Some Combinatorial Results on the Factor Group $K(G)$

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character matrix = $\left(Z_p \right)$ from the character table of Z_p and finding the invariant factors of this matrix, the primary decomposition of the factor group $K(G)$, where $p \geq 2$ is determined. Also, we found the general form of this decomposition for $p \geq 2$.

Keywords: factor group, character table, rational valued character matrix.

1. Introduction

Let G be a finite group, two elements of G are said to be Γ- conjugate if the cyclic subgroups they generate are conjugate in G, and this defines an equivalence relation on G. Its classes are called Γ- classes.

The Z - valued class function on the group G, which is constant on the Γ - classes forms a finitely generated abelian group $cf(G,Z)$ of a rank equal to the number of Γ- classes.

The intersection of $cf(G,Z)$ with the group of all generalized characters of G, is a normal subgroup of $cf(G,Z)$ denoted by $R(G)$,then $cf(G,Z)/\, R(G)$ is a finite abelian factor group which is denoted by $K(G)$.

Each element in $R\,(G)$ can be written as $v_1\phi_1 + v_2\phi_2 + \cdots \ldots + u_s\phi_s$, where l is the

its rows correspond to the $\phi i's$ and its columns correspond to the Γ- classes of G.

The matrix expressing $R(G)$ basis in terms of the $cf(G,Z)$ basis is $\equiv^* (G)$. We can use the theory of invariant factors to obtain the direct sum of the cyclic Z-module of orders the distinct invariant factors of $\equiv^* (G)$ to find the primary decomposition of $K(G)$.

M.S. Kirdar [6]studied the factor group $cf(G,Z)/\, \bar{R}(G)$ for the group of type $Z_2^{(n)}$. M.N.AL-Harere[1] studied the factor group $cf(G,Z)/\, \bar{R}(G)$ for the group of type $Z_3^{(n)}$. M.N.AL-Harere and Fuad A.AL-Heety[2] studied the factor group $cf(G,Z)/\, \bar{R}(G)$ for the group of type $Z_5^{(n)}$.

Here, we determine the primary decomposition of the group $K(Z_p^{(n)})$ and find a general solution for it when $p \geq 2$.

2. Basic Concepts

In this section, we will present an introduction to concepts, representation, characters and characters table of finite group.

2.1 Matrix representation of characters

Definition 2.1.1,[4]

A matrix representation of a group G is a homomorphism $A: G \to GL(n,F)$

n is called the degree of the matrix representation A.

Definition 2.1.2,[4]

Two matrix representations $A_{1(X)}$ and $A_{2(X)}$ are said to be equivalent, if they have the same degree, and if there exists a fixed invertible matrix $T \in GL(n,F)$ such that:

$$A_{1(X)} = T^{-1}A_{2(X)}T \quad \forall\ x \in G.$$

Definition 2.1.3 ,[7]

A matrix representation A of a group G is a reducible representation if it is equivalent to a matrix representation of the form

$$\begin{bmatrix} A_{1(x)} & T(x) \\ 0 & A_{2(x)} \end{bmatrix}, \text{ for all } x \in G,$$

where $A_1(x) and\ A_2(x)$ are representations of G their degree is less than the degree of A.

A reducible representation A is called completely reducible if it is equivalent to a matrix representation of the form

$$\begin{bmatrix} A_{1(x)} & 0 \\ 0 & A_{2(x)} \end{bmatrix}, \text{ For all } x \in G$$

If A is not reducible, then A is said to be an irreducible representation.

Definition 2.1.4,[5]

Let G be a finite group and let $\rho: G \to GL(n,C)$ be a matrix representation of G of degree n given by $\rho(x) = A(x) \ \ \forall \ \ x \in G$, associated with ρ there is a function χ_ρ: $G \to C$ defined by

$\chi_\rho(x) = tr(A(x))$. We call the function χ_ρ the character of the representation ρ.

Proposition 2.1.5,[8]

The characters of G are class functions on G, that is, conjugate elements have the same character.

Definition 2.1.6,[4]

Characters associated with irreducible representations are called irreducible (or simple) characters, and those of reducible representations are compound.

Definition 2.1.7,[4]

A class functions on G of the form $\phi = v_1\chi_1 + v_2\chi_2 + \cdots + v_k\chi_k$ where $\{\chi_1, \chi_2, \dots \chi_k\}$ is the complete set of irreducible characters of G, and $v_1, v_2, \dots, v_k$ are integers, which is called the generalized characters of G.

Theorem 2.1.8,[4]

The number k of distinct irreducible characters of G is equal to the number of its conjugacy classes.

2.2 Character of finite Abelian group: [4]

For a finite group G of order n, complete information about the irreducible characters of G is displayed in a table called the character table of G.

We list the elements of G in the 1^{st} row, we put $\chi_i(x^j) = \chi_i^j$, $1 \le i \le n, 1 \le j \le n-1$. Denoting the number of elements in C_α by $|C_\alpha|$ (α=$1, \dots, n$), we have the class equation $|C_1|$+…+$|C_n|$ =$|G|$.

$\equiv G =$

C_α	1	x	x^2	…	x^{n-1}
$\lvert C_\alpha \rvert$	1	1	1	…	1
χ_1	1	1	1		1
χ_2	1	χ_2^1	χ_2^2		χ_2^{n-1}
⋮	⋮	⋮	⋮	…	⋮
χ_n	1	χ_n^1	χ_n^2	…	χ_n^{n-1}

Table (1)

If $G = Z_n$ the cyclic group of order n, and ω=$e^{2\pi i/n}$ is a primitive n-th root of unity, then the general formula of the character table of Z_n is :

$\equiv Z_n =$

C_α	1	z	z^2	…	z^{n-1}
$\|C_\alpha\|$	1	1	1	…	1
χ_1	1	1	1		1
χ_2	1	ω	$ω^2$	…	$ω^{n-1}$
χ_2	1	$ω^2$	$ω^4$	…	$ω^{n-2}$
⋮	⋮	⋮	⋮	⋮	⋮
χ_n	1	$ω^{n-1}$	$ω^{n-2}$	…	ω

Table (2)

3. The factor group $K(G)$

In this section we study the factor group $K(G)$, for a finite group G.

Definition 3.1,[3]

A rational valued character θ of G is a character whose values are in Z, that is $\theta(x) \in Z$, for all $x \in G$.

Definition 3.2,[6]

Two elements of G are said to be Γ-conjugate if the cyclic subgroups they generate are conjugate in G. This defines an equivalence relation on G, its classes are called Γ-classes of G.

Let G be a finite group and let $\chi_1, \chi_2, \dots \chi_k$ be its distinct irreducible characters. A class function on G is a character if and only if it is a linear combination of the χ_i with non-negative integer coefficients. We will denote $R^+(G)$ the set of these functions, the group generated by $R^+(G)$ is called the group of the generalized characters of G and is denoted by $R(G)$, we have:

$$R(G)=Z\chi_1 \oplus Z\chi_2 \oplus Z\chi_3 \oplus \dots Z\chi_k.$$

An element of $R(G)$ is called a virtual character. Since the product of two characters is a character, R(G) is a subring of the ring $cf(G)$ of C-valued class functions on G.

Let $cf(G,Z)$ be the group of all Z-valued class functions of G which are constant on Q-classes and let $\bar{R}(G)$ be the intersection of $cf(G,Z)$ with $R(G)$, $\bar{R}(G)$ be a ring of Z-valued generalized characters of G.

Let ε_m be a complex primitive m-th root of unity. We know that the Galois group

Gal$(F(\varepsilon_m)/F)$ is a subgroup of the multiplicative group $(Z/mZ)^*$ of invertible elements of Z/mZ.

More precisely, if σ∈Gal $(F(\varepsilon_m)/F)$, there exists a unique element $t \in$ $(Z/mZ)^*$ such that $\sigma(\varepsilon_m) = \varepsilon_m^t$ if $\varepsilon_m = 1$.

We denote by Γ_F for the image of Gal(F(ε_m)/F) in $(Z/m\,Z)^*$, and if $t \in \Gamma_F$, we let σ_t denote the corresponding element of Gal(F(ε_m)/F).

Take as a ground field F the field Q of rational numbers, the Galois group of $Q(\varepsilon_m)$ over Q is the group denoted by Γ.

Proposition 3.3.,[7]

The characters $\phi_1, \phi_2, \dots, \phi_m$ form a basis of $\bar{R}$(G) and their number is equal to the number of conjugate classes of cyclic subgroups of G, where

$\phi_i = \sum_{\phi \in Gal(Q(\chi_i))} \chi_i^\sigma$, and χ_i are irreducible C-characters of G, for all $i = 1,2, \dots m$.

Theorem 3.4,[8]

Let M, X, Y be matrices with entries in a principal domain R. Let P and Q be invertible matrices. Then $D_k(QMP) = D_k(M)$.

Theorem 3.5,[8]

Let M be an $m{\times}n$ matrix with entries in a principal domain R.

Then there exist matrices X, Y, D such that:

1. X and Y are invertible.

2. $YMX = D$.

3. D is diagonal matrix.

4. If we denote D_{ii} by d_i, then there exists a natural number r, $0 \leq r \leq \min(m, n)$ such that $j > r$ implies $d_j = 0$ and $j \leq r$ implies $d_j \neq 0$ and $1 \leq j < r$ implies d_j divides d_{j+1}.

Definition 3.6,[6]

Let M, be a matrix with entries in a principal R, equivalent to a matrix

$D = diag\{d_1, d_2, \dots d_r, 0, \dots, 0\}$ such that d_j/d_{j+1} for $1 \leq j < r$,we call D the invariant factor matrix of M and $d_1, d_2, \dots, d_r$ the invariant factors of M.

Theorem 3.7.,[6]

Let M be a finitely generated module over a principal domain R, then M is the direct sum of cyclic submodules with annihilating ideals

$\langle d_1 \rangle, \langle d_2 \rangle, \dots \langle d_r \rangle$, d_j/d_{j+1} for $j = 1,2, \dots, m - 1$.

Theorem 3.8,[6]

$K(G) = \oplus_{i=1}^{r} Z_{di}$

$d_{i=\pm} D_i(\equiv^* G)/D_{i-1}(\equiv^* G)$.

Lemma 3.9, [8]

Let P and Q be two non-singular matrices of degree n and m respectively, over a principal domain R, and let

$X_1 P Y_1 = D(P) = diag\{d_1(P), d_2(P), \dots, d_n(P)\}$,

$X_2 Q Y_2 = D(Q) = diag\{d_1(\mathrm{Q}), d_2(\mathrm{Q}), \dots, d_n(\mathrm{Q})\}$, be the invariant factor matrices of P and Q. Then $(X_1 \otimes X)(P \otimes Q)(Y_1 \otimes Y_2) = D(P) \otimes D(Q)$, from this lemma, the invariant factor matrix of $P \otimes Q$ can be written as follows:

Let H and L be p_1 and p_2 groups respectively, where p_1 and p_2 are distinct primes, we know that

$$\equiv (H{\times}L) = \equiv (H) \otimes \equiv (L)$$

Since gcd $(p_1, p_2) = 1$, we have the next proposition.

Proposition 3.10, [6]

$$\equiv^* (H{\times}L) = \equiv^* (H) \otimes \equiv^* (L).$$

We consider the case when the order of G is prime; the number of rational valued characters are two which is equal to the number of $\Gamma-$classes.

We have

$$\equiv^* G = \begin{bmatrix} 1 & 1 \\ P-1 & -1 \end{bmatrix}$$

Theorem 3.11,[6]

Let G be a cyclic p-group, then $K(G) = Z_p$.

4. $K(G)$, For G cyclic & elementary abelian group

This section is dedicated to study the rational valued characters table of the group Z_p and to determine the primary decomposition of the finite abelian group $K(G\ )$ where G is cyclic group and the case when G is elementary abelian group.

4.1 The primary decomposition of $K(Z_p^{(n)})$:

In this section we will determine the primary decomposition of $K(Z_p^{(n)})$ according to Lemma 3.9 and the Theorem 3.5 when $n = 1$.

Let $X = \begin{bmatrix} 1-p & 1 \\ 1 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ and according to Proposition 3.10, [6]

$\equiv^* G=$

Γ −classes	[1]	[z]
θ_1	1	1
θ_2	$p-1$	- 1

and

$\equiv^* Z_p = \begin{bmatrix} 1 & 1 \\ p-1 & -1 \end{bmatrix}$, so

$$X \equiv^* Z_p \; Y = \begin{bmatrix} -p & 0 \\ 0 & 1 \end{bmatrix}$$

, by Theorem 3.11 $K\,(Z_p) = Z_p$.

When $n = 2$

$$X \otimes X = \begin{bmatrix} (1-p)^2 & 1-p & 1-p & 1 \\ 1-p & 0 & 1 & 0 \\ 1-p & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$Y \otimes Y = \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$\equiv^* \left(Z_p^2\right) =$
$\equiv^* Z_p \otimes \equiv^* Z_p =$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ p-1 & -1 & p-1 & -1 \\ p-1 & p-1 & -1 & -1 \\ (p-1)^2 & 1-p & 1-p & 1 \end{bmatrix}$$

According to Lemma 3.9 we get

$$X \otimes X \equiv \left(Z_p^2\right) Y \otimes Y = \begin{bmatrix} p^2 & 0 & 0 & 0 \\ 0 & -p & 1 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\equiv \left(Z_p^2\right) \sim diag\ \{p^2, -p, -p, 1\}$

Therefore, $K\,(Z_p^2) = \ Z_{p^2} \oplus Z_p^2$

When $n = 3$

Let

$X \otimes X \otimes X=$

$$\begin{bmatrix} -p^3 & p^2 & p^2 & -p & p^2 & -p & -p & 1 \\ p^2 & 0 & -p & 0 & -p & 0 & 1 & 0 \\ p^2 & -p & 0 & 0 & -p & 1 & 0 & 0 \\ -p & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ p^2 & -p & -p & 1 & 0 & 0 & 0 & 0 \\ -p & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -p & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$Y \otimes Y \otimes$

$$Y = \begin{bmatrix} -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\equiv^* \left(Z_p^3\right) =$
$\equiv^* Z_p \otimes \equiv^* Z_p \otimes \equiv^* Z_p =$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ p-1 & -1 & p-1 & -1 & p-1 & -1 & p-1 & -1 \\ p-1 & p-1 & -1 & -1 & p-1 & p-1 & -1 & -1 \\ (p-1)^2 & 1-p & 1-p & 1 & p^2 & 1-p & 1-p & 1 \\ p-1 & p-1 & p-1 & p-1 & -1 & -1 & -1 & -1 \\ (p-1)^2 & 1-p & (p-1)^2 & 1-p & 1-p & 1 & 1-p & 1 \\ (p-1)^2 & (p-1)^2 & 1-p & 1-p & 1-p & 1-p & 1 & 1 \\ (p-1)^3 & -(p-1)^2 & -(p-1)^2 & p-1 & -(p-1)^2 & p-1 & p-1 & -1 \end{bmatrix}$$

And we obtain

$X \otimes X \otimes X \ \equiv^* \left(Z_p^3\right) \ Y \otimes Y \otimes Y =$

$$\begin{bmatrix} p^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -p^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -p^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -p^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$\equiv^* \left(Z_p^3\right) \sim diag\ \{p^3, -p^2, -p^2, p, -p^2, p, p, -1\}$
Hence

$K\ (Z_p^3) = Z_{p^3} \oplus Z_{p^2}^3 \oplus Z_p^3.$

When $n = 4$, we repeat the same method, we get

$X \otimes X \otimes X \otimes X \equiv^* \left(Z_p^4\right)\ Y \otimes Y \otimes Y \otimes Y =$

$$\begin{bmatrix} p^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -p^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -p^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -p^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence,
K $(Z_p^4) = Z_{p^4} \oplus Z_{p^3}^4 \oplus Z_{p^2}^6 \oplus Z_p^4.$

The general case for p is given by the following.

Theorem 4.1. The primary decomposition of the factor group $K(G)$ for the Z_p^n , $wher\, p$ is a prime number is

K (Z_p^n) $=\oplus_{i=0}^{(n-1)} Z_{p^{n-i}}^{\binom{n}{n-i}}$,

Where p is a prime number, $\binom{n}{i} = \frac{n!}{i!(n-i)}$.

Proof: By induction method assume the statement holds for r factors then,

$$\equiv^* \left(Z_p^r\right) \sim$$
$$diag\,\{\pm p^r; \pm \underbrace{p^{r-1}, \dots, \pm p^{r-1}}_{\binom{r}{r-1}}; \underbrace{\pm\, p^{r-2}, \dots, \pm p^{r-2}}_{\binom{r}{r-2}};$$
$$\dots; \pm \underbrace{p^2, \dots, \pm p^2}_{\binom{r}{2}}; \underbrace{\pm p, \dots, \pm p}_{\binom{r}{1}}; \mp 1\}.$$

And according to Proposition 3.10

$$\equiv^* \left(Z_p^{r+1}\right) = \equiv^* \left(Z_p^r\right) \otimes \equiv^* \left(Z_p\right).$$

Hence, by Lemma 3.9 we have

$$\equiv^* \left(Z_p^{r+1}\right) \sim$$
$$diag\,\{\pm p^{r+1}; \pm \underbrace{p^r, \dots, \pm p^r}_{\binom{r}{r-1}}; \underbrace{\pm\, p^{r-1}, \dots, \pm p^{r-1}}_{\binom{r}{r-2}};$$
$$\dots$$
$$; \pm \underbrace{p^3, \dots, \pm p^3}_{\binom{r}{2}}; \pm \underbrace{p^2, \dots, \pm p^2}_{\binom{r}{1}}; \pm p; \pm p^r;$$
$$\pm \underbrace{p^{r-1}, \dots, \pm p^{r-1}}_{\binom{r}{r-1}}; \dots; \underbrace{p^2, \dots, p^2}_{\binom{r}{2}};$$
$$\underbrace{p, \dots, p}_{\binom{r}{1}}; \mp 1\}$$

$\binom{r}{i} + \binom{r}{i-1} = \binom{r+1}{i}$ when , $0 \leq i \leq r$, which means p^i appears $\binom{r+1}{i}$ times in the matrix above.

Therefore, $K\,(Z_p^{r+1}) = \oplus_{i=1}^{(r+1)} Z_{p^i}^{\binom{r+1}{i}}$.

Example 4.2.

For $n = 4$, K (Z_7^4) $=\oplus_{i=0}^{3} Z_{p^{4-i}}^{\binom{4}{4-i}}$

$= Z_{p^4}^{\binom{4}{4}} \oplus Z_{p^3}^{\binom{4}{3}} \oplus Z_{p^2}^{\binom{4}{2}} \oplus Z_p^{\binom{n}{1}}$

$= Z_{2401} \oplus Z_{343}^4 \oplus Z_{49}^6 \oplus Z_7^4$

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