

LEBESGUE MEASURABLE FUNCTION IN FRACTIONAL DIFFERENTIAL EQUATIONS

Sabah Mahmood Shaker

Mathematics Dept./College of Science / Al_Mustansiriya University

Abstract

Bassam, M.A. [1], proved some existence and uniqueness theorems for the following fractional linear differential equation.

$$Ln_{\alpha}(y) = \sum_{i=0}^n p_i(x) y^{[(n-i)\alpha]}(x) = F(x) \dots 1$$

With the initial conditions

$$y^{(k\alpha-1)}(a) = \mu_k$$

Where $a < x < b$, $0 < \alpha \leq 1$, μ_k are real numbers, $k=1,2,\dots,n$, $p_i(x)$, $F(x)$ are continuous functions defined on (a,b) such that $p_0(x) \neq 0$, $i=0,1,\dots,n$ and $y^{[(n-i)\alpha]}$ denotes the fractional derivative of order $(n-i)\alpha$ for the function y .

In this work we prove some theorems for equation (1), however for $\alpha=1$. Equation (1) is an ordinary differential equation of order n , therefore all the theorems proved here will be reduced to well known result in the theory of ordinary differential equations. Moreover,

We give some examples and an application for equation (1).

Keywords: Ordinary Differential Equations, Lebesgue Measurable Function, Fractional Differential Equations.

1. Introduction

The subject of this work is related to linear differential equations involving fractional derivatives of order α , $0 < \alpha \leq 1$

(see the definitions (1) and (2)).

The concept of differentiation and integration of fractional order is by no means new. The earliest work on fractional derivative was done by Leibniz in 1695, but this work was first published in 1849, [2].

The major study on fractional differentiation started by Liouville (1835) [3] in which he defined the fractional derivative as an infinite exponential series:

$$\frac{d^u y}{dx^u} = \sum A_m e^{mx} m^u,$$

where the order u of differentiation is real or imaginary.

Since then many authors have discussed fractional differentiation and integration and their properties, together with applications in different areas. (A number of important papers have appeared with different approach to the definition of fractional differentiation, see [4].

Fractional derivatives can be regarded as tangible as those of integer order and the derivative of order α when α is negative

integer it is just an indefinite integral and for α is positive integer it is an ordinary derivative, see examples (1).

2. Theoretical Part

In this part we give a collection of definitions, remarks and lemmas which will be used in various places in this work.

Definition (1). (Fractional Integration).

Let f be a lebesgue measurable function defined a.e. (almost everywhere) on $[a, b]$. If $\alpha > 0$, then we define:

$$I_a^\alpha f = \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} f(s) ds$$

Provided that the integral (lebesgue) exists.

Denote $L(a, b)$ to the set of all lebesgue measurable function f defined on measurable set such that:

$$\int_a^b |(b-s)^{\alpha-1} f(s)| ds < \infty$$

Definition(2) (Fractional differentiation).

If $\alpha \leq 0$; n is the smallest positive integer such that:

$\alpha + n > 0$:

$$\text{Then } I_a^\alpha f = \frac{d^n}{dx^n} I_a^r f, \text{ at } x=b,$$

(where $r = \alpha + n$) provided that $I_a^r f$ and its $(n-1)$ first derivatives exist in a segment $|b-x| < h$ where h is appositve real, and the n^{th} derivatives exist at $x=b$.

Remark (1):

Let $\alpha, \beta \in \mathbb{R}; \beta > -1$. If $x > a$, then

$$I_a^\alpha \frac{(t-a)^\beta}{\Gamma(\beta+1)} = \begin{cases} \frac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}, & \alpha+\beta \neq c \\ 0, & \alpha+\beta = c \end{cases}$$

where c is negative integer.

Case 1:

for $\alpha > 0$.

By Definition (1), we have

$$\begin{aligned} I_a^\alpha \frac{(t-a)^\beta}{\Gamma(\beta+1)} &= \frac{1}{\Gamma(\alpha)} \int_a^x \frac{(t-a)}{\Gamma(\beta+1)} (x-t)^{\alpha-1} dt \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta+1)} \int_a^x (t-a)^\beta (x-t)^{\alpha-1} dt \end{aligned}$$

Let $u = \frac{x-t}{x-a}$, $x > a$ then

$$\int_a^x (t-a)^\beta (x-t)^{\alpha-1} dt = \int_0^1 u^{\alpha-1} (1-u)^{(\beta+1)-1} (x-a)^{\alpha+\beta} du$$

It follows from (Betta function) that:

$$\int_a^x (t-a)^\beta (x-t)^{\alpha-1} dt = (x-a)^{\alpha+\beta} \beta(\alpha, \beta+1)$$

and so:

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)\Gamma(\beta+1)} \int_a^x (t-a)^\beta (x-t)^{\alpha-1} dt \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta+1)} (x-a)^{\alpha+\beta} \beta(\alpha, \beta+1) \end{aligned}$$

$$\text{Since } \beta(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

then:

$$\frac{1}{\Gamma(\alpha + \beta + 1)\beta(\alpha, \beta + 1)}(x-a)^{\alpha+\beta} \beta(\alpha, \beta + 1) =$$

If $\alpha + \beta = \text{negative integer}$, then $\Gamma(\alpha + \beta + 1) = \infty$

$$\text{Thus } \frac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha + \beta + 1)} = 0$$

Case 2:

For $\alpha \leq 0$.

By Definition (2), we have

$${}_a^x I^\alpha \frac{(t-a)^\beta}{\Gamma(\beta+1)} = {}_a^x D {}_a^x I^r \frac{(t-a)^\beta}{\Gamma(\beta+1)} =$$

$$D_x^n \frac{(x-a)^{\alpha+n+\beta}}{\Gamma(\alpha+n+\beta+1)} = \frac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}$$

Definition (3):

If $\alpha \in \mathbb{R}$; $f(x)$ is defined a.e. on $[a, b]$ we define:

$${}_a^{(\alpha)} D f(x) = \frac{d^\alpha f}{dx^\alpha} = f^{(\alpha)}(x) = I_a^{-\alpha} f$$

For all $x \in (a, b]$ provide that $I_a^{-\alpha} f$ exists.

Lemma (1) (Linearity)

$$a) {}_a^{(\alpha)} D x(u(x) + v(x)) = {}_a^{(\alpha)} D x u(x) + {}_a^{(\alpha)} D x v(x)$$

Where u, v are two functions defined on (a, b) .

$$b) {}_a^{(\alpha)} D x k u(x) = k {}_a^{(\alpha)} D x u(x), \text{ where } k \text{ is constant}$$

$$c) {}_a^{(\alpha)} D x {}_a^{(\beta)} D x u(x) = {}_a^{(\alpha+\beta)} D x u(x), 0 < \alpha \leq 1, 0 < \beta \leq 1.$$

Lemma (2):

Let:

$$U_\alpha(\lambda; x) = \sum_{n=1}^{\infty} \lambda^{n-1} \frac{x^{n\alpha-1}}{\Gamma(n\alpha)}, \quad \dots(2)$$

Where $\lambda \in \mathbb{R}$, then:

- (i) the series converges for $x \neq 0$ and $\alpha > 0$,
- (ii) the series converges everywhere $\alpha \geq 1$,
- (iii) if $\alpha = 1$ in equation(2), then $U_1(\lambda; x) = e^{\lambda x}$.

Lemma (3)

Let:

$$p_0 y^{(n\alpha)}(x) + p_1 y^{[(n-1)\alpha]}(x) + \dots + p_{n-1} y^{(\alpha)}(x) + p_n y(x) = 0 \quad \dots(3)$$

and

$$\phi(x) = \frac{\mu(x-a)^{\alpha-1}}{\Gamma(\alpha)} + \mu \sum_{k=1}^{\infty} \lambda^k \frac{(x-a)^{[(k+1)\alpha]-1}}{\Gamma[(k+1)\alpha]}$$

where $0 < \alpha \leq 1$, $a < x \leq b$, if $\mu, \lambda, p_i = 0, 1, \dots, n-1$, are constants such that $p_0 \neq 0$ is a solution of equation (3) if and only if:

$$p(\lambda) = p_0 \lambda^n + p_1 \lambda^{n-1} + \dots + p_{n-1} \lambda + p_n = 0,$$

This is called the characteristic of equation (3).

3. Practical Part

Examples (1)

1)

$$\begin{aligned}\frac{d^{0.5}f}{dx^{0.5}} &= f^{(0.5)}(x) = I_a^{x(-0.5)} f = \frac{d}{dx} I_a^{x(0.5)} f \\ &= \frac{d}{dx} \int_a^x \frac{(x-t)^{0.5-1}}{\Gamma(0.5)} f(t) dt \\ &= \frac{d}{dx} \frac{1}{\sqrt{\pi}} \int_a^x \frac{1}{\sqrt{(x-t)}} f(t) dt\end{aligned}$$

$$2) \frac{d^{-0.5}f}{dx^{-0.5}} = f^{(-0.5)}(x) = I_a^{x0.5} f =$$

$$\int_a^x \frac{(x-t)^{-0.5}}{\sqrt{\pi}} f(t) dt = \frac{1}{\sqrt{\pi}} \int_a^x \frac{1}{\sqrt{(x-t)}} f(t) dt$$

3)

$$\frac{d^{-1}f}{dx^{-1}} = f^{(-1)}(x) = I_a^{x1} f = \frac{1}{\Gamma(1)} \int_a^x f(t) dt = \int_a^x f(t) dt$$

$$4) \frac{d^{-2}f}{dx^{-2}} = f^{(-2)}(x) = I_a^{x2} f =$$

$$\int_a^x \frac{(x-t)^{2-1}}{\Gamma(2)} f(t) dt = \int_a^x (x-t) f(t) dt.$$

5)

$$\frac{d^2f}{dx^2} = f^{(2)}(x) = I_a^{x-2} f = \frac{d^3}{dx^3} I_a^{x1} f =$$

$$\frac{d^2}{dx^2} \bullet \frac{d}{dx} \int_a^x \frac{(x-t)^0}{\Gamma(1)} f(t) dt = \frac{d^2f}{dx^2}$$

4. Abel's Integral Equation

Abel was interested in the problem of the tautochrone; that is, determining a curve in the (x,y) plane such that the time required for particle to slide down the curve to its lowest point is independent of its initial placement on the curve (see [5]).

Example (2):

Find the general solution of the following initial value problem:

$$y^{(2a)}(x) - 3y^{(a)}(x) + 2y(x) = 0$$

$$y^{(1-a)}(a) = \mu_1, y^{(2a-1)}(a) = \mu_2,$$

where $a < x < b$ and μ_1, μ_2 are real numbers.

Solution.

By Lemma (3) $p(\lambda) = \lambda^2 - 3\lambda + 2 = 0$ and so $\lambda_1 = 1, \lambda_2 = 2$:

$$\phi_1(x) = \mu \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} + \mu \sum_{k=1}^{\infty} 1^k \frac{(x-a)^{[(k+1)\alpha]-1}}{\Gamma[(k+1)\alpha]},$$

$$\phi_2(x) = \mu \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} + \mu \sum_{k=1}^{\infty} 2^k \frac{(x-a)^{[(k+1)\alpha]-1}}{\Gamma[(k+1)\alpha]}.$$

Define $\phi(x) = c_1 \phi_1(x) + c_2 \phi_2(x)$, $a < x < b$, where c_1, c_2 are constants.

Thus,

$$\phi^{(\alpha-1)}(a) = C_1 \phi_1^{(\alpha-1)}(a) + C_2 \phi_2^{(\alpha-1)}(a) = \mu_1 \dots (4)$$

$$\phi^{(2\alpha-1)}(a) = C_1 \phi_1^{(2\alpha-1)}(a) + C_2 \phi_2^{(2\alpha-1)}(a) = \mu_2 \dots (5)$$

Now, solving equations (4), (5) we get

$$C_1 = \frac{\mu_1 \phi_2^{(2\alpha-1)}(a) - \mu_2 \phi_2^{(\alpha-1)}(a)}{w_{2\alpha}(\phi_1, \phi_2)(a)} \text{ and}$$

$$C_2 = \frac{\mu_1 \phi_1^{(2\alpha-1)}(a) - \mu_2 \phi_1^{(\alpha-1)}(a)}{w_{2\alpha}(\phi_1, \phi_2)(a)}$$

Consequently,

$\phi(x) = C_1 \phi_1(x) + C_2 \phi_2(x)$ is the required solution for the above initial-value problem.

5. Conclusions

We have mentioned in the introduction of this paper that the theory of fractional differentiation and integration goes side by side with the theory of ordinary differentiation and integration, in the sense that they have similar properties and applications. Some of these prosperities were given in this paper (see Lemma (3), Abels integral equation) for the others see [4].

Using this fact, many authors were able to generalize theorems for ordinary differentiation and integration together with their applications to suit the fractional cases.

Thus we expect that many theorems in the theory of ordinary differential equations can be generalized to the fractional cases.

As a future plan to work in this field, we suggest that the results obtained in practical part related to equation (1) can be extended to the cases where the coefficient functions $p_0(x)$ and $F(x)$ are lebesgue integrable functions on some interval I or analytic functions on I.

From the applicator point of view, we also suggest to discuss some more applications of this subject and tackle certain problems that can be solved using fractional differential equation.

References

- [1]. **Bassam, M. A.**, "Some Existence Theorems on Differential Equations of Generalized Order", Euilie Relve and Aynowardt. Matreatio, (1965).
- [2]. **Center, W.**, "On Fractional Differentiation", Cambridge and Dublin Math. J, 4, 21-26.(1849).
- [3]. **Liouville, J.**, "Memoire Surle Changement de is Variable Dans ie Calcui des Different Lelles a Indices Quelconques", Ecole polytech. 15, section 24, 17-45. (1835).
- [4]. **Oldham, Keith B. and Jerome Spanier**, "The Fractional Calculus", Academic Press, Inc. New york and London, 1974.
- [5]. **Davis, H. T.**, "The Application of fractional Operators to Functional Equations", presented to the Indiana Section of the Mathematical Association of America. May 8 1926.

دالة ليببيك القياسية في المعادلات التفاضلية الكسرية

م.م. صباح محمود شاكر

قسم الرياضيات/ كلية العلوم/ الجامعة المستنصرية

مستخلص

برهن بسام محمد علي [1] بعض مبرهنات الوجود والوحدانية للمعادلات التفاضلية الخطية الكسرية الآتية:

$$(1) \dots \left\{ \begin{array}{l} {}^L n_{\alpha}(y) = \sum_{i=0}^n p_i(x) y^{[(n-i)^{\alpha}]}(x) = F(x) \\ \text{ذات الشرط الابتدائي:} \\ y^{(k\alpha-1)}(a) = \mu_k \end{array} \right.$$

حيث ان $0 < \alpha \leq 1$, $a < x < b$ و μ_k عدد حقيقي، $k=1,2,\dots,n$ ، $F(x)$ ، $p_i(x)$ هي دوال مستمرة معرفة على الفترة (a,b) ، حيث ان $p_0(x) \neq 0$, $i=0,1,\dots,n$ و $y^{[(n-i)^{\alpha}]}$ تمثل مشتقات كسرية من الرتبة $(n-i)\alpha$ بالنسبة للدالة y .

في هذا البحث تم اثبات بعض المبرهنات المتعلقة بالمعادلة (١) وخصوصا عند $\alpha=1$.

المعادلة (١) هي معادلة تفاضلية اعتيادية من الرتبة n ، لذلك فان جميع المبرهنات المثبتة هنا سوف تختزل للحصول على نتائج معرفة جيدا في نظرية المعادلات التفاضلية الاعتيادية. وفي النهاية نعطي بعض الامثلة والتطبيقات الخاصة بمعادلة (١).