On Closed Dual Rickart Modules

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Abstract--- *The notion of dual Rickart modules has been studied lately. In this article, we continue investigate and study several properties of closed dual Rickart modules which explain by Ghawi Th.Y. as a proper generalization the idea of the dual Rickart modules and as a dual concept of closed Rickart modules. A right R-module M is called closed dual Rickart if, for each $\varphi \in S = End(M)$, $Im\varphi$ is a closed sub module of M . For a module M, we verify that M is closed dual Rickart and closed simple if and only if M is coquasi-Dedekind and Extending . We also establish that if, M_1 and M_2 are closed simple modules such that $M = M_1 \oplus M_2$ is closed dual Rickart and M_2 is projective, then either $Hom(M_1, M_2) = 0$ or $M_1 \cong M_2$. Furthermore, we give a counter example to show that the direct sums of modules is not closed under closed dual Rickart . We also give a necessity station for a finite direct sum of closed dual Rickart modules to be closed dual Rickart . Other results are provided in this work. Examples to illustrate some results and converses are given.*

Keywords-- c-d-Rickart Modules; d-Rickart Modules; Closed Simple Modules; Epi-retractable Modules.

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1. INTRODUCTION

The notions of closed Rickart and closed dual Rickart modules are appeared in [2]. A right R-module M is called a closed Rickart (closed dual Rickart) module (for short c-Rickart, resp. c-d-Rickart) if, the kernel (resp. the image) in M of any single element of $S = End(M)$ is closed in M . A ring R is c-Rickart (c-d-Rickart) if, R_R is a c-Rickart (resp. c-d-Rickart) module. A right R-module M is called dual Rickart (for short d-Rickart) if, the image in M of any single element φ of $S = End(M)$ is generated by idempotent element of S (i.e. $Im\varphi \leq^{\oplus} M$) [8]. It is evident that every d-Rickart module is c-d-Rickart, but not conversely . The concept of closed Rickart modules has been studied extensively by Hadi I.M-A and Ghawi Th.Y. in [5]. In this article, our objective is to endure studying the notion of c-d-Rickart modules . Moreover, this paper is structured of two section. In section 2, we supplied some characterizations and investigate its properties . It is shown that c-d-Rickart property inherits by its direct summand. A module M is called epi-retractable if, every submodule of M is a homomorphic image of M [3]. We confirm that the concepts c-d-Rickart modules and semisimple modules are coincide under the concept of epi-retractable modules . As a consequence, we obtain that $End(M)$ is an integral domain if, a module M is c-d-Rickart and closed simple . Where, a module M is said to be closed simple if, the trivial submodules are the only closed in M [2]. In section 3, it was shown in (Example 3.1), a direct sum of c-d-Rickart modules is not c-d-Rickart, in general,so we obtain a chara-cterization for an arbitrary direct sum of c-d-Rickart modules to be c-d-Rickart, provided that each sub-module is fully invariant in the direct sum . Other results in this section are provided. Throughout this article, $R$is an associative ring with identity, and M is a unital right R-module . For a right R-module M,$S = End(M)$will denote the endomorphism ring of M . The notations$N \subseteq M, N \leq M$,$N \leq_e M$, $N \leq^c M$, or$N \leq^{\oplus} M$ mean that N is a subset, a sub module, an essential sub module, a closed sub module, or a direct summand of M, respectively .$E(M)$denotes the injective hull of M ,we will denotes$r_R(M) = \{r \in R | Mr = 0\}$,also,$r_S(N) = \{\varphi \in S | \varphi N = 0\}$for $N \leq M$.

2. CLOSERD DUAL RICKART MODULES

In this section, we investigate and study the notion of closed dual Rickart module (c-d-Rickart in short), and obtain some of its fundamental properties. A right R-module M is called c-d-Rickart if, the image in M of any single element of $End(M)$ is a closed sub module in M . Several relations between a c-d-Rickart module and other classes of modules are obtained in this section.

REMARKS AND EXAMPLES 2.1

(i) Let M be a module. If $End(M)$is a von Neumann regular ring, then M is$c$-d-Rickart.

Proof.
Since $S = End(M)$ is a regular ring, then by [11, Th. 4] $Im\varphi \leq^{\oplus} M$, so $Im\varphi$ is closed in M for each $\varphi \in S$, and hence the result is obtained .■

(ii) Every semi simple (simple) module is c-d-Rickart. The Z-module Q is c-d-Rickart (because $End_Z(Q) \cong Q$is a regular ring, so by(i) $Q_Z$is a c-d-Rickart module) but it is neither semi simple nor simple .

(iii) The c-d-Rickart property transforms under an isomorphism .

Proof. Let M,Nbe$R$-modules such that M is c-d-Rickart, and $\varphi: M \longrightarrow N$ is an isomorphism . Assume$\psi \in End(N)$such that $Im\psi \leq_e L$ in N, then$\varphi^{-1}(Im\psi) \leq_e \varphi^{-1}(L)$in M. But$\varphi^{-1}(Im\psi) = \varphi^{-1}(\psi(N)) = \varphi^{-1}\psi(\varphi(M)) = Im(\varphi^{-1}\psi\varphi)$. Since M is a c-d-Rickart module and $\varphi^{-1}\psi\varphi \in End(M)$,then$\varphi^{-1}(Im\psi) = Im(\varphi^{-1}\psi\varphi) \leq^c M$, but$\varphi^{-1}(Im\psi)$ is essential in $\varphi^{-1}(L)$ of M, so $\varphi^{-1}(Im\psi) = \varphi^{-1}(L)$ this implies that $Im\psi = L$, and hence$Im\psi \leq^c N$. ■

The following example reveal the difference between concepts c-d-Rickart and c-Rickart modules.

Example 2.2 By [5] we have that Z-module Z is c-Rickart, but it is easy to see that Z_Z not c-d-Rickart . Also, the Z-module Z_{p^∞} is c-d-Rickart (in fact, for all $\varphi \in End_Z(Z_{p^\infty})$, $Im\varphi = Z_{p^\infty}$),but it is not a c-Rickart Z-module, as the following shows: assume $\varphi \in End_Z(Z_{p^\infty})$ defined by $\varphi(x) = xp$for all$x \in Z_{p^\infty}$, and p is prime, then$(0 \neq)Ker\varphi = \langle \frac{1}{p}+z \rangle \neq Z_{p^\infty}$, but Z_{p^∞} is closed simple, this means $Ker\varphi$ is not closed in Z_{p^∞} .

Example 2.3 By Example 2.2, the Z-module $Z_{p^\infty}$is c-d-Rickart, where p is prime, but the sub module Z_{p^2} is not a c-d-Rickart Z-module . Also, even though Z_{p^2} is not a c-d-Rickart Z-module, but the sub module $pZ_{p^2} \cong Z_p$is a c-d-Rickart Z-module . This example show that the c-d-Rickart property does not always transfer from a module to each of its submodules or conversely .

Proposition 2.4. Every direct summand of a c-d-Rickart module is c-d-Rickart .

Proof. Let M be a c-d-Rickart module, and $N \leq^{\oplus} M$, then$M = N \oplus K$ for some$K \leq M$. Let $\varphi \in End(N)$.Consider the sequence $M \overset{\rho}{\rightarrow} N \overset{\varphi}{\rightarrow} N \overset{i}{\rightarrow} M$ where ρ is a projection map, and i is the inclusion mapping . Since M is c-d-Rickart and$i \circ \varphi \circ \rho \in End(M)$, then $Im(i \circ \varphi \circ \rho)$ is closed in M. But$Im(i \circ \varphi \circ \rho) = Im\varphi$, thus$Im\varphi \leq^c M$, but $Im\varphi \subseteq N$, so by [7, Prop. 6.24(1)]$Im\varphi \leq^c N$. Therefore N is c-d-Rickart. ■

However, every submodule of a semi simple module is c-d-Rickart.

Now, we need to recall of definitions for a module M such as:

(C_1) every submodule of M is essential in a direct summand of M ;

(C_2) if a submodule A of M is isomorphic to a direct summand of M, then A is a direct summand of M ; and

(C_3) if A and B are direct summands of M such that $A \cap B = 0$, $A \oplus B$ is a direct summand of M .

Modules with the C_1 property are called extending (or CS)-modules . A module M is an extending module if and only if every closed sub module of M is a direct summand [1]. A module M is called continuous if M has C_1 and C_2, and quasi-continuous if M has C_1 and C_3 [1]. An R-module M is said to be quasi-injective if, for any sub module L of M, any $\varphi \in Hom_R(L, M)$ can be extended to an endomorphism of M [7].

Corollary 2.5. Let M be an extending R-module . If M is a c-d-Rickart module then so $Im\varphi$ for every $\varphi \in End(M)$.

Proof. Assume that $\varphi \in End(M)$, then $Im\varphi \leq^c M$, as M is a c-d-Rickart module . By extending of M, $Im\varphi \leq^{\oplus} M$. Thus by Proposition 2.4, $Im\varphi$ is a c-d-Rickart module. ■

Corollary 2.6. If R is a c-d-Rickart ring, then so is eR for every $e^2 = e \in R$.

Proof. Obvious. ■

The following Proposition gives a requisite under which the conceptsd-Rickart and c-d-Rickart modules are coincide.

Proposition 2.7. Every extending c-d-Rickart moduleis d-Rickart .

Proof. It is easy to check. ■

Corollary 2.8. Every injective(or quasi-injective, continuous, quasi-continuous) c-d-Rickart module is d-Rickart .

Remark 2.9. If M is a c-d-Rickart module, then $\frac{M}{N}$is c-d-Rickart for each direct summand N of M . In particular, M is a semi simple module implies that$\frac{M}{N}$ is c-d-Rickart for all $N \leq M$.

We will insert a condition which can be considered as a dual concept of($\boldsymbol{I_c}$)condition in [5] called dual ($\boldsymbol{I_c}$) for an R-module M, as following :

- For all $B \leq M$ with $B \cong A \leq^{\oplus} M$, then B is closed in M … dual ($\boldsymbol{I_c}$)

Proposition 2.10. Every c-d-Rickart module satisfy the dual($\boldsymbol{I_c}$). The converse hold, whenever $Im\varphi \cong N \leq^{\oplus} M$ for each $\varphi \in End(M)$.

Proof. If M is a c-d-Rickart module . Let $A, B \leq M$ with $B \cong A \leq^{\oplus} M$, then there exists an isomorphism $\varphi: A \to B$. Consider the sequence $M \xrightarrow{\rho} A \xrightarrow{\varphi} B \xrightarrow{i} M$ where ρ is a projection map, and i is the inclusion mapping. Since M is a c-d-Rickart module and $i \circ \varphi \circ \rho \in End(M)$, then $Im(i \circ \varphi \circ \rho) \leq^c M$. But $Im(i \circ \varphi \circ \rho) = i \circ \varphi(\rho(M)) = \varphi(A) = B$.Thus $B \leq^c M$ and so M satisfies the dual ($\boldsymbol{I_c}$). Conversely, let $\psi \in End(M)$, so by assumption, $Im\psi \cong N \leq^{\oplus} M$. Since M satisfy the dual ($\boldsymbol{I_c}$), then $Im\psi \leq^c M$ and that ends the proof .■

Corollary 2.11. If M is a c-d-Rickart module with$N \cong L \leq^{\oplus} M$for all $N \leq M$, then M is semi simple .

Proof. Since M is a c-d-Rickart module, then by Proposition 2.10 M satisfy the dual ($\boldsymbol{I_c}$) . Let $N \leq M$, so by condition,N is isomorphic to a direct summand of M, hence $N \leq^c M$this mean all its sub modules are closed . Therefore M is semi simple ■

Notice that the Z-module Z does not satisfy the dual ($\boldsymbol{I_c}$), since $2Z \cong Z \leq^{\oplus}$ Z but $2Z$ is not closed in Z (in fact, $2Z \leq_e Z$). This is another reason to make $Z_Z$is not c-d-Rickart.

Recall that a module M is called epi-retractable if, every sub module of M is a homomorphic image of M [3]. An R-module M is said to be coquasi-Dedekindif, every proper sub module N of M, $Hom(M,N) = 0$ (Equivalently, M is a coquasi-Dedekind module if, every nonzero endomorphism of M is an epimorphism [17]. A module M is said to be closed simple if, the trivial sub modules are the only closed sub modules of M [2].

It is well understood that a semi simple module imp lies c-d-Rickart . Now, we give a condition under which these concepts are coincide.

Proposition 2.12. Let M be an epi-retractable module . Then M is c-d-Rickart if and only if M is semi simple .

Proof. Suppose that M is a c-d-Rickart module, $S = End(M)$and$N \leq M$. If $N = 0$then N is closed. Suppose $(0 \neq)N \leq M$, so there exists an$(0 \neq)\varphi \in S$ such that$Im\varphi = N$, by epi-retractablity of M . But M is a c-d-Rickart module, so$Im\varphi = N \leq^c M$, and hence M is a semi simple module .■

Corollary 2.13. If M is an epi-retractable and c-d-Rickart module, then $N$is c-d-Rickart for all $N \leq M$.

Proof. It is easy check. ■

The condition "epi-retractable" in above Corollary is inevitable, for example: it is well known that Q_Z is c-d-Rickart, but it is not semi simple. Note that Q_Z is not epi-retractable, see [3, Rem. 2.12].

Proposition 2.14. The following arrangements are equivalent for an R-module M:

(i) M is a c-d-Rickart and closed simple module ;

(ii) M is a coquasi-Dedekind and extending module.

Proof.$(i) \Longrightarrow (ii)$ It is clear that a closed simple module implies extending . Let $\varphi \in End(M)$ and $\varphi \neq 0$, $Im\varphi \leq^c M$, as M is a c-d-Rickart module . We have M is closed simple and $\varphi \neq 0$implies that $Im\varphi = M$. So M is a coquasi-Dedekind module .

$(ii) \Longrightarrow (i)$ Let $\varphi \in End(M)$, then $Im\varphi = M$, since M coquasi-Dedekind . Thus $Im\varphi \leq^c M$ and so M is a c-d-Rickart module. Furthermore, if$0 \neq N \leq^c M$ then$N \leq^{\oplus} M$, as M is an extending module.Consider the sequence $M \xrightarrow{\rho} N \xrightarrow{i} M$ where ρ is a projection map, and i is the inclusion mapping. So$(0 \neq)i \circ \rho \in End(M)$, again M is a coquasi-Dedekind module, $N = Im(i \circ \rho) = M$. Therefore M is a closed simple module. ■

The condition "closed simple" in Proposition 2.14, is inevitable to prove a module M is coquasi-Dedekind, for example: it is clear that the Z-module$Z_6$is c-d-Rickart, but it is not coquasi-Dedekind (since there is an $\varphi \in End(Z_6)$ defined by $\varphi(\overline{x}) = 3\,\overline{x}$which is not epimorphism) . Note that the Z-module Z_6 is not closed simple.

An R-module M is said to be Co-Hopfian if, every monomorphism $\varphi \in End(M)$ is an isomorphism [16]. Recall that an R-module M is said to be retractable if, for each $(0 \neq)N \leq M$, there exists a nonzero $\varphi \in End(M)$ such that $\varphi(M) \subseteq N$ [6].

Corollary 2.15. Every c-d-Rickart closed simple module is Co-Hopfian .

Proof. Follows directly by Proposition 2.14 . ■

Proposition 2.16. Let M be an R-module . Then M is c-d-Rickart closed simple and retractable if and only if M is simple .

Proof.⇒) Let $(0 \neq)N \leq M$. Since M is retractable, so there exists $(0 \neq)\varphi \in End(M)$ such that $\varphi(M) \subseteq N$. By Proposition 2.14, M is a coquasi-Dedekind module, so $N \supseteq Im\varphi = M$, and hence $N = M$ which impliesthe result .

$\Leftarrow$) Clear. ■

Proposition 2.17. If M is a c-d-Rickart retractable R-module, then every a nonzero$N \leq M$, there exists $(0 \neq)K \leq^{c} M$ such that $K \subseteq N$.

Proof.$\Rightarrow$) Assume M is a retractable R-module and $(0 \neq)N \leq M$, so there exists a nonzero endomorphism$\varphi: M \rightarrow M$ such that $\varphi(M) \subseteq N$ .Since M is a c-d-Rickart R-module, $Im\varphi \leq^{c} M$. By putting $Im\varphi = K$, we get the result. ■

Proposition 2.18. Let M be an R-module and $End(M)$ is a commutative ring . If M is a c-d-Rickart and closed simple R-module, $End(M)$is an integral domain .

Proof. Let $\varphi, \psi \in S = End(M)$ such that $\varphi \circ \psi = 0$. Assume that$\psi \neq 0$, then by Proposition 2.14, M is a coquasi-Dedekind R-module, then $Im\psi = M$. Thus $0 = \varphi \circ \psi(M) = \varphi(Im\psi) = \varphi(M)$, so $\varphi = 0$. Hence $S = End(M)$ is an integral domain.■

The converse of Proposition 2.18, need not be true, in usual, such as example: it is clear that $End(Z)_Z \cong Z$ is an integral domain, also Z_z is closed simple, but it is not c-d-Rickart .

Proposition 2.19. Let$R$ be an integral domain and M be a closed simple faithful R-module . If M is ac-d-Rickart module, M is divisible .

Proof. Let $(0 \neq)r \in R$. Since M is faithful, $Mr \neq 0$. Consider $(0 \neq)\varphi \in End(M)$ defined by $\varphi(m) = mr$ for all $m \in M$. Since M is a c-d-Rickart and closed simple module, so by Proposition 2.14, M is coquasi-Dedekind, and hence$M = Im\varphi = Mr$. ■

Corollary 2.20. Let R be an integral domain and M be a torsion free closed simple R-module . If M is ac-d-Rickart module, M is injective .

Proof. It is easy to check. ■

Moreover, Z_Z is torsion free and closed simple over an integral domain$Z$, while Z_Z neither c-d-Rickart nor injective module.

Recall that an R-module M is said to be scalar if, for all $\varphi \in End(M)$, there exists an $r \in R$ such that $\varphi(m) = mr$ for all $m \in M$ [12].

Proposition 2.21. Let M be a scalar and faithful R-module . Then $S = End(M)$ is c-d-Rickart if and only if R is c-d-Rickart.

Proof. Since M is a scalar R-module, then by [9, Lemma 6.2] we have that $S = End(M) \cong R/r_R(M)$. But M is faithful, so $S = End(M) \cong R$. Hence, the result is follow. ■

Proposition 2.22. Let M be an R-module, and$\overline{R} = R/r_R(M)$. Then M is c-d-Rickart as R-module if only if M is c-d-Rickart as $\overline{R}$-module .

Proof. It is easy to check. ■

3. DIRECT SUMS OF CLOSED DUAL RICKART MODELES

In this section, we show that the conception of c-d-Rickart modules is not closed under direct sums. We focus on when the direct sum of a finite c-d-Rickart modules is also c-d-Rickart . We give a number of results concerning direct sums of c-d-Rickart modules .

Example 3.1. The Z-modules Z_p and Z_{p^∞} are both c-d-Rickart, where p is prime . Let $M = Z_{p^\infty} \oplus Z_p$ as Z-module . Consider the endomorphism $\psi \in End(M)$such that $\psi(\hat{x}, \overline{y}) = (\hat{\overline{y}}, 0)$ for all$\hat{x} \in Z_{p^\infty}$ and $\overline{y} \in Z_p$ .Hence $Im\psi = Z_p \oplus (0)$which is not closed in M. In fact, $Z_p \oplus (0)$is essential in $Z_{p^\infty} \oplus (0)$ of M.

Recall that a submodule N of a module M is called a fully invariant submodule if, $\varphi(N) \subseteq N$ for each $\varphi \in End(M)$ [15].

The following result gives a condition under which the direct sums of c-d-Rickart module is also c-d-Rickart.

Proposition 3.2."Let M_i be a fully invariant submodule of a module $M = \oplus_{i\in I} M_i$for all $i \in (I$ is any index set)". "Then M is c-d-Rickart if and only if M_i is c-d-Rickart, for all$i \in I$".

Proof."The necessity follows from Proposition 2.4". "Conversely, assume that $\varphi \in End(M)$,$\varphi = (\varphi_{ij})$where$\varphi_{ij} \in Hom(M_i, M_j)$". "Since M_i is fullyinvariant (for all $i \in I$), so by [10, Lemma 1.9] $Hom(M_i, M_j) = 0$for$i \neq j$, therefore $Im\varphi = \oplus_{i\in I} Im\varphi_{ii}$". "Since M_i is a c-d-Rickart module and $\varphi_{ii} \in End(M_i)$, then$Im\varphi_{ii} \leq^c M_i$for $i \in I$, implies$\oplus_{i\in I} Im\varphi_{ii} \leq^c \oplus_{i\in I} M_i$, this mean $Im\varphi$ is closed in M, and hence M is a c-d-Rickart module".■

Let M and N be R-modules, then M is called closed dual Rickart relative to N (for short M is N-c-d-Rickart) if, for any $\varphi \in Hom(M, N)$, $Im\varphi$ is closed in N . In especial, M is a c-d-Rickart module if and only if $M$is M-c-d-Rickart [2].

Remarks and Examples 3.3.

(i) Let M and N be R-modules, $T = Hom(M, N)$. If $r_T(M) = T$, then M is N-c-d-Rickart module .

Proof. If $\varphi \in T = Hom(M, N)$, then $\varphi \in r_T(M)$, so$\varphi(M) = 0$ and hence $Im\varphi \leq^c M$. Therefore M is N-c-d-Rickart . ■

(ii) For any right R-module M . If N is a semisimple (simple) module, then M is N-c-d-Rickart module . Notice, if p is a prime number,$Z_Z$is Z_p-c-d-Rickart .

(iii) Let M and N be R-modules with $Hom(M, N) = 0$, then M is N-c-d-Rickart R-module . Moreover, $Q_Z$is Z-c-d-Rickart, in fact$Hom_Z(Q, Z) = 0$.

(iv) If M is acoquasi-Dedekind R-module, then M is N-c-d-Rickart R-module for every a proper submodule N of M .

Proof. Since M is a coquasi-Dedekind R-module, so $Hom(M, N) = 0$ for all a proper submodule N of M . So by (iii), the result is follow. ■

Proposition 3.4. The following arrangement sare equivalent for a module M :

(i) M is a c-d-Rickart module ;

(ii) for $B \leq^{\oplus} M$, B is A-c-d-Rickart, for all $A \leq M$;

(iii) for $B \leq^{\oplus} M$,if $L \leq^c M$ with $\varphi \in Hom(M, L)$, then $Im(\varphi|_B)$ is closed in L ;

(iv) for$L \leq^c M$, M is L-c-d-Rickart module .

Proof.$(i) \Longrightarrow (ii)$ Let$A \leq M$. If $B \leq^{\oplus} M$,$B = eM$ for some $e^2 = e \in End(M)$. Assume $\psi: B \longrightarrow A$is a homomorphism, so$\psi e \in End(M)$. Since M is a c-d-Rickart module, so $\psi(B) = \psi(eM) = \psi e(M) \leq^c M$,
so $Im\psi \leq^c M$, and hence $Im\psi \leq^c A$. Therefore B is A-c-d-Rickart.

$(ii) \Longrightarrow (iii)$Assume that $B \leq^{\oplus} M$ and $L \leq^c M$. Let$\varphi \in Hom(M, L)$, $\varphi|_B \in Hom(B, L)$. By (ii), B is L-c-d-Rickart, implies $Im(\varphi|_B)$ is a closed submodule of L.

$(iii) \Longrightarrow (iv)$Obvious.

$(iv) \Longrightarrow (i)$ Follows directly by taking $L = M$.■

Proposition 3.5. Let M_1,M_2 be R-modules withM_2 is closed simple . If $M_1 \oplus M_2$ is ac-d-Rickart module, then either $Hom(M_1, M_2) = 0$, or every nonzero homomorphism from M_1 to M_2 is an epimorphism .

Proof."Assume $Hom(M_1, M_2) \neq 0$ .So, there exists a nonzero homomorphism $\varphi: M_1 \longrightarrow M_2$. Consider the sequence $M \xrightarrow{\rho} M_1 \xrightarrow{\varphi} M_2 \xrightarrow{i} M$where $M = M_1 \oplus M_2$,ρ is a projection map and i is the inclusion mapping . Since M is c-d-Rickart, $Im\varphi = Im(i \circ \varphi \circ \rho) \leq^c M$, so $Im\varphi \leq^c M$, but $Im\varphi \subseteq M_2$ implies $Im\varphi \leq^c M_2$ by [7, Prop. 6.24(1)]. However, M_2 is a closed simple module and $\varphi \neq 0$, hence $Im\varphi = M_2$. Therefore φ is an epimorphism. ■

Corollary 3.6. Let M_1,M_2 be R-modules with M_2 is closed simple . If $M_1 \oplus M_2$ is a c-d-Rickart R-module, then M_1 is M_2-c-d-Rickart .

Proof. Assume that M is a c-d-Rickart module and M_2 is closed simple . Let $\varphi \in Hom(M_1, M_2)$, so by Proposition 3.5, either $Im\varphi = 0$ or $Im\varphi = M_2$, and from two cases, $Im\varphi \leq^c M_2$. ■

Proposition 3.7. Let M_1 be any R-module and M_2 be a closed simple R-module . If $M_1 \oplus M_2$ is a c-d-Rickart R-module and $Hom(M_1, M_2) \neq 0$, then M_2 is a coquasi-Dedekind R-module .

Proof. Assume M is a c-d-Rickart R-module with $Hom(M_1, M_2) \neq 0$. By Proposition 3.5, the nonzero R-homomorphism $\varphi: M_1 \longrightarrow M_2$ is an epimorphism. Let $\psi \in End(M_2)$ and $\psi \neq 0$, then $(0 \neq)\psi \circ \varphi \in Hom(M_1, M_2)$, again by Proposition 3.5, $\psi \circ \varphi$ is an epimorphism; this mean that $M_2 = \psi \circ \varphi(M_1) = \psi(\varphi(M_1)) = \psi(M_2) = Im\psi$, and hence ψ is an epimorphism, so the result is follow . ■

Corollary 3.8. If $M \oplus M$ is a c-d-Rickart R-module, then M is coquasi-Dedekind, whenever M is a closed simple R-module .

Proof. It is clear . ■

Proposition 3.9. Let M_1,M_2 be two closed simple R-modules . If $M_1 \oplus M_2$ is c-d-Rickart, and M_2 is projective, either $Hom(M_1, M_2) = 0$ or $M_1 \cong M_2$.

Proof. It follows by Proposition 3.5, we have either $Hom(M_1, M_2) = 0$ or every nonzero ohomomorphism $\varphi: M_1 \longrightarrow M_2$ is an epimorphism . By projectivity of M_2, $Ker\varphi \leq^{\oplus} M_1$, thus $Ker\varphi \leq^c M_1$. But M_1 is closed simple and $\varphi \neq 0$, this implies $Ker\varphi = 0$, hence φ is (1-1) and $M_1 \cong M_2$. ■

Following [13], a module M is said to be dual Baer if, for every $N \leq M$, there exists an idempotent $e \in S = End(M)$ such that $D(N) = eS$, where $D(N) = \{\varphi \in S | Im\varphi \subseteq N\}$. It is easy to see that every dual Baer module is dual Rickart .

However, we can introduce the following Proposition .

Proposition 3.10. For a ring R , consider the following arrangements:

(i) R is a dual Baer ring ;

(ii) R is a semisimple ring ;

(iii) every R-module is dual Baer ;

(iv) every R-module is c-d-Rickart ;

(v) every flat R-module is c-d-Rickart ;

(vi) every projective R-module is c-d-Rickart ;

(vii) every free R-module is c-d-Rickart ;

($viii$) every free module $R^{(R)}$ is a c-d-Rickart R-module .

Then $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Longrightarrow (iv) \Longrightarrow (v) \Longrightarrow (vi) \Longrightarrow (vii) \Longrightarrow (viii)$. If R is extending, $(viii) \Leftrightarrow (ii)$.

Proof. $(i) \Leftrightarrow (ii)$, $(ii) \Leftrightarrow (iii)$ By [13, Cor. 2.9 and Cor. 2.10] .

$(iii) \Longrightarrow (iv)$ Since, every dual Baer R-module is dual Rickart, so it is a c-d-Rickart R-module .

$(iv) \Longrightarrow (v) \Longrightarrow (vi) \Longrightarrow (vii) \Longrightarrow (viii)$ Obvious.

$(viii) \Leftrightarrow (ii)$ Let I_R be an ideal, so there exists $\varphi: F \longrightarrow I_R$ is an epimorphism(i.e. $Im\varphi = I_R$) and F is a free R-module . We have F is a direct summand of $R^{(R)}$ (since F is a free R-module), but $R^{(R)}$ is a c-d-Rickart R-module, then by Proposition 2.4 , F_R is a c-d-Rickart module. Consider the sequence $F \overset{\varphi}{\rightarrow} I \overset{i}{\rightarrow} R \overset{j}{\rightarrow} R^{(R)} \overset{\rho}{\rightarrow} F$, where ρ is a projection map, and i, j are the inclusion maps . Since F_R is a c-d-Rickart module and $\rho \circ j \circ i \circ \varphi \in End(F_R)$, then $I_R = Im(\rho \circ j \circ i \circ \varphi) \leq^c F_R$, but $F_R \leq^c R^{(R)}$ (since $F_R \leq^{\oplus} R^{(R)}$), so $I_R \leq^c R^{(R)}$ and hence $I \leq^c R$ (since $I \subseteq R \subseteq$

$R^{(R)}$), thusby extending property of R, we have $I \leq^{\oplus} R$and that ends the proof .■

A ring R is called hereditary if, every factor module of every injective module is also injective [7].

Theorem 3.11. For a ring R,R is hereditary if and only if every injective R-module is c-d-Rickart .

Proof. Assume that M is an injective module over a hereditary ring R such that $\varphi \in End(M)$. So $\frac{M}{Ker\varphi} \cong Im\varphi$, but $\frac{M}{Ker\varphi}$ is an injective R-module, by hereditary of R, thus $Im\varphi$ is an injective sub module of M, so $Im\varphi \leq^{c} M$and hence M is a c-d-Rickart R-module . Conversely, let M be an injective R-module and $N \leq M$. Since $E(\frac{M}{N})$ is injective, then $M \oplus E(\frac{M}{N})$ is so injective, thus by assumption, $M \oplus E(\frac{M}{N})$ is a c-d-Rickart R-module . Also, we have $M \leq^{\oplus} M \oplus E(\frac{M}{N})$ and $E(\frac{M}{N}) \leq M \oplus E(\frac{M}{N})$ this implies M is $E(\frac{M}{N})$-c-d-Rickart R-module, by Proposition 3.4 . Now, consider the homomorphism$\psi: M \longrightarrow E(\frac{M}{N})$ defined by $\psi(m) = m + N$ for all $m \in M$. Thus $\frac{M}{N} = Im\psi \leq^{c} E(\frac{M}{N})$, but $E(\frac{M}{N})$ is injective, then $\frac{M}{N}$ is so injective . Therefore R is a hereditary ring.■

Recall that an R-module M is said to have the summand (closed) sum property (for short SSP(CSP)) if, the sum of any two summand (closed) sub modules of M is again summand (closed) [14] and [4] resp.

However, the author Ghawi, Th.Y. in [2, Cor. 1.5.15] presented the following result .

Corollary 3.12. The following arrangements are equivalent for a ring R :

(i) R is a hereditary ring ;

(ii) all injective R-modules have the SSP ;

(iii) all injective R-modules have the CSP .

By combining Theorem 3.11 and Corollary 3.12, we get the following Corollary .

Corollary 3.13. The following arrangements are equivalent for a ring R :

(i) R is a hereditary ring ;

(ii) every injective R-module is a c-d-Rickart module ;

(iii) every injective R-modules have the SSP ;

(iv) every injective R-modules have the CSP .

Proposition 3.14 If M is a closed simple module, then $M \oplus N$ is not a c-d-Rickart module for every a proper sub module N of M .

Proof. Assume $\grave{M} = M \oplus N$ is a c-d-Rickart module, this means$\grave{M}$ is $\grave{M}$-c-d-Rickart module . But, we have N is a direct summand of $\grave{M}$ and $M \leq \grave{M}$, so by Proposition 3.4, $N$is M-c-d-Rickart module . Consider the inclusion map $\varphi: N \longrightarrow M$, so$Im\varphi \leq^{c} M$, but M is closed simple and $\varphi \neq 0$, hence$N = Im\varphi = M$which is a contradiction . ■

By applying Proposition 3.14,$Q \oplus Z$ is not c-d-Rickart Z-module, since$Q$ as Z-module is closed simple and Z is a proper submodule of Q.

REFERENCES

[1] Dung, N.V.; Huyn, D.V.; Smith, P.F. and Wisbauer, R., Extending Modules, Pitman Research notes in math. Series, 313, Longman Scientific and Technical: Harlow, 1994.

[2] Ghawi, Th.Y., Modules With Closed Intersection (Sum) Property, Ph.D. Thesis, University Of AL-Mustansiriyah, Iraq, 2015.

[3] Ghorbani, A.; Vedadi, M.R.,Epi-Retractable Modules And Some Applications, Bull. Iranian Math. Soc. 35, No.1, 2009, 155-166.

[4] Hadi, I.M-A.; Ghawi, Th.Y., Modules With The Closed Sum Property, Int. Math. Forum, Vol.9, No.32, 2014, 1539-1551.

[5] Hadi, I.M-A.; Ghawi, Th.Y., On Closed Rickart Modules, 2016, to appear.

[6]Khuri, S.M., Endomorphism Rings And Lattice Isomorphisms, J. Algebra, No.59, 1979,401-408.

[7]Lam, T.Y., Lectures On Modules And Rings, Springer-Verlag-Berlin, Heidelberg, Newyork, 1999.

[8]Lee, G.; Rizvi, S.T. and Roman, C.S.,Dual Rickart Modules, Comm. Algebra, 39, 2011, 4036-4058.

[9] Mohamed-Ali, E.A., On Ikeda-Nakayama Modules, Ph.D. Thesis, University Of Baghdad, Iraq, 2006.
[10] Ozcan, A.C.; Harmanci, A. and Smith, P.F., Duo Modules, Glasgow math. J, 48, 2006,533-545.
[11] Rangaswamy, K. M., Abelian Groups With Endomorphic Images Of Special Types, J. Algebra,6, 1967, 271–280.
[12] Shihab, B.N., Scalar Reflexive Modules, Ph.D. Thesis, University Of Baghdad, Iraq, 2004.
[13] Tutuncu, D.K.; Tribak, R., On Dual Baer Modules, Glasgow Math. J., 52(2), 2010, 261-269.
[14] Valcan, D.; Napoca, C.,Modules With The Summand Sum Property, Czechoslovak Math. J., Vol.53, No.2, 2003, 277-287.
[15] Wisbauer, R., Foundations Of Module And Ring Theory, reading, Gordon and Breach Science Publishers, 1991.
[16] Xue, W., Hopfian And Co-Hopfian Modules, Comm. In Algebra, 23(4), 1995, 1219-1229.
[17] Yaseen, S.M., Coquasi-Dedekind Modules, Ph.D. Thesis, University of Baghdad, Iraq, 2003.