_u = 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (5)$$

$$H = {\emptyset_1}' + \frac{\widetilde{\emptyset}_2' g}{\epsilon} + \frac{\widetilde{\emptyset}_3' h}{\epsilon^2} - F$$

$\emptyset_1, \widetilde{\emptyset}_2, \widetilde{\emptyset}_3$ –are conjugate.

Variables, which mean transposition .The condition of periodic for $\emptyset_1, \widetilde{\emptyset}_2, \widetilde{\emptyset}_3$

derived from periodic condition of transposition.

From (5) we have :

$$u = \psi(x, y, z, \emptyset_1, \frac{\widetilde{\emptyset}_2}{\epsilon}, \frac{\widetilde{\emptyset}_3}{\epsilon^2}, t)$$

By substitution u in 2,3,,4 and put $\widetilde{\emptyset}_2 = \epsilon\phi_2, \widetilde{\emptyset}_3 = \epsilon^2\phi_3$ we obtain the formula :

$$\left.\begin{aligned}\dot{X} &= hh(X,Y,Z,t) = I_n H_x, X = \begin{pmatrix} x \\ \phi_1 \end{pmatrix}, hh = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \\ \dot{Y} &= q(X,Y,Z,t) = I_m H_y, Y = \begin{pmatrix} y \\ \phi_2 \end{pmatrix} \quad, q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \\ \dot{Z} &= p(X,Y,Z,t) = I_k H_z, Z = \begin{pmatrix} z \\ \phi_3 \end{pmatrix} \quad, p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}\end{aligned}\right\} \dots\dots\dots (6)$$

$$\left.\begin{aligned}x(0,\epsilon) = x(T,\epsilon), \phi_1(0,\epsilon) = \phi_1(T,\epsilon) \\ y(0,\epsilon) = y(T,\epsilon), \phi_2(0,\epsilon) = \phi_2(T,\epsilon) \\ z(0,\epsilon) = z(T,\epsilon), \phi_3(0,\epsilon) = \phi_3(T,\epsilon)\end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots\dots (7)$$

Such that $I_n = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$.

Asymptotic expansion of periodic problem (6),(7) will find by using direct method . i.e. series in powers of ϵ with coefficients depend on t .

$$\text{Put } w = \bar{w}_0 + \epsilon\bar{w}_0 + \cdots \quad , \quad w = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \dots\dots\dots\dots\dots (8)$$

Here we confine our construction on the first two terms in (8) and then substitution (8) in (6),(7) .

We have :

$$\left.\begin{aligned}\dot{\bar{x}}_0 = f(\bar{x}_0, \bar{y}_0, \bar{z}_0, \psi(\bar{x}_0, \bar{y}_0, \bar{z}_0, \bar{\phi}_{10}, \bar{\phi}_{20}, \bar{\phi}_{30}, t), t) = \bar{f}_0 \\ \dot{\bar{\phi}}_{10} = -\bar{H}_x \quad ,\end{aligned}\right\} \dots\dots\dots (9)$$

$$\left.\begin{aligned}0 = g(\bar{x}_0, \bar{y}_0, \bar{z}_0, \psi(\bar{x}_0, \bar{y}_0, \bar{z}_0, \bar{\phi}_{10}, \bar{\phi}_{20}, \bar{\phi}_{30}, t), t) = \bar{g}_0 \\ 0 = -\bar{H}_y \quad ,\end{aligned}\right\} \dots\dots (10)$$

$$\left.\begin{aligned}0 = h(\bar{x}_0, \bar{y}_0, \bar{z}_0, \psi(\bar{x}_0, \bar{y}_0, \bar{z}_0, \bar{\phi}_{10}, \bar{\phi}_{20}, \bar{\phi}_{30}, t), t) = \bar{h}_0 \\ 0 = -\bar{H}_z \quad ,\end{aligned}\right\} \dots\dots (11)$$

$$\bar{x}_0(0) = \bar{x}_0(T, x_0^0, \phi_{10}^0), \bar{\phi}_{10}(0) = \bar{\phi}_{10}(T, x_0^0, \phi_{10}^0) \dots\dots (12)$$

Solution of 10,11,and 12 defines as a functions of x_0^0, ϕ_{10}^0.

$$\phi_1^0 = \phi_1(0,\epsilon) = \phi_{10}^0 + \epsilon\phi_{11}^0 + \epsilon^2\phi_{12}^0 + \cdots,$$

$$x^0 = x(0,\epsilon) = x_0^0 + \epsilon x_1^0 + \epsilon^2 x_2^0 + \cdots.$$

Form (7) we have :

$$Q_1 = \bar{x}_0(0) - \bar{x}_0(T, x_0^0, \phi_{10}^0) = 0,$$

$$Q_2 = \bar{\phi}_{10}(0) - \bar{\phi}_{10}(T, x_0^0, \phi_{10}^0) = 0, \dots (13)$$

We note that the equations :

$$\begin{aligned}q(X,Y,Z,t) = 0 \\ p(X,Y,Z,t) = 0\end{aligned} \text{ must have a solutions}$$

$$\begin{aligned}Y = L_1(X,t) \\ Z = L_2(X,t)\end{aligned}$$

The equation:

$$\frac{d}{dt}\Delta=\frac{d}{dx}hh(X,L_1(X,t),L_2(X,t),t)\Big|_{\substack{x=\bar{x}_0\\ y=\bar{y}_0\\ z=\bar{z}_0\\ \phi_1=\bar{\phi}_{10}\\ \phi_2=\bar{\phi}_{20}\\ \phi_3=\bar{\phi}_{30}}}$$

has not trivial solution with period T. Thus we have the determine :

$$\begin{vmatrix}\frac{\partial Q_1}{\partial Q_{10}^0} & \frac{\partial Q_1}{\partial x_0^0}\\ \frac{\partial Q_2}{\partial Q_{10}^0} & \frac{\partial Q_2}{\partial x_0^0}\end{vmatrix}\neq 0$$

And therefore eq(13) have a unique solution (x_0^0,ϕ_{10}^0)

Now we consider the problem:

$$\hat{J}(\hat{y},\hat{z},\hat{u})=\int_0^T F(\hat{y},\hat{z},\hat{u},t)dt\to$$
$$min\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

$$\hat{x}=f(\hat{x},\hat{y},\hat{z},\hat{u},t),\hat{x}(0)=\hat{x}(T)$$

$$\left.\begin{array}{l}\hat{x}=f(\hat{x},\hat{y},\hat{z},\hat{u},t),\hat{x}(0)=\hat{x}(T)\\ 0=g(\hat{x},\hat{y},\hat{z},\hat{u},t)\\ 0=h(\hat{x},\hat{y},\hat{z},\hat{u},t)\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

Such that controls are $\hat{y},\hat{z},\hat{u}$.

By maximum principle we have:

$$\frac{\partial\hat{H}}{\partial\hat{y}}=0,\frac{\partial\hat{H}}{\partial\hat{z}}=0,\frac{\partial\hat{H}}{\partial\hat{u}}=0\ldots\ldots\ldots\ldots\ldots\ldots.(16)$$

Such that :

$$\left.\begin{array}{l}\hat{H}=\hat{\phi}_1'f(\hat{x},\hat{y},\hat{z},\hat{u},t)+\hat{\phi}_2'g(\hat{x},\hat{y},\hat{z},\hat{u},t)+\\ +\hat{\phi}_3'h(\hat{x},\hat{y},\hat{z},\hat{u},t)-F(\hat{x},\hat{y},\hat{z},\hat{u},t)\\ \dot{\hat{\phi}}_1'=-\frac{\partial\hat{H}}{\partial\hat{x}},\hat{\phi}_1(T)=\hat{\phi}_1(0)\end{array}\right\}\ldots\ldots\ldots(17)$$

Must put the condition:

The system of equations 15-17 has the unique solution $(\hat{y},\hat{z},\hat{u})$.

Since $\frac{\partial\hat{H}}{\partial\hat{u}}=0$ then $\hat{u}=\hat{\alpha}(\hat{x},\hat{y},\hat{z},\hat{\phi}_1,\hat{\phi}_2,\hat{\phi}_3,t)$, by substitution $\hat{u}$ in equations 15-17 we obtain type of equations as the equations 9-10.

Thus by uniqueness of solutions of equations 14,15 we have :

$$\hat{x}=\bar{x}_0,\quad \hat{y}=\bar{y}_0,\quad \hat{z}=\bar{z}_0$$
$$\hat{u}=\bar{u}_0$$
$$\hat{\phi}_1=\bar{\phi}_{10},\hat{\phi}_2=\bar{\phi}_{20},\hat{\phi}_3=\bar{\phi}_{30}$$

IV. Construction of first expansion:

$$\left.\begin{array}{l}\dot{x}_1=\bar{f}_x(t)\bar{x}_1+\bar{f}_y(t)\bar{y}_1+\bar{f}_z(t)\bar{z}_1+\bar{f}_u(t)\bar{u}_1\\ \dot{\phi}_{11}=-\bar{H}_{xx}(t)\bar{x}_1-\bar{H}_{x\phi_1}(t)\bar{\phi}_{11}-\bar{H}_{xy}(t)\bar{y}_1-\bar{H}_{x\phi_2}(t)\bar{\phi}_{21}-\\ \bar{H}_{xz}(t)\bar{z}_1-\bar{H}_{x\phi_3}(t)\bar{\phi}_{31}-\bar{H}_{xu}(t)\bar{u}_1\end{array}\right\}.(18)$$

$$\left.\begin{array}{l}\dot{y}_0=\bar{g}_x(t)\bar{x}_1+\bar{g}_y(t)\bar{y}_1+\bar{g}_z(t)\bar{z}_1+\bar{g}_u(t)\bar{u}_1\\ \dot{\phi}_{21}=-\bar{H}_{yx}(t)\bar{x}_1-\bar{H}_{y\phi_1}(t)\bar{\phi}_{11}-\bar{H}_{yy}(t)\bar{y}_1-\bar{H}_{y\phi_2}(t)\bar{\phi}_{21}-\\ \bar{H}_{yz}(t)\bar{z}_1-\bar{H}_{y\phi_3}(t)\bar{\phi}_{31}-\bar{H}_{yu}(t)\bar{u}_1\end{array}\right\}.(19)$$

$$\left.\begin{array}{l}\dot{z}_0=\bar{h}_x(t)\bar{x}_1+\bar{h}_y(t)\bar{y}_1+\bar{h}_z(t)\bar{z}_1+\bar{h}_u(t)\bar{u}_1\\ \dot{\phi}_{31}=-\bar{H}_{zx}(t)\bar{x}_1-\bar{H}_{z\phi_1}(t)\bar{\phi}_{11}-\bar{H}_{zy}(t)\bar{y}_1-\bar{H}_{z\phi_2}(t)\bar{\phi}_{21}-\\ \bar{H}_{zz}(t)\bar{z}_1-\bar{H}_{z\phi_3}(t)\bar{\phi}_{31}-\bar{H}_{zu}(t)\bar{u}_1\end{array}\right\}.(20)$$

$$\left.\begin{array}{l}0=-\bar{H}_{ux}(t)\bar{x}_1-\bar{H}_{u\phi_1}(t)\bar{\phi}_{11}-\bar{H}_{uy}(t)\bar{y}_1-\bar{H}_{u\phi_2}(t)\bar{\phi}_{21}-\\ \bar{H}_{uz}(t)\bar{z}_1-\bar{H}_{u\phi_3}(t)\bar{\phi}_{31}-\bar{H}_{uu}(t)\bar{u}_1\end{array}\right\}\ldots(21)$$

Such that:

$$\bar{u}_1=\frac{\partial\bar{\alpha}}{\partial x}\bar{x}_1+\frac{\partial\bar{\alpha}}{\partial\phi_1}\bar{\phi}_{11}+\frac{\partial\bar{\alpha}}{\partial y}\bar{y}_1+\frac{\partial\bar{\alpha}}{\partial\phi_2}\bar{\phi}_{21}+\frac{\partial\bar{\alpha}}{\partial z}\bar{z}_1+\frac{\partial\bar{\alpha}}{\partial\phi_3}\bar{\phi}_{31}$$

From (7) we have :

$$\bar{x}_1(0)=\bar{x}_1(T,x_1^0,\phi_{11}^0),\bar{\phi}_{11}(0)=\bar{\phi}_{11}(T,x_1^0,\phi_{11}^0)\ldots\ldots\ldots\ldots.(22)$$

Solution of equation(18) is function of two parameters x_1^0,ϕ_{11}^0.From equation (21) we have:

$$\bar{H}_{u\phi_1}=\bar{f}_u,\bar{H}_{u\phi_2}=\bar{g}_u,\bar{H}_{u\phi_3}=\bar{h}_u$$

$$\dot{x}_1=A_1\bar{x}_1+A_2\bar{y}_1+A_3\bar{z}_1+S_1\bar{\phi}_{11}+S_2\bar{\phi}_{21}+S_3\bar{\phi}_{31}\ldots.(23)$$
$$\dot{\bar{\phi}}_{11}=Q_1\bar{x}_1+Q_2\bar{y}_1+Q_3\bar{z}_1+T_1\bar{\phi}_{11}+T_2\bar{\phi}_{21}+T_3\bar{\phi}_{31}.(24)$$
$$A_1=\bar{f}_x-\bar{f}_u\bar{H}_{uu}^{-1}\bar{H}_{ux},A_2=\bar{f}_y-\bar{f}_u\bar{H}_{uu}^{-1}\bar{H}_{uy},A_3=\bar{f}_z-\bar{f}_u\bar{H}_{uu}^{-1}\bar{H}_{uz}$$
$$s_1=-\bar{f}_u\bar{H}_{uu}^{-1}\bar{f}_u^{-1},s_2=-\bar{f}_u\bar{H}_{uu}^{-1}\bar{g}_u^{-1},s_3=-\bar{f}_u\bar{H}_{uu}^{-1}\bar{h}_u^{-1}$$
$$Q_1=-\bar{H}_{xx}+\bar{H}_{xu}\bar{H}_{uu}^{-1}\bar{H}_{ux},Q_2=-\bar{H}_{xy}+\bar{H}_{xu}\bar{H}_{uu}^{-1}\bar{H}_{uy},Q_1=-\bar{H}_{xz}+\bar{H}_{xu}\bar{H}_{uu}^{-1}\bar{H}_{uz}$$

$$T_1 = -\bar{f}_x + \bar{H}_{xu}\bar{H}_{uu}^{-1}\bar{f}_u, T_2 = -\bar{g}_x + \bar{H}_{xu}\bar{H}_{uu}^{-1}\bar{g}_u, T_3 = -\bar{h}_x + \bar{H}_{xu}\bar{H}_{uu}^{-1}\bar{h}_u$$

$$\dot{y}_0 = A_4\bar{x}_1 + A_5\bar{y}_1 + A_6\bar{z}_1 + S_4\bar{\phi}_{11} + S_5\bar{\phi}_{21} + S_6\bar{\phi}_{31} .. (25)$$

$$\dot{\bar{\phi}}_{20} = Q_4\bar{x}_1 + Q_5\bar{y}_1 + Q_6\bar{z}_1 + T_4\bar{\phi}_{11} + T_5\bar{\phi}_{21} + T_6\bar{\phi}_{31} \dots (26)$$

$$A_4 = \bar{g}_x - \bar{g}_u\bar{H}_{uu}^{-1}\bar{H}_{ux}, A_5 = \bar{g}_y - \bar{g}_u\bar{H}_{uu}^{-1}\bar{H}_{uy}, A_6 = \bar{g}_z - \bar{g}_u\bar{H}_{uu}^{-1}\bar{H}_{uz}$$

$$s_4 = -\bar{g}_u\bar{H}_{uu}^{-1}\bar{f}_u^{-1}, s_5 = -\bar{g}_u\bar{H}_{uu}^{-1}\bar{g}_u^{-1}, s_6 = -\bar{g}_u\bar{H}_{uu}^{-1}\bar{h}_u^{-1}$$

$$Q_4 = -\bar{H}_{yx} + \bar{H}_{yu}\bar{H}_{uu}^{-1}\bar{H}_{ux}, Q_5 = -\bar{H}_{yy} + \bar{H}_{yu}\bar{H}_{uu}^{-1}\bar{H}_{uy}, Q_6 = -\bar{H}_{yz} + \bar{H}_{yu}\bar{H}_{uu}^{-1}\bar{H}_{uz}$$

$$T_4 = -\bar{f}_y + \bar{H}_{yu}\bar{H}_{uu}^{-1}\bar{f}_u, T_5 = -\bar{g}_y + \bar{H}_{yu}\bar{H}_{uu}^{-1}\bar{g}_u, T_6 = -\bar{h}_y + \bar{H}_{yu}\bar{H}_{uu}^{-1}\bar{h}_u$$

$$\dot{z}_0 = A_7\bar{x}_1 + A_8\bar{y}_1 + A_9\bar{z}_1 + S_7\bar{\phi}_{11} + S_8\bar{\phi}_{21} + S_9\bar{\phi}_{31} \dots\dots\dots\dots (27)$$

$$\dot{\bar{\phi}}_{30} = Q_7\bar{x}_1 + Q_8\bar{y}_1 + Q_9\bar{z}_1 + T_7\bar{\phi}_{11} + T_8\bar{\phi}_{21} + T_9\bar{\phi}_{31}. (28)$$

$$A_7 = \bar{h}_x - \bar{h}_u\bar{H}_{uu}^{-1}\bar{H}_{ux}, A_8 = \bar{h}_y - \bar{h}_u\bar{H}_{uu}^{-1}\bar{H}_{uy}, A_9 = \bar{h}_z - \bar{h}_u\bar{H}_{uu}^{-1}\bar{H}_{uz}$$

$$s_7 = -\bar{h}_u\bar{H}_{uu}^{-1}\bar{f}_u^{-1}, s_8 = -\bar{h}_u\bar{H}_{uu}^{-1}\bar{g}_u^{-1}, s_9 = -\bar{h}_u\bar{H}_{uu}^{-1}\bar{h}_u^{-1}$$

$$Q_7 = -\bar{H}_{zx} + \bar{H}_{zu}\bar{H}_{uu}^{-1}\bar{H}_{ux}, Q_8 = -\bar{H}_{zy} + \bar{H}_{zu}\bar{H}_{uu}^{-1}\bar{H}_{uy}, Q_9 = -\bar{H}_{zz} + \bar{H}_{zu}\bar{H}_{uu}^{-1}\bar{H}_{uz}$$

$$T_7 = -\bar{f}_z + \bar{H}_{zu}\bar{H}_{uu}^{-1}\bar{f}_u, T_8 = -\bar{g}_z + \bar{H}_{zu}\bar{H}_{uu}^{-1}\bar{g}_u, T_9 = -\bar{h}_z + \bar{H}_{zu}\bar{H}_{uu}^{-1}\bar{h}_u$$

Put $\bar{y}_0 = 0, \bar{\phi}_{20} = 0, \bar{z}_0 = 0, \bar{\phi}_{30} = 0$ in equations 25,26,27 and 28 then solve these equations , we have:

$$\left.\begin{aligned} \bar{y}_1 &= E_1(\bar{x}_1, \bar{\phi}_{11}) \\ \bar{z}_1 &= E_2(\bar{x}_1, \bar{\phi}_{11}) \\ \bar{\phi}_{21} &= E_3(\bar{x}_1, \bar{\phi}_{11}) \\ \bar{\phi}_{31} &= E_4(\bar{x}_1, \bar{\phi}_{11}) \end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots\dots\dots (29)$$

By substitution (29) in equations 23,24 we have:

$$\left.\begin{aligned} \dot{x}_1 &= G_1(\bar{x}_1, \bar{\phi}_{11}) \\ \dot{\bar{\phi}}_{11} &= G_2(\bar{x}_1, \bar{\phi}_{11}) \end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots\dots\dots (30)$$

Solve the system (30) we obtain a solution $(\bar{x}_1, \bar{\phi}_{11})$, that means we have a solution $(\bar{x}_1, \bar{y}_1, \bar{z}_1)$.

References

[1] Kokotovic P. V., Khalil N. K., O'Reilly J., Singular perturbation methods in control, Academic Press. London, 1986.

[2] Panovko Ya. G. ,Fundamentals of the Applied Theory of Elastic Oscillations, M .: Engineering،1967

[3] Pervozvansky AA ,Course of the theory of automatic control, M .: Nauka, 1986.

[4] Plotnikov V.A., Asymptotic methods of optimal control problem, M. .: Nauka of physic-mathematic,Moscow,1980.

[5] Sobolev V. A., Integral manifolds and decomposition of singularly perturbed systems , System of Control Lett. 1984, no. 5. P. 169-279.

[6] Sobolev V.A. , Singular perturbations in a linearly quadratic optimal control problem, Moscow: Nauka, 1991. No. 2. P. 53-64.

[7] Strygin VV, Sobolev VA, Separation of motions by the method of integral manifolds, Moscow: Nauka, 1988.

[8] Vasilieva AB, Dmitriev MG ,Singular perturbations in optimal control problems, Moscow: VINITI, 1982. P. 3-78.

[9] Yu. A. Mitropolsky and OB Lykova, Integral manifolds in nonlinear mechanics, Moscow: Nauka, 1973.