Generalized Bivariate Fibonacci Polynomials and a Related Class of Bi-Univalent Functions

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Abstract— *In this study, we propose and analyze a new subclass of analytic bi-univalent functions defined on U=ζϵC:|ζ|<1. The functions considered are characterized by subordination to a generating function constructed via generalized bivariate Fibonacci polynomials. In particular, we derive bounds for the initial coefficients in the Taylor–Maclaurin series expansion of functions belonging to this subclass. Furthermore, we obtain estimates for the Fekete–Szegö functional. The results presented here generalize and extend several known results in the theory of bi-univalent functions. The relevance of the main results and their relationship to previously defined subclasses are illustrated with examples and comments..*

Keywords: Bi-univalent functions, generalized bivariate Fibonacci polynomials, regular functions, Subordination

I. Preliminaries

Geometric Function Theory (GFT), a central domain within complex analysis, has witnessed significant recent developments, particularly through the integration of special functions in the study of univalent and bi-univalent function classes, leading to deeper insights into coefficient bounds, subordination results, and functional inequalities.

Let $\mathbb{C}$ denote the complex plane, and define $\mathfrak{U}=\{\zeta\in\mathbb{C}:|\zeta|<1\}$. Let $\mathcal{A}$ denote the class of regular (i.e., holomorphic) functions in $\mathfrak{U}$. Each element $\phi\in\mathcal{A}$ is of the form:

$$\phi(\zeta)=\zeta+d_2\zeta^2+d_3\zeta^3+\cdots=\zeta+\sum_{j=2}^{\infty}d_j\zeta^j,\quad \zeta\in\mathfrak{U}, \tag{1.1}$$

and let $\mathcal{S}$ denote the set of all functions $\phi\in\mathcal{A}$ that are univalent in $\mathfrak{U}$. The Bieberbach conjecture, proposed in [4], posits that $|d_j|\le j$ for all $j\ge 2$ and for each $\phi\in\mathcal{S}$. To approach this conjecture, researchers introduced numerous subclasses of $\mathcal{S}$, leading to extensive investigations into coefficient bounds. After decades of effort, the conjecture was conclusively resolved by Louis de Branges in [10] for all $j\ge 2$.

In parallel, the Fekete–Szegö functional (FSF) concerning the estimation of $|d_3-\xi d_2^2|$, where $\xi\in\mathbb{R}$, and $\phi\in\mathcal{S}$ has also attracted considerable interest, with many results obtained for specific subclasses of $\mathcal{S}$; see [13].

A notable direction in the study of such classical problems is the class of bi-univalent functions. The function $\phi\in\mathcal{A}$ is said to be bi-univalent in $\mathfrak{U}$ if $\phi\in\mathcal{S}$ and ϕ^{-1} has univalent analytic continuation to the disc $\mathfrak{U}$. Let σ denote the class of bi-univalent functions in $\mathfrak{U}$ of the form (1.1). Lewin in [21] investigated the class σ. The analysis of initial coefficient bounds for members of σ has emerged as a natural extension of classical univalent theory, with particular focus on analogues of the FSF, and Hankel determinant problems in the bi-univalent setting.

The classical theorem of Koebe (see [12]) guarantees that any function $\phi\in\mathcal{S}$ written in the form (1.1) has an inverse given by

$$\phi^{-1}(w)=w-d_2w^2+(2d_2^2-d_3)w^3-(5d_2^3-5d_2d_3+d_4)w^4+\cdots=\psi(w) \tag{1.2}$$

satisfying $\zeta = \psi(\phi(\zeta)), \zeta \in \mathfrak{U}$ and $w = \phi(\psi(w)), r_0(\phi) > |w|, r_0(\phi) \geq 1/4, w \in \mathfrak{U}$. The class σ is non-empty, as it contains normalized functions whose inverses are also univalent in $\mathfrak{U}$. Notable examples include $\phi_1(\zeta) = \frac{1}{2}\log\left(\frac{1+\zeta}{1-\zeta}\right) = \zeta + \frac{\zeta^3}{3} + \cdots$ with inverse $\phi_1^{-1}(w) = tanh(w) = w - \frac{w^3}{3} + \frac{2w^5}{15} + \cdots$, and $\phi_2(\zeta) = \frac{\zeta}{\sqrt{1-\zeta^2}} = \zeta + \frac{\zeta^3}{2} + \cdots$, with inverse $\phi_2^{-1}(w) = \frac{w}{\sqrt{1+w^2}} = w - \frac{w^3}{2} + \frac{3w^5}{8} + \cdots$; both functions and their inverses are univalent $\mathfrak{U}$, providing explicit, non-trivial members of σ. On the other hand, the functions $\zeta - \frac{\zeta^2}{2}$, and the Koebe function $\frac{\zeta}{(1-\zeta)^2}$, while belonging to $\mathcal{S}$, do not lie in σ. For a concise overview and a discussion of various properties of the σ family, refer to [5, 6, 23, 33]. The recent renewed interest in the study of functions $\in \sigma$ was largely initiated by the work of Srivastava and his collaborators [27]. Since then, numerous researchers have explored several interesting subclasses within σ; for further details, see [7, 8, 11, 14, 15, 22, 34] as well as the references cited therein.

Fibonacci polynomials and their extensions play a significant role in combinatorics, number theory, physics, and engineering. Motivated by their wide-ranging applications, various generalizations have been introduced (see [20]). In particular, the Generalized Bivariate Fibonacci Polynomials (GBFP) have recently attracted attention within the framework of GFT. Their structural properties and recurrence relations provide a valuable analytic tool for defining and studying new subclasses of σ via subordination, facilitating the investigation of coefficient problems and functional inequalities in GFT.

The Fibonacci numbers are well-known to satisfy the recurrence relation

$$\mathcal{F}_j = \mathcal{F}_{j-1} + \mathcal{F}_{j-2}, \quad j \geq 2,$$

with $\mathcal{F}_0 = 0$ and $\mathcal{F}_1 = 1$. Extending this concept, the Fibonacci polynomials are defined by the relation (see [3])

$$\mathcal{F}_j(\varkappa) = \varkappa\mathcal{F}_{j-1}(\varkappa) + \mathcal{F}_{j-2}(\varkappa), \quad j \geq 2,$$

with $\mathcal{F}_0(\varkappa) = 0$ and $\mathcal{F}_1(\varkappa) = 1$. Assuming that $k(\varkappa, y)$ and $l(\varkappa, y)$ have real coefficients, an extended generalization of $\mathcal{F}_j(\varkappa)$, denoted by $\mathcal{F}_j(\varkappa, y)$, was introduced in [19]. It is defined by

$$\mathcal{F}_j(\varkappa, y) = k(\varkappa, y)\mathcal{F}_{j-1}(\varkappa, y) + l(\varkappa, y)\mathcal{F}_{j-2}(\varkappa, y), \quad j \geq 2, \tag{1.3}$$

with initial values $\mathcal{F}_0(\varkappa, y) = 0$, $\mathcal{F}_1(\varkappa, y) = 1$ and $k^2(\varkappa, y) + 4l(\varkappa, y) > 0$.

As shown in [19], the generating function of this GBFP sequence is

$$\mathcal{F}(\varkappa, y, z) = \sum_{j=0}^{\infty} \mathcal{F}_j(\varkappa, y)z^j = \frac{z}{1 - k(\varkappa, y)z - l(\varkappa, y)z^2}. \tag{1.4}$$

The paper [9] and its references provide a brief history and a wealth of information for readers interested in GBFP.

For the sake of conciseness, we state that $k(\varkappa, y) = k$ and $l(\varkappa, y) = l$. Clearly

$$\mathcal{F}_2(\varkappa, y) = k, \mathcal{F}_3(\varkappa, y) = k^2 + l, \cdots \tag{1.5}$$

are evident from (1.3).

The current investigation focuses on a particular subclass of σ, consisting of functions that are subordinate to well-known sequences of numbers or polynomials. A substantial body of work has examined coefficient estimates and the FSF given by $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$, within such families of associated with classical polynomials and numerical sequences (see [2, 16, 18, 24, 26, 28-30, 32, 35] for detailed studies). More recently, research has shifted toward subclasses of σ involving functions governed by GBFP. Noteworthy contributions on coefficient bounds and FSF estimates for these GBFP-related classes can be found in [1, 17, 25, 31].

The use of GBFP in GFT is motivated by their rich algebraic and analytic structures, which align naturally with key tools in GFT, particularly subordination theory, coefficient estimation, and operator theory. GBFP are defined through recurrence relations involving two parameters, offering greater flexibility and generality compared to classical Fibonacci polynomials. This flexibility allows for the construction of new subclasses of σ by embedding these polynomials into subordination conditions. Moreover, the generating functions and orthogonality relations associated with GBFP provide a framework for handling coefficient problems within the unit disk. These polynomials also serve as kernels or coefficient sequences in the formulation of convolution (Hadamard product) operators, enabling deeper investigations into the geometric and growth properties of analytic functions.

In essence, the GBFP offer a unifying structure that bridges combinatorial identities, recurrence sequences, and analytic function theory, making them a powerful tool for generalizing and extending classical results in GFT.

Remark 1.1 By assigning specific values to the parameters k and l, a variety of well-known polynomial sequences can be derived from the GBFP framework (see [36]). For instance:

(i) Setting $k = \varkappa$ and $l = y$ yields the bivariate Fibonacci polynomials;

(ii) The Pell polynomials arise when $k = 2\varkappa$ and $l = 1$;

(iii) Choosing $k = 1$ and $l = \varkappa$ leads to the Jacobsthal polynomials;

(iv) The Fermat polynomials are obtained for $k = 3\varkappa$ and $l = -2$;

(v) Taking $k = 2\varkappa$ and $l = -1$ gives the second kind Chebyshev polynomials;

and so on.

For $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathcal{A}$ regular in $\mathfrak{U}$, $\mathfrak{a}_1$ is subordinate to $\mathfrak{a}_2$, if there is a function $\theta(\zeta)$ that is regular in $\mathfrak{U}$ with $\theta(0) = 0$ and $|\theta(\zeta)| < 1$, such that $\mathfrak{a}_1(\zeta) = \mathfrak{a}_2(\theta(\zeta)), \zeta \in \mathfrak{U}$. This is denoted as $\mathfrak{a}_1 \prec \mathfrak{a}_2$ or $\mathfrak{a}_1(\zeta) \prec \mathfrak{a}_2(\zeta)$. In case, if $\mathfrak{a}_2 \in \mathcal{S}$, then

$$\mathfrak{a}_1(\zeta) \prec \mathfrak{a}_2(\zeta) \;\Leftrightarrow\; \mathfrak{a}_1(0) = \mathfrak{a}_2(0) \quad and \quad \mathfrak{a}_1(\mathfrak{U}) \subset \mathfrak{a}_2(\mathfrak{U}).$$

This paper uses the the inverse function $\phi^{-1}(w) = \psi(w)$ defined as in (1.2), and generating function $\mathcal{F}(\varkappa, y, z)$ stated as in (1.4), unless otherwise noted.

Building upon earlier observations concerning coefficient bounds within subclasses of σ, we introduce a new subfamily, denoted by $\mathfrak{T}_\sigma(\varrho, \nu, F)$, consisting of functions in σ that are subordinate to a GBFP.

Definition 1.1 If $\phi \in \sigma$ satisfies

$$\frac{1}{2}\left(\left(\frac{\left(\zeta\phi'(\zeta)+\varrho\zeta^2\phi''(\zeta)\right)'}{\phi'(\zeta)}\right) + \left(\frac{\left(\zeta\phi'(\zeta)+\varrho\zeta^2\phi''(\zeta)\right)'}{\phi'(\zeta)}\right)^{\frac{1}{\nu}}\right) \prec F(\zeta) = \frac{\mathcal{F}(\varkappa,y,\zeta)}{\zeta}, \zeta \in \mathfrak{U}, \tag{1.6}$$

and

$$\frac{1}{2}\left(\left(\frac{(w\psi'(w)+\varrho w^2\psi''(w))'}{\psi'(w)}\right) + \left(\frac{(w\psi'(w)+\varrho w^2\psi''(w))'}{\psi'(w)}\right)^{\frac{1}{\nu}}\right) \prec F(w) = \frac{\mathcal{F}(\varkappa,y,w)}{w}, \tag{1.7}$$

$w \in \mathfrak{U}$, then we say that $\phi \in \mathfrak{T}_\sigma(\varrho, \nu, F)$, where $\varrho \geq 0$, $0 < \nu \leq 1$, and

$$F(z) = \frac{1}{1-kz-lz^2}, k^2 + 4l > 0. \tag{1.8}$$

Remark 1.2 $\mathfrak{T}_\sigma(\varrho, 1, F)$ was considered in [25].

The content is arranged as follows: Section 2 focuses on obtaining estimates for $|d_2|$, $|d_3|$, and $|d_3 - \xi d_2^2|$ for $\xi \in \mathbb{R}$, for functions in the class $\mathfrak{T}_\sigma(\varrho, \nu, F)$, as well as on discussing relevant links between specific instances and the principal results. The study is concluded with some remarks in Section 3.

II. Key findings

In Section 2, we begin by establishing bounds on $|d_2|$, $|d_3|$, and $|d_3 - \xi d_2^2|$ for $\xi \in \mathbb{R}$ for functions within the class $\mathfrak{T}_\sigma(\varrho, \nu, F)$.

Theorem 2.1 *Let $\varrho \geq 0, \xi \in \mathbb{R}, 0 < \nu \leq 1$, and $F(\zeta)$ be defined as in (1.8). If $\phi \in \sigma$ is a member of $\mathfrak{T}_\sigma(\varrho, \nu, F)$, then*

$$|d_2| \leq \nu|k|\sqrt{\frac{|k|}{|(\nu(\nu+1)(1+5\varrho)+(1-\nu)(1+2\varrho)^2)k^2-(\nu+1)^2(1+2\varrho)^2(k^2+l)|}}, \tag{2.1}$$

$$|d_3| \leq \frac{\nu^2k^2}{(\nu+1)^2(1+2\varrho)^2} + \frac{\nu|k|}{3(\nu+1)(1+3\varrho)}, \tag{2.2}$$

and

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{\nu|k|}{3(\nu+1)(1+3\varrho)} \\ \frac{\nu^2|k|^3\,|1-\xi|}{|(\nu(\nu+1)(1+5\varrho)+(1-\nu)(1+2\varrho)^2)k^2-(\nu+1)^2(1+2\varrho)^2(k^2+l} \end{cases}$$

(2.3)

where

$$\daleth = \left|\frac{(\nu(\nu+1)(1+5\varrho)+(1-\nu)(1+2\varrho)^2)k^2-(\nu+1)^2(1+2\varrho)^2(k^2+l)}{3\nu(\nu+1)(1+3\varrho)k^2}\right|. \tag{2.4}$$

Proof. Let $\phi \in \mathfrak{T}_\sigma(\varrho, \nu, F)$. Then, from the subordinations given in (1.6) and (1.7), we deduce

$$\frac{1}{2}\left(\left(\frac{(\zeta\phi'(\zeta)+\varrho\zeta^2\phi''(\zeta))'}{\phi'(\zeta)}\right) + \left(\frac{(\zeta\phi'(\zeta)+\varrho\zeta^2\phi''(\zeta))'}{\phi'(\zeta)}\right)^{\frac{1}{\nu}}\right) = F(\mathfrak{u}(\zeta)), \tag{2.5}$$

and

$$\frac{1}{2}\left(\left(\frac{(w\psi'(w)+\varrho w^2\psi''(w))'}{\psi'(w)}\right) + \left(\frac{(w\psi'(w)+\varrho w^2\psi''(w))'}{\psi'(w)}\right)^{\frac{1}{\nu}}\right) = F(\mathfrak{v}(w)), \tag{2.6}$$

where

$$\mathfrak{u}(\zeta) = \sum_{j=1}^{\infty} \mathfrak{u}_j\zeta^j = \mathfrak{u}_1\zeta + \mathfrak{u}_2\zeta^2 + \mathfrak{u}_3\zeta^3 + \cdots, \quad \zeta \in \mathfrak{U},$$

and

$$\mathfrak{v}(w) = \sum_{j=1}^{\infty} \mathfrak{v}_jw^j = \mathfrak{v}_1w + \mathfrak{v}_2w^2 + \mathfrak{v}_3w^3 + \cdots, \quad w \in \mathfrak{U},$$

satisfy (see[12])

$$|\mathfrak{u}_j| \leq 1, and\ |\mathfrak{v}_j| \leq 1\ (j \in \mathbb{N}). \tag{2.7}$$

Equation (2.5) may be reformulated as follows through basic mathematical procedures

$$\frac{1}{2}\left(\left(\frac{(\zeta\phi'(\zeta)+\varrho\zeta^2\phi''(\zeta))'}{\phi'(\zeta)}\right)+\left(\frac{(\zeta\phi'(\zeta)+\varrho\zeta^2\phi''(\zeta))'}{\phi'(\zeta)}\right)^{\frac{1}{\nu}}\right)=1+\left(\frac{\nu+1}{\nu}\right)(1+2\varrho)d_2\zeta$$

$$+\left(\left(\frac{\nu+1}{\nu}\right)(3(1+3\varrho)d_3-2(1+2\varrho)d_2^2)+\left(\frac{1-\nu}{\nu^2}\right)(1+2\varrho)^2d_2^2\right)\zeta^2+\cdots \quad (2.8)$$

and

$$F(\mathfrak{u}(\zeta))=1+\mathcal{F}_2(\varkappa,y)\mathfrak{u}_1\zeta+[\mathcal{F}_2(\varkappa,y)\mathfrak{u}_2+\mathcal{F}_3(\varkappa,y)\mathfrak{u}_1^2]\zeta^2+\cdots \quad (2.9)$$

Applying elementary algebraic operations, we can express equations (2.6) in the following form:

$$\frac{1}{2}\left(\left(\frac{(w\psi'(w)+\varrho w^2\psi''(w))'}{\psi'(w)}\right)+\left(\frac{(w\psi'(w)+\varrho w^2\psi''(w))'}{\psi'(w)}\right)^{\frac{1}{\nu}}\right)=1-\left(\frac{\nu+1}{\nu}\right)(1+2\varrho)d_2w$$

$$+\left(\left(\frac{\nu+1}{\nu}\right)(2(2+7\varrho)d_2^2-3(1+3\varrho)d_3)+\left(\frac{1-\nu}{\nu^2}\right)(1+2\varrho)^2d_2^2\right)w^2+\cdots \quad (2.10)$$

and

$$F(\mathfrak{v}(w))=1+\mathcal{F}_2(\varkappa,y)\mathfrak{v}_1w+[\mathcal{F}_2(\varkappa,y)\mathfrak{v}_2+\mathcal{F}_3(\varkappa,y)\mathfrak{v}_1^2]w^2+\cdots. \quad (2.11)$$

Owing to (2.5), a comparison of corresponding terms in (2.8) and (2.9) leads to the following conclusions.

$$\left(\frac{\nu+1}{\nu}\right)(1+2\varrho)d_2=\mathcal{F}_2(\varkappa,y)\mathfrak{u}_1, \quad (2.12)$$

$$\left(\frac{\nu+1}{\nu}\right)(3(1+3\varrho)d_3-2(1+2\varrho)d_2^2)+\left(\frac{1-\nu}{\nu^2}\right)(1+2\varrho)^2d_2^2=\mathcal{F}_2(\varkappa,y)\mathfrak{u}_2+\mathcal{F}_3(\varkappa,y)\mathfrak{u}_1^2, \quad (2.13)$$

Analogously, by virtue of the equality in (2.6), a comparison of like-degree terms in (2.10) and (2.11) yields

$$-\left(\frac{\nu+1}{\nu}\right)(1+2\varrho)d_2=\mathcal{F}_2(\varkappa,y)\mathfrak{v}_1, \quad (2.14)$$

and

$$\left(\frac{\nu+1}{\nu}\right)(2(2+7\varrho)d_2^2-3(1+3\varrho)d_3)+\left(\frac{1-\nu}{\nu^2}\right)(1+2\varrho)^2d_2^2=\mathcal{F}_2(\varkappa,y)\mathfrak{v}_2+\mathcal{F}_3(\varkappa,y)\mathfrak{v}_1^2. \quad (2.15)$$

From (2.12) and (2.14), we get

$$\mathfrak{u}_1=-\mathfrak{v}_1, \quad (2.16)$$

and

$$2\left(\frac{\nu+1}{\nu}\right)^2(1+2\varrho)^2d_2^2=\mathcal{F}_2^2(\varkappa,y)(\mathfrak{u}_1^2+\mathfrak{v}_1^2). \quad (2.17)$$

The sum of (2.13) and (2.15) produces

$$\left(\left(\frac{\nu+1}{\nu}\right)(1+5\varrho)+\left(\frac{1-\nu}{\nu^2}\right)(1+2\varrho)^2\right)2d_2^2=\mathcal{F}_2(\varkappa,y)(\mathfrak{u}_2+\mathfrak{v}_2)+\mathcal{F}_3(\varkappa,y)(\mathfrak{u}_1^2+\mathfrak{v}_1^2). \quad (2.18)$$

Replacing $\mathfrak{u}_1^2+\mathfrak{v}_1^2$ from (2.17) in (2.18) we get

$$2d_2^2=\frac{\mathcal{F}_2^3(\varkappa,y)(\mathfrak{u}_2+\mathfrak{v}_2)}{(\nu(\nu+1)(1+5\varrho)+(1-\nu)(1+2\varrho)^2)\mathcal{F}_2^2(\varkappa,y)-(\nu+1)^2(1+2\varrho)^2\mathcal{F}_3(\varkappa,y)}. \quad (2.19)$$

Applying the expressions given in (1.5) to $\mathcal{F}_2(\varkappa,y)$ and $\mathcal{F}_3(\varkappa,y)$, along with the application of (2.7) to $\mathfrak{u}_2$ and $\mathfrak{v}_2$, yields (2.1).

The derivation of equation (2.1) follows from the application of (1.5) to $\mathcal{F}_2(\varkappa,y)$ and $\mathcal{F}_3(\varkappa,y)$ and the use of (2.7) for $\mathfrak{u}_2$ and $\mathfrak{v}_2$.

Taking the difference between (2.13) and (2.15), we get the bound on $|d_3|$:

$$d_3=d_2^2+\frac{\nu\mathcal{F}_2(\varkappa,y)(\mathfrak{u}_2-\mathfrak{v}_2)}{6(\nu+1)(1+3\varrho)}. \quad (2.20)$$

If we replace d_2^2 from (2.17) in (2.20) we get

$$d_3=\frac{\nu^2\mathcal{F}_2^2(\varkappa,y)(\mathfrak{u}_1^2+\mathfrak{v}_2^2)}{2(\nu+1)^2(1+2\varrho)^2}+\frac{\nu\mathcal{F}_2(\varkappa,y)(\mathfrak{u}_2-\mathfrak{v}_2)}{6(\nu+1)(1+3\varrho)}. \quad (2.21)$$

We derive (2.2) from (2.21) by utilizing (1.5) and (2.7).

Finally, we determine the bound on $|d_3-\xi d_2^2|$ using the values of d_2^2 and d_3 from (2.19) and (2.20), respectively. Consequently, we have

$$|d_3-\xi d_2^2|=|\nu\mathcal{F}_2(\varkappa,y)|\left|\left(\frac{1}{6(\nu+1)(1+3\varrho)}+\mathcal{B}(\xi,F)\right)\mathfrak{u}_2-\left(\frac{1}{6(\nu+1)(1+3\varrho)}-\mathcal{B}(\xi,F)\right)\mathfrak{v}_2\right|,$$

where

$$\mathcal{B}(\xi,F)=\frac{(1-\xi)\nu\mathcal{F}_2^2(\varkappa,y)}{(\nu(\nu+1)(1+5\varrho)+(1-\nu)(1+2\varrho)^2)\mathcal{F}_2^2(\varkappa,y)-(\nu+1)^2(1+2\varrho)^2\mathcal{F}_3(\varkappa,y)}.$$

Clearly

$$|d_3-\xi d_2^2|\le\begin{cases}\frac{|\nu||\mathcal{F}_2(\varkappa,y)|}{3(\nu+1)(1+3\varrho)} & ; \ |\mathcal{B}(\xi,F)|\le\frac{1}{6(\nu+1)(1+3\varrho)}\\ 2|\nu||\mathcal{F}_2(\varkappa,y)||\mathcal{B}(\xi,F)| & ; \ |\mathcal{B}(\xi,F)|\ge\frac{1}{6(\nu+1)(1+3\varrho)}.\end{cases} \quad (2.22)$$

Using (2.22) and the definition of $\daleth$ in (2.4), we derive (2.3).

Remark 2.1 Theorem 3.4 in [25] follows directly from Theorem 2.1 when $\nu=1$.

Theorem 2.1, when evaluated at $\xi=1$, gives:

Corollary 2.1 Let $0<\nu\le1$, and $F(\varsigma)$ be defined as in (1.8). If $\phi\in\sigma$ is a member of $\mathfrak{T}_\sigma(\varrho,\nu,F)$, then $|d_3-d_2^2|\le\frac{|\nu k|}{3(\nu+1)(1+3\varrho)}$.

Remark 2.2 From Corollary 2.1, we obtain Corollary 3.7 in [25] by letting $\nu=1$.

III. Special cases

Using the above definition, several subfamilies of bi-univalent functions associated with GBFP can be generated by choosing specific values of the parameters ϱ and ν. The corresponding consequences of our main results then follow directly from the findings established in this paper. In this section, we discuss a few representative cases in detail.

$Case$ 1. Setting $\varrho = 0$ in the class $\mathfrak{T}_\sigma(\varrho,\nu,F)$ defines the subclass $\mathfrak{C}_\sigma(\nu,F) := \mathfrak{T}_\sigma(0,\nu,F)$, which consists of functions $\phi \in \sigma$ satisfying

$$\frac{1}{2}\left(\left(\frac{(\zeta\phi'(\zeta))'}{\phi'(\zeta)}\right)+\left(\frac{(\zeta\phi'(\zeta))'}{\phi'(\zeta)}\right)^{\frac{1}{\nu}}\right) \prec F(\zeta) = \frac{\mathcal{F}(\varkappa,y,\zeta)}{\zeta}, \zeta \in \mathfrak{U},$$

and

$$\frac{1}{2}\left(\left(\frac{(w\psi\prime(w))\prime}{\phi\prime(w)}\right)+\left(\frac{(w\psi\prime(w))\prime}{\psi\prime(w)}\right)^{\frac{1}{\nu}}\right) \prec F(w) = \frac{\mathcal{F}(\varkappa,y,w)}{w}, w \in \mathfrak{U},$$

where $0 < \nu \le 1$, and $F(\zeta)$ is as in (1.8).

Corollary 3.1 Let $\xi \in \mathbb{R}, 0 < \nu \le 1$, and $F(\varsigma)$ be defined as in (1.8). If $\phi \in \sigma$ is a member of $\mathfrak{C}_\sigma(\nu,F)$, then

$$|d_2| \le \nu|k|\sqrt{\frac{|k|}{|2\nu k^2+(\nu+1)^2 l|}}, \quad |d_3| \le \frac{\nu^2k^2}{(\nu+1)^2}+\frac{\nu|k|}{3(\nu+1)},$$

and

$$|d_3-\xi d_2^2| \le \begin{cases} \frac{\nu|k|}{3(\nu+1)} & ; |1-\xi| \le \left|\frac{2\nu k^2+(\nu+1)^2 l)}{3\nu(\nu+1)k^2}\right| \\ \frac{\nu^2|k|^3\,|1-\xi|}{|2\nu k^2+(\nu+1)^2 l|} & ; |1-\xi| \ge \left|\frac{2\nu k^2+(\nu+1)^2 l)}{3\nu(\nu+1)k^2}\right|. \end{cases}$$

Remark 3.1 From Corollary 3.1, we get Corollary 3 and Corollary 7 in [36] by letting $\nu = 1$.

Corollary 3.1, when evaluated at $\xi = 1$, gives:

Corollary 3.2 Let $0 < \nu \le 1$, and $F(\varsigma)$ be defined as in (1.8). If $\phi \in \sigma$ is a member of $\mathfrak{C}_\sigma(\nu,F)$, then $|d_3 - d_2^2| \le \frac{\nu|k|}{3(\nu+1)}$.

$Case$ 2. Setting $\varrho = 1$ in the class $\mathfrak{T}_\sigma(\varrho,\nu,F)$ defines the subclass $\mathfrak{F}_\sigma(\nu,F) := \mathfrak{T}_\sigma(1,\nu,F)$, which consists of functions $\phi \in \sigma$ satisfying

$$\frac{1}{2}\left(\frac{\zeta\phi''(\zeta)}{\phi'(\zeta)}\left(1+\frac{(\zeta^2\phi''(\varsigma))'}{\zeta\phi''(\zeta)}\right)+\left(\frac{\zeta\phi''(\zeta)}{\phi'(\zeta)}\left(1+\frac{(\zeta^2\phi''(\zeta))'}{\zeta\phi''(\zeta)}\right)\right)^{\frac{1}{\nu}}\right) \prec F(\zeta) = \frac{\mathcal{F}(\varkappa,y,\zeta)}{\zeta},$$

and

$$\frac{1}{2}\left(\frac{w\phi''(w)}{\phi'(w)}\left(1+\frac{(w^2\phi''(w))'}{w\phi''(w)}\right)+\left(\frac{w\phi''(w)}{\phi'(w)}\left(1+\frac{(w^2\phi''(w))'}{w\phi''(w)}\right)\right)^{\frac{1}{\nu}}\right) \prec F(w) = \frac{\mathcal{F}(\varkappa,y,w)}{w},$$

where $\zeta, w \in \mathfrak{U}$, $0 < \nu \le 1$, and $F(\zeta)$ is as in (1.8).

Corollary 3.3 Let $\xi \in \mathbb{R}, 0 < \nu \le 1$, and $F(\varsigma)$ be defined as in (1.8). If $\phi \in \sigma$ is a member of $\mathfrak{F}_\sigma(\nu,F)$, then

$$|d_2| \le \nu|k|\sqrt{\frac{|k|}{3|\nu(\nu+7)k^2+3(\nu+1)^2 l|}}, \quad |d_3| \le \frac{\nu^2k^2}{9(\nu+1)^2}+\frac{\nu|k|}{12(\nu+1)},$$

and

$$|d_3-\xi d_2^2| \le \begin{cases} \frac{\nu|k|}{12(\nu+1)} & ; |1-\xi| \le \left|\frac{\nu(\nu+7)k^2+3(\nu+1)^2 l}{4\nu(\nu+1)k^2}\right| \\ \frac{\nu^2|k|^3\,|1-\xi|}{3|\nu(\nu+7)k^2+3(\nu+1)^2 l|} & ; |1-\xi| \ge \left|\frac{\nu(\nu+7)k^2+3(\nu+1)^2 l}{4\nu(\nu+1)k^2}\right|. \end{cases}$$

Corollary 3.3, when evaluated at $\xi = 1$, gives:

Corollary 3.4 Let $0 < \nu \le 1$, and $F(\varsigma)$ be defined as in (1.8). If $\phi \in \sigma$ is a member of $\mathfrak{F}_\sigma(\nu,F)$, then $|d_3 - d_2^2| \le \frac{\nu|k|}{12(\nu+1)}$.

$Case$ 3. Setting $\nu = 1$ in the class $\mathfrak{F}_\sigma(\nu,F)$ defines the subclass $\mathfrak{G}_\sigma(F) := \mathfrak{F}_\sigma(1,F)$, which consists of functions $\phi \in \sigma$ satisfying

$$\frac{\zeta\phi''(\zeta)}{\phi'(\zeta)}\left(1+\frac{(\zeta^2\phi''(\zeta))'}{\zeta\phi''(\zeta)}\right) \prec F(\zeta) = \frac{\mathcal{F}(\varkappa,y,\zeta)}{\zeta},$$

and

$$\frac{w\phi''(w)}{\phi'(w)}\left(1+\frac{(w^2\phi''(w))'}{w\phi''(w)}\right) \prec F(w) = \frac{\mathcal{F}(\varkappa,y,w)}{w},$$

where $\zeta, w \in \mathfrak{U}$, and $F(\zeta)$ is as in (1.8).

Corollary 3.5 Let $\xi \in \mathbb{R}$, and $F(\varsigma)$ be defined as in (1.8). If $\phi \in \sigma$ is a member of $\mathfrak{G}_\sigma(F)$, then

$$|d_2| \le \nu|k|\sqrt{\frac{|k|}{12|2k^2+3l|}}, \quad |d_3| \le \frac{\nu^2k^2}{36}+\frac{\nu|k|}{24},$$

and

$$|d_3-\xi d_2^2| \le \begin{cases} \frac{\nu|k|}{24} & ; |1-\xi| \le \left|\frac{2k^2+3l}{2k^2}\right| \\ \frac{\nu^2|k|^3\,|1-\xi|}{12|2k^2+3l|} & ; |1-\xi| \ge \left|\frac{2k^2+3l}{2k^2}\right|. \end{cases}$$

Corollary 3.5, when evaluated at $\nu = 1$, gives:

Corollary 3.6 Let $F(\zeta)$ be defined as in (1.8). If $\phi \in \sigma$ is a member of $\mathfrak{G}_\sigma(F)$, then $|d_3 - d_2^2| \leq \frac{|k|}{24}$.

IV. Conclusion

In this work, we introduced and studied a new subclass of σ subordinate to the generating function of GBFP, denoted by $\mathfrak{T}_\sigma(\varrho, \nu, F)$. For functions belonging to $\mathfrak{T}_\sigma(\varrho, \nu, F)$, we derived estimates for the initial Maclaurin coefficients $|d_2|$ and $|d_3|$, and also evaluated FSF of the form $|d_3 - \xi d_2^2|$ for a real parameter ξ.

These results may further be extended to the (p, q)-derivative based operator classes (such as those in [17]) under suitable selections of the parameters. Through appropriate parameter selections, the obtained estimates not only subsume several known results as special cases but also provide new coefficient bounds for previously uninvestigated subclasses. The relationships between the present findings and earlier contributions have been outlined to underline both the relevance and the novelty of the subclass under consideration. As a possible direction for future research, this subclass may be studied in the setting of higher-order Hankel and Toeplitz determinant problems, including, for instance, fourth and higher orders.

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