A Fuzzy ∧- ideal of a $\mathcal{BH}$-algebra

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Abstract--- *In this paper, we introduce the notions of ∧ - ideal of a $\mathcal{BH}$-algebra in ordinary and fuzzy senses. Also, we give some properties of them and link these notions with some types of ideals of $\mathcal{BH}$- algebra in ordinary and fuzzy senses .The image and preimage of fuzzy ∧- ideal under a $\mathcal{BH}$ homomorphism are investigated.*

المستخلص ـــ *قدمنا في هذا البحث مفهوم مثالي جديد وهو مثالي ideal - ∧ الضبابي في جبر- $\mathcal{BH}$, كما درسنا المثالي ideal - ∧ الاعتيادي وربطناها مع بعض المفاهيم والمثاليات في جبر - $\mathcal{BH}$ وأعطينا بعض خواص هذا المثالي في الحالة الاعتيادية والحالة الضبابية في جبر – $\mathcal{BH}$. وناقشنا الصورة والصورة العكسية لهذا المثالي تحت التشاكل في جبر - $\mathcal{BH}$.*

Keywords: ***$\mathcal{BH}$-algebra, ideal of $\mathcal{BH}$-algebra, fuzzy ideal of $\mathcal{BH}$-algebra.***

I. INTRODUCTION

The concept of $\mathcal{BH}$-algebra and the notion of ideal of a $\mathcal{BH}$-algebra have been introduced by $Y. B.$ Jun, $E. H.$ Roh and $H.$ S. Kim in 1998, [10]. In 2001, Q.Zhang, $Y. B.$ Jun and $E. H.$ Roh introduced a normal $\mathcal{BH}$-algebra [9]. Then, Y. $B.$ Jun, $E. H.$ Roh, $H. S.$ Kim and Q.Zhang discussed more properties on $\mathcal{BH}$-algebras [9]. In 2012 , $H. H.$ Abbass and $H. A.$ Dahham introduced the notion of completely closed ideal of a $\mathcal{BH}$-algebra[5].

And the fuzzy set concept has been introduced by L. $A.$ Zadeh In 1965, [6]. In 2001, Q.Zhang, $E. H.$ Roh and $Y. B.$ Jun studied the fuzzy theory in $\mathcal{BH}$-algebras [8]

II. Preliminaries

Some essential notations and definitions of $\mathcal{BH}$-algebras and ideals of $\mathcal{BH}$ – algebra in fuzzy and ordinary senses that we need in our work has been introduced in this section.

Definition (2.1) [2]:

A nonempty set $\mathcal{K}$ with a binary operation"$*$" and a constant 0 satisfying the following conditions:

1) $\varkappa * \varkappa=0, \forall \varkappa \in \mathcal{K}$.

2) $\varkappa * \gamma=0$ and $\gamma * \varkappa=0$ infer $\varkappa=\gamma$, $\forall \varkappa, \gamma \in \mathcal{K}$.

3) $\varkappa * 0=\varkappa, \forall \varkappa \in \mathcal{K}$

is called A $\mathcal{BH}$-algebra.

Definition (2.2) [5]:

An associative $\mathcal{BH}$ -algebra is a $\mathcal{BH}$ -algebra $\mathcal{K}$ satisfying the following:

$(\varkappa * \gamma) * z=\varkappa *(\gamma * z), \forall \varkappa, \gamma, z \in \mathcal{K}$.

Definition (2.3) [2]:

A nonempty subset I of a $\mathcal{BH}$-algebra $\mathcal{K}$ is known as an ideal of $\mathcal{K}$ if it fulfills:

1) $0 \in$ I.

2) $\varkappa * \gamma \in$ I and $\gamma \in$ I infer $\varkappa \in$ I.

Definition (2.4) [5]:

A completely closed ideal I of a $\mathcal{BH}$-algebra $\mathcal{K}$ is an ideal of $\mathcal{K}$ satisfying $x * \gamma \in$ I, $\forall \varkappa, \gamma \in$ I.

Definition (2.5) [1]:

A normal ideal I of a $\mathcal{BH}$-algebra $\mathcal{K}$ is an ideal of $\mathcal{K}$ satisfying the following :

$\varkappa *(\varkappa * \gamma) \in$ I infers $\gamma *(\gamma * \varkappa) \in$ I, for all $\varkappa, \gamma \in \mathcal{K}$.

Definition (2.6) [5]:
Let $\mathcal{K}$ a $\mathcal{BH}$ -algebra and I be a subset of $\mathcal{K}$, then I is known as a $\mathcal{BH}$ -ideal of $\mathcal{K}$ if satisfying the following conditions:

1) $0 \in I$
2) $\varkappa \divideontimes \gamma \in I$ and $\gamma \in I \Longrightarrow \varkappa \in I$,
3) $\varkappa \in I$ and $\gamma \in \mathcal{K} \Longrightarrow \varkappa \divideontimes \gamma \in I$, $I \divideontimes \mathcal{K} \subseteq I$.

Remark (2.7)[11]:

Let $(\mathcal{K}, \divideontimes, 0)$and $(\Upsilon, \divideontimes', 0')$are $\mathcal{BH}$-algebras. A mapping $F: \mathcal{K} \rightarrow \Upsilon$ is known as a homomorphism if $F(\varkappa \divideontimes \gamma) = F(\varkappa) \divideontimes' F(\gamma)$ $\forall\ \varkappa, \gamma \in \mathcal{K}$. A homomorphism F is known as epimorphism if it is surjective. For any homomorphism $F: \mathcal{K} \rightarrow \Upsilon$, the set $\{\varkappa \in \mathcal{K}: F(\varkappa) = 0'\}$is called the kernel of F, meant by $ker(F)$,if S is a subset of $\mathcal{K}$ then the set $\{F(s): s \in \mathcal{K}\}$ is known as the image of S indicated by F(S) and if H is a subset of 𝕐 then the set $\{\varkappa \in \mathcal{K}: F(\varkappa) \in H\}$ is known as the preimage of the set H. Notice that $F(0) = 0'$.

Proposition(2.8) [5]:
In a $\mathcal{BH}$-algebra, every $\mathcal{BH}$-ideal is a completely closed ideal.

Proposition (2.9) [5]:
Let $\mathcal{K}$ and Υ are a $\mathcal{BH}$ -algebra and I is an ideal of $\mathcal{K}$ and $F: \mathcal{K} \rightarrow \Upsilon$ be a $\mathcal{BH}$-epimorphism. Then F(I) is an ideal of Υ.

Proposition (2.10) [5]:
Let $\mathcal{K}$ and Υ are a $\mathcal{BH}$ -algebra and I is an ideal of Υ and $F: \mathcal{K} \rightarrow \Upsilon$ be a $\mathcal{BH}$-epimorphism. Then F^{-1}(I) is an ideal of $\mathcal{K}$.

Proposition (2.11) [5]:
If $\mathcal{K}$ is a $\mathcal{BH}$-algebra and $\{I_j, j \in \lambda\}$ be an ideals family of $\mathcal{K}$.Then $\bigcap_{j\in\lambda} I_j$ is an ideal of $\mathcal{K}$.

Proposition (2.12) [5]:
The associative $\mathcal{BH}$-algebra $\mathcal{K}$ is satisfying the following properties:

1) $0 \divideontimes \varkappa = \varkappa \quad \forall \varkappa \in \mathcal{K}$.

2) $\varkappa \divideontimes \gamma = \gamma \divideontimes \varkappa \quad \forall \varkappa, \in \mathcal{K}$.

Now, we give a survey about the fuzzy concepts of a $\mathcal{BH}$-algebra that we need later

Definition (2.13) [6]:
A fuzzy set (fuzzy subset) $\mathcal{A}$ in a nonempty set $\mathcal{K}$ is a function from $\mathcal{K}$ as the domain into the unit closed interval of real numbers [0,1] as the range of $\mathcal{A}$.

Definition (2.14) [3]:
For any two fuzzy set $\mathcal{A}$ and$\mathfrak{B}$ in $\mathcal{K}$ we have:
$(\mathcal{A} \cap \mathfrak{B})\ (\varkappa) = \min\{\mathcal{A}(\varkappa), \mathfrak{B}\ (\varkappa)\}, \forall \varkappa \in \mathcal{K}$.
$(\mathcal{A} \cup \mathfrak{B})\ (\varkappa) = \max\{\mathcal{A}(\varkappa), \mathfrak{B}(\varkappa)\}, \forall \varkappa \in \mathcal{K}$.
$\mathcal{A} \cap \mathfrak{B}$ and $\mathcal{A} \cup \mathfrak{B}$ are fuzzy sets in $\mathcal{K}$. generally speaking, if $\{\mathcal{A}_\alpha, \alpha \in \Lambda\}$ is a family of fuzzy sets in $\mathcal{K}$, then :
$(\bigcap_{j\in\Gamma} \mathcal{A}_j)\ (\varkappa) = \inf\{\mathcal{A}_j\ (\varkappa), j \in \Gamma\}, \forall \varkappa \in \mathcal{K}$
$(\bigcup_{j\in\Gamma} \mathcal{A}_j)\ (\varkappa) = \sup\{\mathcal{A}_j\ (\varkappa), j \in \Gamma\}, \forall \varkappa \in \mathcal{K}$.
Which are also fuzzy sets in $\mathcal{K}$.

Definition (2.15) [7]:
Let $\mathcal{A}$ be a fuzzy subset of $\mathcal{K}$, for all $\alpha \in [0,1]$. The set $\mathcal{A}_\alpha = \{\varkappa \in \mathcal{K}, \mathcal{A}(\varkappa) \geq \alpha\}$ is known as a level subset of $\mathcal{A}$. Note that: $\mathcal{A}_\alpha$ is a subset of $\mathcal{K}$ in the ordinary sense.

Definition (2.16) [2]:
A fuzzy ideal is a fuzzy subset $\mathcal{A}$ of a $\mathcal{BH}$-algebra $\mathcal{K}$ satisfying the following:
1) $\mathcal{A}(0) \geq \mathcal{A}(\varkappa) \quad \forall\ \varkappa \in \mathcal{K}$.
2) $\mathcal{A}(\varkappa) \geq \min\{\mathcal{A}(\varkappa \divideontimes \gamma), \mathcal{A}(\gamma)\}\ \forall \varkappa, \gamma \in \mathcal{K}$.

Definition (2.17): [5]
A fuzzy completely closed ideal $\mathcal{A}$ of a $\mathcal{BH}$-algebra $\mathcal{K}$ is a fuzzy ideal satisfying the following
$\mathcal{A}(\varkappa \divideontimes \gamma) \geq \min\{\mathcal{A}(\varkappa), \mathcal{A}(\gamma)\}, \forall\ \varkappa, \gamma \in \mathcal{K}$

Definition (2.18): [4]
Let $\mathcal{K}$ and Υ be any two sets, $\mathcal{A}$ be any fuzzy set in $\mathcal{K}$ and $F: \mathcal{K} \rightarrow \Upsilon$ be a function. The set $F^{-1}(\gamma) = \{\varkappa \in \mathcal{K} \mid F(\varkappa) = \gamma\}, \forall \gamma \in \Upsilon$. The fuzzy set $\mathfrak{B}$ in Υ defined by $\mathfrak{B}(\gamma) = \begin{cases} \sup\{\mathcal{A}(x) \mid x \in f^{-1}(\gamma)\}; & \text{if } f^{-1}(\gamma) \neq \emptyset \\ 0; & \text{otherwise} \end{cases}$, $\forall \gamma \in \Upsilon$, is called the image of $\mathcal{A}$ under F and is denoted by $F(\mathcal{A})$.

Definition (2.19) [4]:
Let $\mathcal{K}$ and Υ are any two sets, $F: \mathcal{K} \rightarrow \Upsilon$ be any function and $\mathfrak{B}$ be a fuzzy subset of $F(\mathcal{A})$. The fuzzy subset $\mathcal{A}$ of $\mathcal{K}$ defined by: $\mathcal{A}(\varkappa) = \mathfrak{B}(F(\varkappa)), \forall \varkappa \in \mathcal{K}$ is called the preimage of $\mathfrak{B}$ under F and is denoted by $F^{-1}(\mathfrak{B})$.

Theorem (2.20) [5]:
Let $\mathcal{A}$ be a fuzzy ideal of a $\mathcal{BH}$-algebra $\mathcal{K}$. Then the set $\mathcal{K}_{\mathcal{A}}=\{ \varkappa \in \mathcal{K} : \mathcal{A}(\varkappa)= \mathcal{A}(0)\}$ is an ideal of $\mathcal{K}$.

III. The Main Results:

We define in this section, the notions of a $\wedge$ - ideal of a $\mathcal{BH}$-algebra ordinary and fuzzy senses. Some properties of them are investigated. Also, relations among them and other structure of a $\mathcal{BH}$ – algebra is given.

Definition (3.1) :
Let $\mathcal{K}$ be a $\mathcal{BH}$-algebra. The ideal I of $\mathcal{K}$ is said to be $\wedge$ - ideal if it is satisfying: $\varkappa \wedge \gamma = \gamma \divideontimes (\gamma \divideontimes \varkappa) \in$ I $\forall\, \varkappa, \gamma \in$ I

Example (3.2) [6]:
Let $\mathcal{K} = \{0,\beta,\delta,\sigma\}$ be a $\mathcal{BH}$- algebra with binary operation defined by :

$\divideontimes$	0	β	δ	σ
0	0	β	0	δ
β	β	0	σ	0
δ	δ	δ	0	σ
σ	σ	σ	σ	0

The ideal I=$\{0,\beta\}$ of $\mathcal{K}$ is $\wedge$ - ideal since
$0 \wedge 0 = 0 \divideontimes (0 \divideontimes 0) = 0 \divideontimes 0=0$
$0 \wedge \beta = \beta \divideontimes (\beta \divideontimes 0) = \beta \divideontimes \beta = 0$
$\beta \wedge 0 = 0 \divideontimes (0 \divideontimes \beta) = 0 \divideontimes \beta = \beta$
$\beta \wedge \beta = \beta \divideontimes (\beta \divideontimes \beta) = \beta \divideontimes 0= \beta$
Then I is a $\wedge$ - ideal of $\mathcal{K}$.

Proposition (3.3):
Let $\mathcal{K}$ be a $\mathcal{BH}$-algebra, Then every completely closed ideal of $\mathcal{K}$ is a $\wedge$- ideal of $\mathcal{K}$.

Proof :
Let I be a completely closed ideal and $\varkappa, \gamma \in$ I
Then $\gamma \divideontimes \varkappa \in I$ [since I is a completely closed ideal of $\mathcal{K}$]
Now γ, $\gamma \divideontimes \varkappa \in$ I implies that $\gamma \divideontimes (\gamma \divideontimes \varkappa) = \varkappa \wedge \gamma \in$ I.
Then I is a $\wedge$ - ideal of $\mathcal{K}$.

Proposition (3.4):
Let $\mathcal{K}$ be associative $\mathcal{BH}$- algebra. Then every ideal in $\mathcal{K}$ is a $\wedge$ - ideal of $\mathcal{K}$.

Proof:
Let $\mathcal{K}$be an associative $\mathcal{BH}$- algebra and I is an ideal of $\mathcal{K}$ with $\varkappa$, $\gamma \in I$. Now,

$\gamma \divideontimes (\gamma \divideontimes \varkappa) = (\gamma \divideontimes \gamma) \divideontimes \varkappa = 0 \divideontimes \varkappa = \varkappa \in$ I [By proposition 2.12]

Then I is a $\wedge$ - ideal of $\mathcal{K}$.

Corollary (3.5):
Every $\mathcal{BH}$ – ideal is a $\wedge$- ideal.

Proof:

Is directly from propositions (2.8) and (3.3).

Proposition (3.6):
Every normal ideal of associative $\mathcal{BH}$-algebra $\mathcal{K}$ is a $\wedge$ - ideal.

Proof:

If I is a normal ideal of $\mathcal{K}$ and let $\varkappa, \gamma \in$ I.

Now,

$\varkappa \divideontimes (\varkappa \divideontimes \gamma) = (\varkappa \divideontimes \varkappa) \divideontimes \gamma = 0 \divideontimes \gamma = \gamma \in$ I

[By proposition 2.12]

Thus $\varkappa \divideontimes (\varkappa \divideontimes \gamma) \in$ I $\Rightarrow \gamma \divideontimes (\gamma \divideontimes \varkappa) \in$ I

[If I is a normal ideal]

$\Longrightarrow$ I is a $\wedge$ - ideal of $\mathcal{K}$.

Proposition (3.7):
If I is a $\wedge$ - ideal of $\mathcal{K}$ and $F\colon \mathcal{K} \rightarrow \Upsilon$ be a $\mathcal{BH}$ – epimorphism. Then F(I) is a $\wedge$- ideal of Υ.

Proof:
$\because$ I is an $\wedge$ - ideal of $\mathcal{K}$.
$\Rightarrow$ I is an ideal $\Rightarrow$ F (I) is an ideal of Υ [Using proposition (2.9)]
Now, let $\varkappa, \gamma \in F$ (I)
Then $\exists\, \alpha, \beta \in$ I such that

$F(\alpha) = \varkappa, F(\beta) = \gamma$

$\Rightarrow \beta \divideontimes (\beta \divideontimes \alpha) \in$ I *[since I is $\wedge$ - ideal]*

$\Rightarrow F(\beta \divideontimes (\beta \divideontimes \alpha)) \in F$ (I)

$\Rightarrow F(\beta) \divideontimes F(\beta \divideontimes \alpha) \in F$ (I)

$\Rightarrow F(\beta) \divideontimes (F(\beta) \divideontimes F(\alpha)) \in F$ (I)

$\Rightarrow \gamma \divideontimes (\gamma \divideontimes \varkappa) \in F$ (I)

Then F (I) *is a* $\wedge$ - ideal *of* Υ.

Proposition (3.8):
Let $: \mathcal{K} \rightarrow \Upsilon$ be a $\mathcal{BH}$ – epimorphism and I be $\wedge$ - ideal of Υ .Then F^{-1} (I) is a $\wedge$ - ideal of $\mathcal{K}$.

Proof :
$\because$ I is a $\wedge$ - ideal of Υ
$\Rightarrow$ F^{-1} (I) is an ideal in $\mathcal{K}$
[Proposition 2.10]

Now, let $\varkappa, \gamma \in F^{-1}(\mathrm{I})$

$\Rightarrow F(\varkappa), F(\gamma) \in \mathrm{I}$

$\Rightarrow F(\gamma) \circledast' (F(\gamma) *' F(\varkappa)) \in \mathrm{I}$

[Since I is an $\wedge$ - ideal of Υ]

$\Rightarrow F(\gamma) \circledast' F(\gamma \circledast \varkappa) \in \mathrm{I}$

$\Rightarrow F(\gamma \circledast (\gamma \circledast \varkappa)) \in \mathrm{I}$

$\Rightarrow \gamma \circledast (\gamma \circledast \varkappa) \in F^{-1}(I)$

Then F^{-1} (I) is a $\wedge$ - ideal of $\mathcal{K}$.

Proposition (3.9):

Let $F: \mathcal{K} \rightarrow \Upsilon$ is a $\mathcal{BH}$ – homomorphism. Then ker (F) is a $\wedge$ - ideal of .

Proof:

It is clear that ker (F) is an ideal of $\mathcal{K}$.

Now, let $\varkappa, \gamma \in ker\,(F)$

$\Rightarrow F(\varkappa) = 0', F(\gamma) = 0'$

$\Rightarrow F(\gamma \circledast (\gamma \circledast \varkappa)) = F(\gamma) \circledast' (F(\gamma \circledast \varkappa))$

$= F(\gamma) \circledast' (F(\gamma) *' F(\varkappa))$

$= 0' \circledast' (0' *' 0')$

Then $\gamma \circledast (\gamma \circledast \varkappa) \in ker\,(F)$

Then ker (F) is a $\wedge$ - ideal of $\mathcal{K}$.

Proposition (3.10):

Let $\{\mathrm{I}_j, j \in \lambda\}$ be a family of $\wedge$ - ideals of a $\mathcal{BH}$- algebra $\mathcal{K}$.Then$\bigcap_{j\in\lambda} \mathrm{I}_j$ is a $\wedge$ - ideal of $\mathcal{K}$.

Proof:

$\because \mathrm{I}_j$ is a $\wedge$ - ideal of $\mathcal{K}, \forall\ j \in \lambda$

$\Rightarrow \mathrm{I}_j$ is an ideal of $\mathcal{K}$, $\forall\, j \in \lambda$

$\Rightarrow \bigcap_{j\in\lambda} \mathrm{I}_j$ is an ideal
[using proposition (2.11)]

Now, let $x, \gamma \in \bigcap_{j\in\lambda} \mathrm{I}_j$

$\Rightarrow \varkappa, \gamma \in \mathrm{I}_j$, $for\ each\ j \in \lambda$

$\Rightarrow \gamma \circledast (\gamma \circledast \varkappa) \in \mathrm{I}_j, \forall\, j \in \lambda$
[since I_j is $\wedge$ - ideal]

$\Rightarrow \gamma \circledast (\gamma \circledast \varkappa) \in \bigcap_{j\in\lambda} \mathrm{I}_j$

Therefore, $\bigcap_{j\in\lambda} \mathrm{I}_j$ is a $\wedge$ - ideal of $\mathcal{K}$.

Proposition (3.11):

In a $\mathcal{BH}$- algebra $\mathcal{K}$ if $\{\mathrm{I}_j, j\in \lambda\}$ is a chain of $\wedge$ - ideals. Then $\bigcup_{j\in\lambda} \mathrm{I}_j$ is a $\wedge$ - ideal of $\mathcal{K}$.

Proof :

Since I_j is $\wedge$ - ideal of $\mathcal{K}$, $\forall\ j \in \lambda$

$\Rightarrow \mathrm{I}_j$ is an ideal of $\mathcal{K}$, $\forall\, j\in \lambda$

Therefore, $\bigcup_{j\in\lambda} \mathrm{I}_j$ is an ideal of $\mathcal{K}$.

Now, let $\varkappa, \gamma \in \bigcup_{j\in\lambda} \mathrm{I}_j$

$\Rightarrow \mathrm{I}_j, \mathrm{I}_k \in \{\mathrm{I}_j\}_{j\in\lambda}, \exists\, \varkappa \in \mathrm{I}_j, \gamma \in \mathrm{I}_j$

Then either $\mathrm{I}_k \subseteq \mathrm{I}_j$ or $\mathrm{I}_j \subseteq \mathrm{I}_k$

If $\mathrm{I}_j \subseteq \mathrm{I}_k$

$\Rightarrow \varkappa, \gamma \in \mathrm{I}_k$

$\Rightarrow \gamma \circledast (\gamma \circledast \varkappa) \in \mathrm{I}_k$

Similarly,

If $I_k \subseteq \mathrm{I}_j$

$\Rightarrow \gamma \circledast (\gamma \circledast \varkappa) \in \bigcup_{j\in\lambda} \mathrm{I}_j$

Therefore, $\bigcup_{j\in\lambda} \mathrm{I}_j$ is a $\wedge$ - ideal of $\mathcal{K}$.

Definition (3.12):

Let $\mathcal{K}$ be a $\mathcal{BH}$-algebra. A fuzzy ideal $\mathcal{A}$ of $\mathcal{K}$is known as a fuzzy $\wedge$ - ideal of $\mathcal{K}$ if it is satisfies $\mathcal{A}(\gamma \circledast(\gamma \circledast \varkappa)) \geq \min\{\mathcal{A}(\varkappa),\mathcal{A}(\gamma)\}, \forall \varkappa,\gamma \in \mathcal{K}$.

Theorem (3.13):

In a $\mathcal{BH}$- algebra $\mathcal{K}$ if $\gamma \circledast (\gamma \circledast \varkappa) = z, z \in \{0, \varkappa, \gamma\}\ \forall\, \varkappa, \gamma \in \mathcal{K}$. Then every fuzzy ideal is a fuzzy $\wedge$ - ideal of $\mathcal{K}$

Proof:

Let $\mathcal{A}$ be a fuzzy ideal and $\varkappa, \gamma \in \mathrm{I}$
If $\gamma \circledast (\gamma \circledast \varkappa) = 0$
$\Rightarrow \mathcal{A}(\gamma \circledast(\gamma \circledast \varkappa)) = \mathcal{A}(0) \geq \min\{\mathcal{A}(\varkappa), \mathcal{A}(\gamma)\}$

If $\gamma \circledast (\gamma \circledast \varkappa) = \varkappa$

$\Rightarrow \mathcal{A}(\gamma \circledast (\gamma \circledast \varkappa)) = \mathcal{A}(\varkappa) \geq \min\{\mathcal{A}(\varkappa), \mathcal{A}(\gamma)\}$
If $\gamma \circledast (\gamma \circledast \varkappa) = \gamma$
$\Rightarrow \mathcal{A}(\gamma \circledast (\gamma \circledast \varkappa)) = \mathcal{A}(\gamma)$
$\geq \min\{\mathcal{A}(\varkappa), \mathcal{A}(\gamma)\}$

Then $\mathcal{A}$ is a fuzzy $\wedge$ - ideal of $\mathcal{K}$.

Proposition (3.14):

In associative $\mathcal{BH}$-algebra $\mathcal{K}$. Every fuzzy ideal is a fuzzy $\wedge$ - ideal.

Proof:

Let $\mathcal{A}$ be a fuzzy ideal**,** and $\varkappa, \gamma \in \mathcal{K}$

$\mathcal{A}(\gamma \circledast (\gamma \circledast \varkappa)) = \mathcal{A}((\gamma \circledast \gamma) \circledast \varkappa)$
$= \mathcal{A}(0 \circledast \varkappa) = \mathcal{A}(\varkappa)$
[$0 \circledast \varkappa = \varkappa$ by proposition (2.12)]

$\Rightarrow \mathcal{A}(\gamma \circledast (\gamma \circledast \varkappa)) \geq \min\{\mathcal{A}(\varkappa), \mathcal{A}(\gamma)\}$

Then $\mathcal{A}$ is a fuzzy ∧- ideal of $\mathcal{K}$.

Theorem (3.15):

In a $\mathcal{BH}$- algebra $\mathcal{K}$, $\mathcal{A}$ is a fuzzy ∧ - ideal if and only if $\mathcal{A}_\alpha$ is a ∧ - Ideal of $\mathcal{K}$, $for\ all\ \ \alpha \in [0, 1], \mathcal{A}(0) = sup_{\varkappa \in \mathcal{K}}\ \mathcal{A}(\varkappa)$.

Proof:

It is clear that $\mathcal{A}_\alpha$ is an ideal of $\mathcal{K}$.

Now, Let $\varkappa, \gamma \in \mathcal{A}_\alpha$ .Then $\mathcal{A}(\varkappa) \geq \alpha$ *and* $\mathcal{A}(\gamma) \geq \alpha$ implies that $min\{\mathcal{A}(\varkappa), \mathcal{A}(\gamma)\} \geq \alpha$.

But, $\mathcal{A}(\gamma * (\gamma * \varkappa)) \geq min\{\mathcal{A}(\varkappa), \mathcal{A}(\gamma)\}$
[since $\mathcal{A}$ is a fuzzy ∧ - ideal]

$\Rightarrow \mathcal{A}(\gamma * (\gamma * \varkappa)) \geq \alpha$
$\Rightarrow \gamma * (\gamma * \varkappa) \in \mathcal{A}_\alpha$

Then $\mathcal{A}_\alpha$ is a ∧ - ideal of $\mathcal{K}$.

Conversely,

It is clear that $\mathcal{A}$ is a fuzzy ideal of $\mathcal{K}$.

Let $\varkappa, \gamma \in \mathcal{K} \Rightarrow \mathcal{A}(\varkappa), \mathcal{A}(\gamma) \in [0, \mathcal{A}(0)]$

Now, let $\alpha = min\{\mathcal{A}(\varkappa), \mathcal{A}(\gamma)\}$

Then $\mathcal{A}(\varkappa) \geq \alpha$ *and* $\mathcal{A}(\gamma) \geq \alpha$ implies that $\varkappa, \gamma \in \mathcal{A}_\alpha$.Thus $\gamma * (\gamma * \varkappa) \in \mathcal{A}_\alpha$
[since $\mathcal{A}_\alpha$ is ∧ - ideal of $\mathcal{K}$]

Hence
$\mathcal{A}(\gamma * (\gamma * \varkappa)) \geq \alpha = min\{\mathcal{A}(\varkappa), \mathcal{A}(\gamma)\}$.
Therefore, $\mathcal{A}$ is a fuzzy ∧ - ideal of $\mathcal{K}$.

Theorem (3.16):

Let $\mathcal{A}$ be a fuzzy ∧ - ideal of a $\mathcal{BH}$- algebra $\mathcal{K}$.Then the set

$\mathcal{K}_\mathcal{A} = \{\varkappa \in \mathcal{K}: \mathcal{A}(\varkappa) = \mathcal{A}(0)\}$is a ∧ - ideal of $\mathcal{K}$.

Proof:

If $\mathcal{A}$ is a fuzzy ∧ - ideal of $\mathcal{K}$.

$\Rightarrow \mathcal{A}$ is a fuzzy ideal of $\mathcal{K}$.

$\Rightarrow \mathcal{K}_\mathcal{A}$ is an ideal [By Theorem (2.20)]

Now *let* $\varkappa, \gamma \in \mathcal{K}_\mathcal{A}$

$\Rightarrow \mathcal{A}(\varkappa) = \mathcal{A}(\gamma) = \mathcal{A}(0)$

$\Rightarrow min\{\mathcal{A}(\varkappa), \mathcal{A}(\gamma)\} = \mathcal{A}(0)$

But,

$\mathcal{A}(\gamma * (\gamma * \varkappa)) \geq min\{\mathcal{A}(\varkappa) \mathcal{A},(\gamma)\} = \mathcal{A}(0)$

[Since $\mathcal{A}$ is a fuzzy ∧ - ideal]

$\mathcal{A}(\gamma * (\gamma * \varkappa)) = \mathcal{A}(0)$

Then, $\gamma * (\gamma * \varkappa) \in \mathcal{K}_\mathcal{A}$

Therefore, $\mathcal{K}_\mathcal{A}$ is a ∧ - ideal of $\mathcal{K}$.

Theorem (3.17):

Let $\mathcal{K}$ be a $\mathcal{BH}$-algebra and $\mathcal{A}$ be a fuzzy set of $\mathcal{K}$.Then $\mathcal{A}$ is a fuzzy ∧ - ideal of $\mathcal{K}$ if and only if $\bar{\mathcal{A}}(\varkappa) = \mathcal{A}(\varkappa) + 1 - \mathcal{A}(0)$ is a fuzzy ∧-ideal of $\mathcal{K}$.

Proof:

If $\mathcal{A}$ is a fuzzy ∧ - ideal of $\mathcal{K}$
$\Rightarrow \mathcal{A}$ is a fuzzy ideal of $\mathcal{K}$. It is clear that $\bar{\mathcal{A}}$ is a fuzzy ideal of $\mathcal{K}$.
Now, let $\varkappa, \gamma \in \mathcal{K}$

$\bar{\mathcal{A}}(\gamma * (\gamma * \varkappa)) = \mathcal{A}(\gamma * (\gamma * \varkappa)) + 1 - \mathcal{A}(0)$

$\Rightarrow \bar{\mathcal{A}}(\gamma * (\gamma * \varkappa))$

$\geq min\{\mathcal{A}(\varkappa), \mathcal{A}(\gamma)\} + 1 - \mathcal{A}(0)$

$\Rightarrow \bar{\mathcal{A}}(\gamma * (\gamma * \varkappa))$

$\geq min\{\mathcal{A}(\varkappa) + (1 - \mathcal{A}(0)), \mathcal{A}(\gamma) + (1 - \mathcal{A}(0))\}$

$\Rightarrow \bar{\mathcal{A}}(\gamma * (\gamma * \varkappa)) \geq min\{\bar{\mathcal{A}}(\varkappa) \bar{\mathcal{A}}(\gamma)\}$

Then $\bar{\mathcal{A}}$ is a fuzzy ∧ - ideal of $\mathcal{K}$.

Conversely,

Let $\bar{\mathcal{A}}$ be a fuzzy ∧ - ideal of $\mathcal{K}$

It is clear that $\mathcal{A}$ is a fuzzy ideal of $\mathcal{K}$.

Now let $\varkappa, \gamma \in \mathcal{K}$

$\mathcal{A}(\gamma * (\gamma * \varkappa)) = \bar{\mathcal{A}}(\gamma * (\gamma * \varkappa)) - 1 + \mathcal{A}(0)$

$\Rightarrow \mathcal{A}(\gamma * (\gamma * \varkappa)) \geq$
$min\{\bar{\mathcal{A}}(\varkappa), \bar{\mathcal{A}}(\gamma)\} - 1 + \mathcal{A}(0)$

$\Rightarrow \mathcal{A}(\gamma * (\gamma * \varkappa)) \geq$
$min\{\bar{\mathcal{A}}(\varkappa) - 1 + \mathcal{A}(0), \bar{\mathcal{A}}(\gamma) - 1 + \mathcal{A}(0)\}$

$\Rightarrow \mathcal{A}(\gamma * (\gamma * \varkappa)) \geq min\{\mathcal{A}(\varkappa), \mathcal{A}(\gamma)\}$

Then is a fuzzy ∧ - ideal of $\mathcal{K}$.

Proposition (3.18):

Let $\mathcal{K}$ be $\mathcal{BH}$-algebra, and $\{\mathcal{A}_j: j \in \Gamma\}$ be a family of fuzzy ∧ - ideal of $\mathcal{K}$. Then $(\bigcap_{j \in \Gamma} \mathcal{A}j)$ is a fuzzy ∧ - ideal of $\mathcal{K}$.

Proof:

It is clear that $(\bigcap_{j \in \Gamma} \mathcal{A}_j)$ is a fuzzy ideal of $\mathcal{K}$.

Now, let $\varkappa, \gamma \in \mathcal{K}$

$(\bigcap_{j \in \Gamma} \mathcal{A}_j)(\gamma * (\gamma * \varkappa))$

$= inf\{\mathcal{A}j(\gamma * (\gamma * \varkappa)), j \in \Gamma\}$

$\geq inf\{min\{\mathcal{A}_j(\varkappa), \mathcal{A}_j(\gamma)\}, j \in \Gamma\}$

$\geq min\ \{inf\ \mathcal{A}_j\ (\varkappa), inf\ \mathcal{A}_j(\gamma)\},\ j\ \in \Gamma\ \}$

$\geq\ min\ \{\ (\ \bigcap_{j\in\Gamma} \mathcal{A}_j\ )\ (\varkappa), (\ \bigcap_{j\in\Gamma} \mathcal{A}_j\ )\ (y)\}$

$\Rightarrow (\bigcap_{j\in\Gamma} \mathcal{A}_j)\ (\gamma * (\gamma * \varkappa))$
$\geq \min\{(\bigcap_{j\in\Gamma} \mathcal{A}_j)\ (x), (\bigcap_{j\in\Gamma} \mathcal{A}_j)\ (\gamma)\ \}$
$\forall \varkappa, \gamma \in \mathcal{K}$

Therefore, $(\ \bigcap_{j\in\Gamma} \mathcal{A}_j\ )$ is a fuzzy $\wedge$ - ideal of $\mathcal{K}$.

Proposition (3.19):
Let $\mathcal{K}$ be a $\mathcal{BH}$-algebra, and $\{\mathcal{A}_j\!: j\ \in\Gamma\}$ be a chain of fuzzy $\wedge$ - ideals of $\mathcal{K}$.

Then $(\ \bigcup_{j\in\Gamma} \mathcal{A}_j\ )$ is a fuzzy $\wedge$ - ideal of $\mathcal{K}$.

Proof:
It is clear that $(\ \bigcup_{j\in\Gamma} \mathcal{A}_j\ )$ is a fuzzy ideal of $\mathcal{K}$.

Now, let $\varkappa, \gamma\ \in \mathcal{K}$

$(\ \bigcup_{j\in\Gamma} \mathcal{A}_j\ )(\gamma * (\gamma * \varkappa))$

$=sup\{\ \mathcal{A}_j(\ \gamma * (\gamma * \varkappa))\ ,\ j\ \in\Gamma\ \}$

$\geq sup\{\ min(\mathcal{A}_j\ (\varkappa), \mathcal{A}_j\ (\gamma), j\ \in\Gamma\}$
[Since $\mathcal{A}_j$ is a fuzzy $\wedge$ - ideal, $\forall j\ \in\Gamma$] .
[By definition (2.16)]

$\geq min\ \{\ sup(\mathcal{A}_j\ (\varkappa), sup\ \mathcal{A}_j\ (\gamma)),\ j\ \in\Gamma\ \}$

$\geq min\ \{\ (\ \bigcup_{j\in\Gamma} \mathcal{A}_j\ )\ (\varkappa), (\ \bigcup_{j\in\Gamma} \mathcal{A}_j\ )\ (\gamma)\}$

$\Rightarrow (\ \bigcup_{j\in\Gamma} \mathcal{A}_j\ )\ (\gamma * (\gamma * \varkappa))$
$\geq min\{\ (\bigcup_{j\in\Gamma} \mathcal{A}_j\ )\ (\varkappa), (\ \bigcup_{j\in\Gamma} \mathcal{A}_j\ )\ (\gamma)\}$
$\forall \varkappa\ , \gamma\ \in \mathcal{K}$

Therefore, $(\ \bigcup_{j\in\Gamma} \mathcal{A}_j\ )$ is a fuzzy $\wedge$ - ideal of $\mathcal{K}$.

Theorem (3.20):
Let $(\mathcal{K},*,0)$and$(\Upsilon,*',0')$are two $\mathcal{BH}$-algebra, $F\!: \mathcal{K}\rightarrow \Upsilon$ be a $\mathcal{BH}$- epimorphism. If $\mathcal{A}$ is a fuzzy $\wedge$ - ideal of $\mathcal{K}$, then $F(\mathcal{A})$ is a fuzzy $\wedge$ - ideal of Υ.

Proof:
It is clear that $F(\mathcal{A})$ is a fuzzy ideal of $\mathbb{Y}$

Now, let $\gamma_1,\ \gamma_2 \in \Upsilon$.

We have $(F(\mathcal{A}))(\ \gamma_1 *'\ (\gamma_1 *'\ \gamma_2))$

$=sup\ \{\mathcal{A}\ (\varkappa_1 * (\varkappa_1 * \varkappa_2))\ |\ \varkappa_1 \in\ F^{-1}(\gamma_1),$

$\varkappa_2 \in\ F^{-1}(\gamma_2)\ and\ (\varkappa_1 * (\varkappa_1 * \varkappa_2))$

$\in F^{-1}(\gamma_1 *'(\gamma_1 *'\ \gamma_2))\}$ [By definition (2.18)]

$\geq sup\{min\{\ \mathcal{A}(\varkappa_1) : \varkappa_1 \in\ F^{-1}(\gamma_1),$
$min\{\ \mathcal{A}(\varkappa_2) : \varkappa_2 \in\ F^{-1}(\gamma_2)\}$

[Since $\mathcal{A}$ is a fuzzy $\wedge$-ideal of]

$=min\ \{\ sup\mathcal{A}(\varkappa_1) : \varkappa_1 \in\ F^{-1}(\gamma_1),$
$min\{\ \mathcal{A}(x_2) : \varkappa_2 \in\ F^{-1}(\gamma_2)\}$

$=min\ \{\ F\ (\mathcal{A})\ (y_1), F\ (\mathcal{A})(y_2)\}.$

Then $(\ F(\mathcal{A}))(\ \gamma_1 *'(\gamma_1\ *'\ \gamma_2))$

$\geq min\ \{\ F\ (\mathcal{A})\ (\gamma_1), F\ (\mathcal{A})(\ \gamma_2)\}$

Therefore, $F(\mathcal{A})$ is a fuzzy $\wedge$ - ideal of Υ.

Theorem (3.21):
If $(\mathcal{K},*,0)$ and$(\ \Upsilon,*',0'\ )$are two $\mathcal{BH}$-algebra, $F\!:\!\mathcal{K}\rightarrow \Upsilon$ be a $\mathcal{BH}$-homomorphism. If $\mathfrak{B}$ is a fuzzy $\wedge$ - ideal of Υ, then $F^{-1}(\mathfrak{B})$ is a fuzzy $\wedge$ - ideal of $\mathcal{K}$.

Proof
It is clear that $F^{-1}(\mathfrak{B})$ is a fuzzy ideal of $\mathcal{K}$.

Now, let $\varkappa_1$, $\varkappa_2 \in \mathcal{K}$. Then

$(F^{-1}(\mathfrak{B}))(\ \varkappa_1 * (\varkappa_1 * \ \varkappa_2))$

$=\mathfrak{B}\ (F(\varkappa_1 * (\varkappa_1 * \ \varkappa_2))$

$=\{\ \mathfrak{B}\ (F(\varkappa_1) *'\ (F\ (\varkappa_1) *'\ F\ (\varkappa_2))\}$

[Since F is a homomorphism]

$\geq min\ \{\ \mathfrak{B}(\ F\ (\varkappa_1)), \mathfrak{B}\ (F\ (\varkappa_2))\}.$

[Since ($\mathfrak{B}$ be a fuzzy $\wedge$ - ideal)]

$=min\ \{\ F^{-1}(\mathfrak{B})(\ \varkappa_1), F^{-1}(\mathfrak{B})(\ \varkappa_2)\}.$

So, $(F^{-1}(\mathfrak{B}))(\ \varkappa_1 * (\ \varkappa_1 * \ \varkappa_2))$
$\geq min\ \{\ F^{-1}(\mathfrak{B})(\ \varkappa_1), F^{-1}(\mathfrak{B})(\ \varkappa_2)\}.$

Thus, $F^{-1}(\mathfrak{B})$ is a fuzzy $\wedge$ - ideal of $\mathcal{K}$.

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