Analytical Solutions of the One-Dimensional Volterra Integro-Differential Equations within Local Fractional Derivative

Hassan Kamil Jassim
Department of Mathematics
Thi-Qar University
Nasiriyah, Iraq
hassan.kamil@yahoo.com

Ammar Ali Neamah
Department of Mathematics
Kufa University
Najaf, Iraq,
ammara.meamah@uokufa.edu.iq

Abstract—*The one-dimensional Volterra Integro-Differential Equations of the second kind associated with local fractional derivative operators are investigated. Analytical solutions are obtained by using the local fractional variational Iteration method (LFVIM). The method in general is easy to implement and yields good results. Illustrative examples are included to demonstrate the validity and applicability of the new technique.*

Keywords—Integral equations; Local fractional Volterra integro-differential equation; Local fractional variational iteration method; Local fractional operator; Analytical solutions.

1. INTRODUCTION

The theory of local fractional calculus is one of useful tools to process the fractal and continuously non differentiable functions [1-8]. It was successfully applied in local fractional Fokker–Planck equation [1], local fractional telegraph equation [2], local fractional wave equation [3], fractal-time dynamical systems [4], fractal elasticity [5], local fractional diffusion equation [8], local fractional Laplace equation [9], local fractional integral equations [10-12].

Recently, the local fractional variational iteration method [16] is derived from local fractional operators [6-15]. The method, which accurately computes the solutions in a local fractional series form or in an exact form, presents interest to applied sciences for problems where the other methods cannot be applied properly.This paper is organized as follows. In Section 2, the basic mathematical tools are reviewed. Section 3 presents the local fractional variational iteration method based on local fractional operator. Illustrative examples are shown in section 4. Conclusions are in Section 5.

2. MATHEMATICAL FUNDAMENTALS

In this section we present some basic definitions and notations of the local fractional operators (see [16–21]).

Definition 1. Suppose that there is the relation

$$\left|f(x)-f(x_0)\right|<\varepsilon^{\alpha} \ ,0<\alpha\leq 1 (2.1)$$

with $\left|x-x_0\right|<\delta$, for $\varepsilon,\delta>0$ and $\varepsilon,\delta\in R$, then the function $f(x)$ is called local fractional continuous at $x=x_0$ and it is denoted by $\lim_{x\to x_0} f(x)=f(x_0)$.

Definition 2. In fractal space, let $f(x)\in C_{\alpha}(a,b)$, local fractional derivative of $f(x)$ of order α at $x=x_0$ is given by

$$D_x^{\alpha} f(x_0)=f^{(\alpha)}(x)=\lim_{x\to x_0}\frac{\Delta^{\alpha}(f(x)-f(x_0))}{(x-x_0)^{\alpha}},$$

(2.2)

Where

$$\Delta^{\alpha}(f(x)-f(x_0))\cong\Gamma(\alpha+1)(f(x)-f(x_0)).$$

Definition 3. A partition of the interval $[a,b]$ is denoted as (t_j, t_{j+1}), $j = 0,...,N-1$, and $t_N = b$ with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_0, \Delta t_1, ...\}$. The local fractional integral of $f(x)$ in the interval $[a,b]$ is given by

$$ {}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha $$

$$ = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha . \qquad (2.3) $$

Definition 4. In fractal space, the Mittage Leffler function, sine function and cosine function are defined as

$$ E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(1+k\alpha)} \text{ , } 0 < \alpha \le 1 (2.4) $$

$$ \sin_\alpha(x^\alpha) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{(2k+1)\alpha}}{\Gamma[1+(2k+1)\alpha]} \text{ , } 0 < \alpha \le 1 $$
(2.5)

$$ \cos_\alpha(x^\alpha) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k\alpha}}{\Gamma[1+2k\alpha]} \text{ , } 0 < \alpha \le 1 $$

(2.6)

3. ANALYSIS OF THE METHOD

The standard $k\alpha$ order local fractional Volterra integro-differential equation of the second kind is given by

$$ u^{(k\alpha)}(x) = f(x) + \frac{1}{\Gamma(1+\alpha)} \int_0^x K(x,t) u(t)(dt)^\alpha \text{ ,(3.1)} $$

where $u^{((m-1)\alpha)}(0) = a_{m-1}$, $m = 1,2,...,k$ are the initial conditions.

According to the rule of local fractional variational iteration method, the correction local fractional functional for Eq. (3.1) is given by

$$ u_{n+1}(x) = u_n(x) + \frac{1}{\Gamma(1+\alpha)} $$
$$ \int_0^x \frac{\lambda(\zeta)^\alpha}{\Gamma(1+\alpha)} \left[u_n^{(k\alpha)}(\zeta) - f(\zeta) - \frac{1}{\Gamma(1+\alpha)} \int_0^\zeta K(r) u_n(r)(dr)^\alpha \right] $$
$$ (d\zeta)^\alpha \text{ (3.2)} $$

where $\frac{\lambda^\alpha}{\Gamma(1+\alpha)}$ is a general fractal Lagrange's multiplier.

Here, we can construct a correction functional as follows:

$$ u_{n+1}(x) = u_n(x) + \frac{1}{\Gamma(1+\alpha)} $$
$$ \int_0^x \frac{\lambda(\zeta)^\alpha}{\Gamma(1+\alpha)} \left[u_n^{(k\alpha)}(\zeta) - f(\zeta) - \frac{1}{\Gamma(1+\alpha)} \int_0^\zeta K(r) \tilde{u}_n(r)(dr)^\alpha \right] $$

$$ (d\zeta)^\alpha \text{ , (3.3)} $$

Where $\tilde{u}_n$ is considered as a restricted local fractional variation; that is, $\delta^\alpha \tilde{u}_n = 0,$ we obtain the following fractal Lagrange multiplier

$$ \frac{\lambda^\alpha}{\Gamma(1+\alpha)} = (-1)^k \frac{(\zeta - x)^{(k-1)\alpha}}{\Gamma[1+(k-1)\alpha]} . \text{(3.4)} $$

Therefore,

$$ u_{n+1}(x) = u_n(x) + \frac{1}{\Gamma(1+\alpha)} $$
$$ \int_0^x (-1)^k \frac{(\zeta - x)^{(k-1)\alpha}}{\Gamma(1+(k-1))} \left[u_n^{(k\alpha)}(\zeta) - f(\zeta) - \frac{1}{\Gamma(1+\alpha)} \int_0^\zeta K(r) u_n(r)(dr)^\alpha \right] $$
$$ (d\zeta)^\alpha \text{ ,} $$
(3.5)

Finally, the solution is

$$ u(x) = \lim_{n \to \infty} u_n(x) . $$
(3.6)

4. ILLUSTRATIVE EXAMPLES

In this section three examples for the local fractional Volterra integro-differential equation is presented in order to demonstrate the simplicity and the efficiency of the above method.

Example 1. we consider the local fractional Volterra integro-differential equation

$$ u^{(\alpha)}(x) = 1 + \frac{1}{\Gamma(1+\alpha)} \int_0^x u(t)(dt)^\alpha \quad ,u(0) = 1 \text{ .(4.1)} $$

The correction functional for this equation is given by:

$$ u_{n+1}(x) = u_n(x) - $$
$$ \frac{1}{\Gamma(1+\alpha)} \int_0^x \left[u_n^{(\alpha)}(\zeta) - 1 - \frac{1}{\Gamma(1+\alpha)} \int_0^\zeta u_n(r)(dr)^\alpha \right] (d\zeta)^\alpha $$
… (4.2)

Where we used $\frac{\lambda(\zeta)^{\alpha}}{\Gamma(1+\alpha)}=-1$ for first-order integro-differential equation as shown in (3.4).

We can use the initial condition to select $u_0(x)=u(0)=1$. Using this selection into the correction functional gives the following successive approximations

$$u_0(x)=1 \qquad (4.3)$$

$$u_1(x)=u_0(x)-\frac{1}{\Gamma(1+\alpha)}\int_0^x\left[u_0^{(\alpha)}(\zeta)-1-\frac{1}{\Gamma(1+\alpha)}\int_0^{\zeta}u_0(r)(dr)^{\alpha}\right](d\zeta)^{\alpha}$$

$$=1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)} \quad (4.4)$$

$$u_2(x)=u_1(x)-\frac{1}{\Gamma(1+\alpha)}\int_0^x\left[u_1^{(\alpha)}(\zeta)-1-\frac{1}{\Gamma(1+\alpha)}\int_0^{\zeta}u_1(r)(dr)^{\alpha}\right](d\zeta)^{\alpha}$$

$$=1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{x^{4\alpha}}{\Gamma(1+4\alpha)} \quad (4.5)$$

$$u_3(x)=u_2(x)-\frac{1}{\Gamma(1+\alpha)}\int_0^x\left[u_2^{(\alpha)}(\zeta)-1-\frac{1}{\Gamma(1+\alpha)}\int_0^{\zeta}u_2(r)(dr)^{\alpha}\right](d\zeta)^{\alpha}$$

$$=1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}+\frac{x^{5\alpha}}{\Gamma(1+5\alpha)}+\frac{x^{6\alpha}}{\Gamma(1+6\alpha)} \qquad (4.6)$$

And so on.

$$u_n(x)=1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+\cdots+\frac{x^{2n\alpha}}{\Gamma(1+2n\alpha)}.$$

(4.7)

$$u(x)=\lim_{n\to\infty}u_n(x)$$

$$=1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}+\cdots$$

that gives the exact solution

$$u(x)=E_{\alpha}(x^{\alpha}). \qquad \ldots (4.8)$$

Example 2. we consider the local fractional Volterra integro-differential equation

$$u^{(2\alpha)}(x)=1+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(x-t)^{\alpha}}{\Gamma(1+\alpha)}u(t)(dt)^{\alpha},$$

(4.9)

with the initial conditions

$u(0)=1,\ u^{(\alpha)}(0)=0.$

The correction functional for this equation is given by

$$u_{n+1}(x)=u_n(x)+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(\zeta-x)^{\alpha}}{\Gamma(1+\alpha)}\left[u_n^{(2\alpha)}(\zeta)-1-\frac{1}{\Gamma(1+\alpha)}\int_0^{\zeta}\frac{(\zeta-r)^{\alpha}}{\Gamma(1+\alpha)}u_n(r)(dr)^{\alpha}\right]$$

(4.10)

where we used $\frac{\lambda(\zeta)^{\alpha}}{\Gamma(1+\alpha)}=\frac{(\zeta-x)^{\alpha}}{\Gamma(1+k\alpha)}$ for second-order integro-differential equation as shown in Eq. (3.4).

We can use the initial condition to select $u_0(x)=u(0)+\frac{x^{\alpha}}{\Gamma(1+\alpha)}u^{(\alpha)}(0)=1$. Using this selection into the correction functional gives the following successive approximations:

$$u_0(x)=1 \quad \ldots(4.11)$$

$$u_1(x)=u_0(x)+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(\zeta-x)^{\alpha}}{\Gamma(1+\alpha)}\left[u_0^{(2\alpha)}(\zeta)-1-\frac{1}{\Gamma(1+\alpha)}\int_0^{\zeta}\frac{(\zeta-r)^{\alpha}}{\Gamma(1+\alpha)}u_0(r)(dr)^{\alpha}\right]$$

$$=1+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{4\alpha}}{\Gamma(1+4\alpha)} \ (4.12)$$

$$u_2(x)=u_1(x)+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(\zeta-x)^{\alpha}}{\Gamma(1+\alpha)}\left[u_1^{(2\alpha)}(\zeta)-1-\frac{1}{\Gamma(1+\alpha)}\int_0^{\zeta}\frac{(\zeta-r)^{\alpha}}{\Gamma(1+\alpha)}u_1(r)(dr)^{\alpha}\right]$$

$$=1+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}+\frac{x^{6\alpha}}{\Gamma(1+6\alpha)}+\frac{x^{8\alpha}}{\Gamma(1+8\alpha)}$$
…(4.13)

$$u_3(x)=u_2(x)+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(\zeta-x)^{\alpha}}{\Gamma(1+\alpha)}\left[u_2^{(2\alpha)}(\zeta)-1-\frac{1}{\Gamma(1+\alpha)}\int_0^{\zeta}\frac{(\zeta-r)^{\alpha}}{\Gamma(1+\alpha)}u_2(r)(dr)^{\alpha}\right]$$

$$=1+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}+\frac{x^{6\alpha}}{\Gamma(1+6\alpha)}+\frac{x^{8\alpha}}{\Gamma(1+8\alpha)}+\frac{x^{10\alpha}}{\Gamma(1+10\alpha)}+\frac{x^{12\alpha}}{\Gamma(1+12\alpha)}\ (4.14)$$

$$\vdots$$

$$u_n(x)=1+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}+\cdots+\frac{x^{4n\alpha}}{\Gamma(1+4n\alpha)}$$
…(4.15)

$$u(x)=\lim_{n\to\infty}u_n(x)$$

$$=1+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}+\frac{x^{6\alpha}}{\Gamma(1+6\alpha)}+\frac{x^{8\alpha}}{\Gamma(1+8\alpha)}+\cdots$$

that gives the exact solution

$$u(x)=\cosh_{\alpha}(x^{\alpha}).\qquad(4.16)$$

Example 3. We consider the local fractional Volterra integro-differential equation

$$u^{(3\alpha)}(x)=1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(x-t)^{\alpha}}{\Gamma(1+\alpha)}u(t)(dt)^{\alpha},$$
…(4.17)

with the initial conditions:

$u(0)=1$, $u^{(\alpha)}(0)=0$, $u^{(2\alpha)}(0)=1$.

The correction functional for this equation is given by

$$u_{n+1}(x)=u_n(x)-\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(\zeta-x)^{2\alpha}}{\Gamma(1+2\alpha)}\left[\begin{array}{l}u_n^{(3\alpha)}(\zeta)-1-\frac{\zeta^{\alpha}}{\Gamma(1+\alpha)}-\frac{\zeta^{3\alpha}}{\Gamma(1+3\alpha)}\\-\frac{1}{\Gamma(1+\alpha)}\int_0^{\zeta}\frac{(\zeta-r)^{\alpha}}{\Gamma(1+\alpha)}u_n(r)(dr)^{\alpha}\end{array}\right](d\zeta)^{\alpha}$$
…(4.18)

Where we used $\frac{\lambda(\zeta)^{\alpha}}{\Gamma(1+\alpha)}=-\frac{(\zeta-x)^{2\alpha}}{\Gamma(1+2\alpha)}$ for third-order integro-differential equation as shown in (3.4). Now, we have

$$u_0(x)=1+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\quad…(4.19)$$

$$u_1(x)=u_0(x)-\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(\zeta-x)^{2\alpha}}{\Gamma(1+2\alpha)}\left[\begin{array}{l}u_0^{(3\alpha)}(\zeta)-1-\frac{\zeta^{\alpha}}{\Gamma(1+\alpha)}-\frac{\zeta^{3\alpha}}{\Gamma(1+3\alpha)}\\-\frac{1}{\Gamma(1+\alpha)}\int_0^{\zeta}\frac{(\zeta-r)^{\alpha}}{\Gamma(1+\alpha)}u_0(r)(dr)^{\alpha}\end{array}\right](d\zeta)^{\alpha}$$

$$=1+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}+\frac{x^{5\alpha}}{\Gamma(1+5\alpha)}+\frac{x^{6\alpha}}{\Gamma(1+6\alpha)}+\frac{x^{7\alpha}}{\Gamma(1+7\alpha)}\ (4.20)$$

$$u_2(x)=u_1(x)-\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(\zeta-x)^{2\alpha}}{\Gamma(1+2\alpha)}\left[\begin{array}{l}u_1^{(3\alpha)}(\zeta)-1-\frac{\zeta^{\alpha}}{\Gamma(1+\alpha)}-\frac{\zeta^{3\alpha}}{\Gamma(1+3\alpha)}-\\\frac{1}{\Gamma(1+\alpha)}\int_0^{\zeta}\frac{(\zeta-r)^{\alpha}}{\Gamma(1+\alpha)}u_1(r)(dr)^{\alpha}\end{array}\right](d\zeta)^{\alpha}$$

$$=1+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}+\frac{x^{5\alpha}}{\Gamma(1+5\alpha)}+\frac{x^{6\alpha}}{\Gamma(1+6\alpha)}+\frac{x^{7\alpha}}{\Gamma(1+7\alpha)}+\cdots\ (4.25)$$

and so on.

$$u_n(x)=1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+\cdots+\frac{x^{n\alpha}}{\Gamma(1+n\alpha)}-\frac{x^{\alpha}}{\Gamma(1+\alpha)}\ (4.26)$$

$$u(x) = \lim_{n\to\infty} u_n(x)$$

$$= 1 + \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots - \frac{x^\alpha}{\Gamma(1+\alpha)}$$

that gives the exact solution

$$u(x) = E_\alpha(x^\alpha) - \frac{x^\alpha}{\Gamma(1+\alpha)}. \qquad \ldots(4.27)$$

CONCLUSIONS

In this work the Volterra integro-differential equations within the local fractional differential operator had been analyzed using the local fractional variational iteration method. The obtained solutions are nondifferentiable functions, which are Cantor functions and they discontinuously depend on the local fractional derivative. It is shown that the local fractional variational iteration method is an efficient and simple tool for solving local fractional integral equations.

ACKNOWLEDGMENT

The authors are grateful to the referees for their invaluable suggestions and comments for the improvement of the paper.

REFERENCES

[1] K. Kolwankar, A. D. Gangal, Local fractional Fokker Planck equation, Phys. Rev. Lett., vol. 80, pp.214- 217, 1998.

[2] J. H. He, A new fractal derivation, Thermal Science, vol.15, pp.145-147, 2011.

[3] J. H. He, S. K. Elagan, Z. B. Li, Geometrical explanation of the fractional complex and derivative chain rule for fractional calculus, Phy. Lett. A, vol.376, pp.257- 259, 2012.

[4] A. Parvate, A. D. Gangal, Calculus on fractal subsets of real line I: formulation, Fractals, 17(1) (2009) 53-81.

[5] A. Carpinteri, B. Cornetti, K. M. Kolwankar, Calculation of the tensile and flexural strength of disordered materials using fractional calculus, Chaos, Solitons, Fractals, 21 (2004) 623 632.

[6] X. J Yang, Local Fractional Integral Transforms, Progr. Nonlinear Sci.,vol. 4, pp. 1-225, 2011.

[7] X. J Yang, Local Fractional Functional Analysis and Its Applications, Asian Academic publisher Limited, Hong Kong, 2011.

[8] H. K. Jassim, C. Ünlü, S. P. Moshokoa, C. M. Khalique, Local Fractional Laplace Variational Iteration Method for Solving Diffusion and Wave Equations on Cantor Sets within Local Fractional Operators, Mathematical Problems in Engineering,vol.2015Article ID 309870, pp. 1-9, 2015.

[9] X. J. Yang, Local fractional partial differential equations with fractal boundary problems, ACMA, vol.1, pp. 60-63, 2012.

[10] X. J. Yang, Local fractional Kernel transform fractal space and its applications, ACMA, vol.1, pp. 86-93, 2012.

[11] X. J. Yang, Local fractional variational iteration method and its algorithms, ACMA, vol.1, pp. 139-145, 2012.

[12]] X. J. Yang, Local fractional integral equations and their applications, ACSA, vol.1, pp. 234- 239, 2012.

[13] H. Jafari and H. K. Jassim, Numerical solutions of telegraph and Laplace equations on Cantor sets using local fractional Laplace decomposition method , International Journal of Advances in Applied Mathematics and Mechanics, vol. 2, pp. 144-151, 2015.

[14] H. K. Jassim, Local Fractional Laplace Decomposition Method for Nonhomogeneous Heat Equations Arising in Fractal Heat Flow with Local Fractional Derivative, International Journal of Advances in Applied Mathematics and Mechanics, vol.2, pp. 1-7, 2015.

[15] H. Jafari, and H. K. Jassim, Application of the Local fractional Adomian Decomposition and Series Expansion Methods for Solving Telegraph Equation on Cantor Sets Involving Local Fractional Derivative Operators, Journal of Zankoy Sulaimani-Part A, vol. 17, pp. 15-22 , 2015.

[16] X. J. Yang and D. Baleanu, “Fractal heat conduction problem solved by local fractional variation iteration Method,” ThermalScience, vol.17pp. 34-38, 2012.

[17] W.-H. Su, D. Baleanu, X.-J. Yang, and H. Jafari, “Damped wave equation and dissipative wave equation in fractal strings within the local fractional variational iteration method, ” Fixed Point Theory and Applications, vol.2013, pp.1-6, 2013.

[18] Y.-J. Yang, D. Baleanu and X.J. Yang, "A Local Fractional Variational Iteration Method for Laplace Equation within Local Fractional Operators", Abstract and Applied Analysis Vol. 2013, Article ID 202650, pp. 1-6, 2013

[19] S. P. Yan, H. Jafari, and H. K. Jassim, " Local Fractional Adomian Decomposition and Function Decomposition Methods for Solving Laplace Equation within Local Fractional Operators," Advances in Mathematical Physics, Vol. 2014, Article ID 161580, pp.1-7, 2014.

[20] S. Q. Wang, Y. J. Yang, and H. K. Jassim, "Local Fractional Function Decomposition Method for Solving Inhomogeneous Wave Equations with Local Fractional Derivative," Abstract and Applied Analysis, vol. 2014, Article ID 176395, pp.1-7, 2014.

[21] X. J. Yang, Advanced Local Fractional Calculus and Its Applications, World Science, New York, NY, USA, 2012.