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NEGATIVE THEOREM FOR L_p,0<P<1 MONOTONE APPROXIMATION

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ABSTRACT. *For a given nonnegative integer number n, we can find a monotone function f depending on n, defined on the interval I=[-1,1], and an absolute constant c>0, satisfying the following relationship:*

$$\frac{2E_n(\acute{f})_p}{(n+1)^3} \le E^1_{n+1}(f)_p \le cE_n(\acute{f})_p,$$

where $E^1_{n+1}(f)_p$ *is the degree of the best* L_p *monotone approximation of the function f by algebraic polynomial of degree not exceeding n+1.* $E_n(\acute{f})_p$ *is the degree of the best* L_p *approximation of the function* $\acute{f}$ *by algebraic polynomial of degree not exceeding n.*

Keywords. Monotone approximation, algebraic polynomial, best approximation.

1. INTRODUCTION

For a non negative integer n, we denote by Π_n , be the space of all algebraic polynomials of degree not exceeding n.

For 0<p<1 we denoted by $L_p(I)$, the space of all functions $f: I \longrightarrow \mathbb{R}$, satisfying that $|f|^p$ is Lebesgue integrable.

Define the following quasi norm on the space $L_p(I)$, as follows:

$$\|f\|_{Lp(I)} = \left(\int_I |f|^p\right)^{\frac{1}{p}} \quad \text{if } 0<p<1$$

$$= \sup_{x\in I}|f| \quad = \|f\|_I \text{ if } p = \infty.$$

We define the degree of best unconstrained approximation of a function in $L_p(I)$, space as,

$$E_n(f)_p = \inf_{P\in\Pi_n} \|f-P\|_{Lp(I)}.$$

It is well known that the following inequality is true

$E_{n+1}(f) \quad \le c/(n+1)E_n(\acute{f})$, (1)

for a continuously differentiable function.

While the degree of of best constrained monotone approximation is defined by:

$$E^1_n(f)_p = \inf_{P\in\Delta^1} \|f-P\|_{Lp(I)},$$

where Δ^1 is the set of all monotone polynomials of Lp(I).

[1] contains a proof of the result

$E^1_n(f)_p \le c(p)n^{-1}E_{n-1}(\acute{f})_p$,(2)

in the case 0<p<1.

In this article we strength inequality (1) for f in Lp(I), and for $E^1_{n+1}(f)_p$. This result is an answer to a question asked by Koniagin and Shvedov during a conference for functional approximation theory in 1998. It mean we introduce a negative theorem to strength (1) for functions in Lp(I), and for the degree $E^1_{n+1}(f)_p$, for 0<p<1.

2. AUXILIARY RESULTS

Throughout our work we need the following lemmas:

Lemma 2.1. [1]

For any polynomial of degree not exceeding v and any f in Lp(I), we have for p>0, that

$$\|p_v\|_I \le C|I|^{-\frac{1}{p}} \|p_v\|_{Lp(I)}, \|f\|_{Lp(I)} \le 2^{1/p} \|f\|_I,$$

where C is an absolute constant depending on v only.

Let us define the Chebyshev polynomials as follows:

$\rho_n(x) = \text{cosncos} - 1x, \text{for } |x| \le 1,$

$= \frac{1}{2}(\left(x+(x^2-1)^{\frac{1}{2}}\right)^n + \left(x-(x^2-1)^{\frac{1}{2}}\right)^n), \quad \text{for } |x| > 1.$

By

$x_k = \cos\left(\frac{n-k}{n}\pi + \frac{\pi}{2n}\right), k = 1,2,\dots,n,$

Denotes the Chebyshev zeros.

Let c be an absolute constant in (0,1), and let $y = \frac{x+1-c}{c}$, and $A_n(x) = P_n(y)$. Also let $J_k = [-1 + c + cx_k, -1 + c + cx_{k+1}]$. Define the function f on [-1,1] as follows

$$f(x) = \begin{cases} \int_{-1+c+cx_k}^{x} A_n(u)du, & -1 + c + cx_k \leq x \leq 1 \\ 0, & o.w. \end{cases}.$$

(3)

Define

$$D_{n+1}(x) = \int_{-1}^{x} A_n(u)du,$$

it is clear that

$$\sup_{x\in[-1,1]}|D_{n+1}(x)| \leq \frac{1}{n+1},$$

Define

3. THE MAIN RESULTS

In this section we shall prove our main result. We show that we can find a function g in Lp[-1,1], satisfying

$$E_n(\acute{f})_p \sim E_{n+1}(f)_p.$$

Theorem 3.1.

For a given nonnegative integer number n, we can find a monotone function f in Lp(I), depending on n, defined on the interval I=[-1,1], and an absolute constant C>0, satisfying the following relationship:

$$\frac{2E_n(\acute{f})_p}{(n+1)^3} \leq E_{n+1}^1(f)_p \leq CE_n(\acute{f})_p.$$

Proof.

Using (3) we have for $-1 + c + cx_k \leq x \leq 1$, that

$\acute{f}(x) = A_n(x) - A_n(-1 + c + cx_k) \geq 0$,

which implies f is monotone nondecreasing on [-1,1].

Now turn the light to the equivalence result,

$$\begin{aligned} E_n(\acute{f})_p^p &\leq \|\acute{f} - A_n\|_p^p \\ &= \int_{-1}^{-1+c+cx_k}|\acute{f} - A_n|^p + \int_{-1+c+cx_k}^{1}|\acute{f} - A_n|^p \\ &= \int_{-1}^{-1+c+cx_k}|\acute{f} - A_n|^p + \int_{-1}^{-1+c+cx_k}|A_n|^p \\ &\leq \int_{-1}^{-1+2c}|A_n|^p \\ &= \int_{-1}^{-1+2c}\left|P_n(\frac{x+1-c}{c})\right|^p dx \\ &= \int_{-1}^{-1}|P_n(x)|^p dx \\ &\leq 2. \end{aligned}$$

Thus

$$E_n(\acute{f})_p \leq 2^{\frac{1}{p}}.$$

Let σ_{n+1} be a polynomial of degree not exceeding n+1, with first derivative greater than or equal to zero on [-1,1].

First let us prove the inequality

$$\|f - \sigma_{n+1}\|_p \geq \frac{2^{\frac{1}{p}}}{200(n+1)}.$$

(4)

Define

$$C_{n+1}(x) = \sigma_{n+1}(x) + \int_{-1}^{x} A_n(u)du.$$

Assume m is the roots of the polynomial $\acute{C}_{n+1}$, with the property that these roots containing their multiplicities, that are also belong to the interval $< -1 + c + cx_1, -1 + c + cx_k)$, < used for the even n and (used for the odd n. Clearly m≤n.

For the proof of (4) we have two cases, m<n/2 and m≥n/2.

For the first case we have from the assumption that σ_{n+1} be a polynomial of degree not exceeding n+1, with first derivative greater than or equal to zero on Jk. We obtain

$$\acute{\sigma}_{n+1}(x) > A_n(x) \geq 0, \quad \text{for x in } J_k$$

We shall assume k is in ω if n-k is an even number, and k is in ω^0 if the polynomial $\acute{C}_{n+1}$ has even numbers of roots on J_k.

$$\|f - \sigma_{n+1}\|_{L_p(I)} > \left(\int_{-1+c+cx_1}^{-1+c+cx_k}|f(x) - \sigma_{n+1}(x)|^p dx\right)^{\frac{1}{p}}$$
$$= \left(\int_{-1+c+cx_1}^{-1+c+cx_k}|\sigma_{n+1}(x)|^p dx\right)^{\frac{1}{p}}.$$

Using Markov inequality to obtain

$$\begin{aligned} \|f - \sigma_{n+1}\|_{L_p(I)} &> \frac{2^{\frac{1}{p}}}{n+1}\|\sigma_{n+1}\|_{L_p[-1+c+cx_1,-1+c+cx_k]} \\ &> \frac{2c}{\pi}\frac{2^{\frac{1}{p}}}{n+1}\sin\frac{\pi}{8}\sin\frac{\pi}{10} \\ &> \frac{2^{\frac{1}{p}}}{200(n+1)}. \end{aligned}$$

Using the same lines as in [2] and [3] we can get,

$$\begin{aligned} \|f - D_{n+1}\|_{L_p[0,1]} &= \|f - D_{n+1}\|_{L_p[-1,-1+c+cx_n]} \\ &= \|D_{n+1}\|_{L_p[-1,-1+c+cx_n]}. \end{aligned}$$

Lemma 2.1 implies

$$\begin{aligned} \|f - D_{n+1}\|_{L_p[0,1]} &\leq (c + cx_n)^{1/p}\|D_{n+1}\|_{[-1,-1+c+cx_n]} \\ &\leq \frac{c^2(1+x_n)^{1/p}}{n-1} \\ &\leq \frac{2^{1/p}}{n-1}. \end{aligned}$$

Using Lemma 2.1 again to obtain

$$\begin{aligned} \|C_{n+1}\|_{L_p[-1,1]} &\geq \frac{2^{\frac{1}{p}}}{n+1}\|C_{n+1}\|_{[-1,1]} \\ &\geq \frac{2^{\frac{1}{p}}}{(n+1)^3}\left\|\acute{C}_{n+1}\right\|_{[-1,1]} \\ &\geq \frac{2^{\frac{1}{p}+1}}{(n+1)^3}\left(\frac{7}{6}\right)^{\frac{n}{2}}. \end{aligned}$$

Then

$$\begin{aligned} \|f - \sigma_{n+1}\|_{L_p[-1,1]}^p &\geq \|C_{n+1}\|_{L_p[-1,1]}^p - \|f - D_{n+1}\|_{L_p[-1,1]}^p \\ &\geq \left(\frac{2^{\frac{1}{p}+1}}{(n+1)^3}\left(\frac{7}{6}\right)^{\frac{n}{2}}\right)^p - \frac{2}{(n-1)^p} \\ &\geq \left(\frac{2}{(n+1)^3}\right)^p. \end{aligned}$$

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مبرهنة عكسية للتقريب الرتيب للدوال في الفضاءت
$L_p, 0<p<1$

المستخلص. لكل عدد طبيعي n *يمكن ان نجد دالة رتيبية تعتمد عليه و معرفة على الففترة* [-1,1] *و كذلك يمكن ان نجد ثابت موجب* c *يحقق العلاقة*

$$\frac{2E_n(\acute{f})_p}{(n+1)^3} \le E^1_{n+1}(f)_p \le cE_n\left(\acute{f}\right)_p,$$

حيث ان $E^1_{n+1}(f)_p$ *يمثل درجة التقريب الرتيب الأفضل للدالة* f *في الفضاء* L_p *باستخدام متعددات الحدود الجبرية الرتيبية من الدرجة اقل او يساوي* n+1 .
اما $E_n\left(\acute{f}\right)_p$ *يمثل درجة التقريب الأفضل للدالة* $\acute{f}$ *في الفضاء* L_p *باستخدام متعددات الحدود الجبرية من الدرجة اقل او يساوي* n .

الكلمات المفتاحية. التقريب الرتيب. متعددات الحدود الجبرية. التقريب الافضل .