A FUZZY SEMI_ESSENTIAL SUBMODULE OF A FUZZY MODULE

Asst. Prof. Hussein H. Abbas
DEPARTMENT OF MATHMATICS ,COLLAGE OF EDUCATION OF GIRLS ,
UNIVERSITY OF KUFA
E-mail: Msc_hussien@yahoo.com

SHUKOR NEAMAH AL-AEASHI
DEPARTMENT OF MATHMATICS COLLAGE OF EDUCATION OF GIRLS
UNIVERSITY OF KUFA
E-mail: sh_aeashi@yahoo.com

Abstract

In this paper, we introduce the concept of fuzzy semi-essential (large) submodule , and we study a necessary and sufficient condition for a fuzzy submodule of a fuzzy module to be a fuzzy semi-essential (large) submodule , also fuzzy images and fuzzy inverse-images of generalized fuzzy semi-essential (large) submodule are studied.

Index Terms(key words)—Fuzzy set, Fuzzy module , Fuzzy semi-essential (large) submodule.

Introduction

The concept of fuzzy sets was introduced by Zadeh in 1965 [9]. It was first applied to the theory of groups by Rosenfeld in 1971 [8] . Since then, many authors introduced fuzzy subring and fuzzy ideals [5],[2]The concept of fuzzy module was introduced by Negoita and Relescu in 1975 [10]. Since then several authors have studied fuzzy modules. For examples see ([4], [5], [3], [7]). we recall the definitions of fuzzy module, fuzzy submodule with some properties of them, which are needed later . In this paper we give the definition of fuzzy semi-essential (large) submodule of fuzzy module and we study some of its propositions.

1-Fuzzy sets

In this section, we shall give the concept of fuzzy set with some basic definitions and some of its properties .

Definition 1.1 [9] :

Let S be a non-empty set and I be the closed interval [0, 1] of the real line (real numbers). A fuzzy set A of S (a fuzzy subset of S) is a function from S into I.

Definition 1.2 [9] :

A fuzzy set A of a set S is called a fuzzy constant if $A(x) = t$ for all $x \in S$ where $t \in [0, 1]$.

Definition 1.3 [10] :

Let $x_t : S \to [0, 1]$ be a fuzzy set of S, where $x \in S$, $t \in [0, 1]$ defined by:

$$x_t(y) = \begin{cases} t & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \quad \text{, for all } y \in S.$$

x_t is called a fuzzy singleton or fuzzy point in S.

Proposition 1.4[7] :

Let a_t and b_k be two fuzzy singletons of a set S. If $a_t = b_k$, then a = b and t = k , where t, k $\in$ [0, 1].

Definition 1.5 [11] :

Let A and B be two fuzzy sets in S, then:

(1) A=B if and only if A (x) = B (x), for all x $\in$ S.

(2) A $\subseteq$ B if and only if A (x) $\leq$ B (x), for all x $\in$ S . If A $\subset$ B and there exists x $\in$ S such that A (x) < B (x), then:

A is called a proper fuzzy subset of B and written A $\subset$ B.

Definition 1.6 [10] :

Let A and B be two fuzzy sets in S, then :

(1) (A $\cap$ B) (x) = min {A (x), B (x)}, for all x $\in$ S.

(2) (A $\cup$ B) (x) = max {A (x), B (x)}, for all x $\in$ S.

A $\cap$ B and A $\cup$ B are fuzzy sets in S,

In general . if $\{A_\alpha , \alpha \in \Lambda\}$ is a family of fuzzy sets in S, then :

$\left(\bigcap_{\alpha\in\Lambda} A_\alpha\right)$ (x) = inf { A_α (x), $\alpha \in \Lambda$}, $\left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)$ (x) = sup { A_α (x), $\alpha \in \Lambda$}, for all x $\in$ S.

which are also fuzzy sets in S.

Definition 1.7 [4], [5] :

Let A be a fuzzy set in S, for all t $\in$ [0, 1], the set A_t = { x $\in$ S, A (x) $\geq$ t} is called a level subset of A.

Note that, A_t is a subset of S in the ordinary sense.

Remark 1.8 [9] :

The following properties of level subsets hold for each t $\in$ [0, 1].

(1) $(A \cap B)_t = A_t \cap B_t$,

(2) $(A \cup B)_t = A_t \cup B_t$ and

(3) A = B if and only if $A_t = B_t$, for all t $\in$ [0, 1].

Now, we give the definitions of image and inverse image of fuzzy sets.

Definition 1.9 [9] :

Let f be a mapping from a set M into a set N, let A be a fuzzy set in M and B be a fuzzy set in N.

The image of A denoted by f (A) is the fuzzy set in N defined by:

$$f(A)(y)=\begin{cases}\sup\{A(z)\,|\, z\in f^{-1}(y)\}\, if\ f^{-1}(y)\neq\phi,\ for all\ y\in N,\\ 0 \quad otherwise\end{cases}$$

where f^{-1} (y) = { x $\in$ M, f (x) = y}.

And the inverse image of B, denoted by f^{-1} (B) is the fuzzy set in M defined by:

f^{-1} (B) (x) = B (f(x)), for all x $\in$ M.

Definition 1.10 [2] :

Let f be a function from a set M into a set N . A fuzzy subset A of M is called f-invariant if A (x) = A (y) whenever f (x) = f (y) , where x, y $\in$ M.

Proposition 1.11 [2] :

If f is a function defined on a set M, A_1 and A_2 are fuzzy subsets of M, B_1 and B_2 are fuzzy subsets of f (M). The following are true:

(1) $A_1 \subseteq f^{-1}(f(A_1))$.

(2) $A_1 = f^{-1}(f(A_1))$, whenever A_1 is f-invariant.

(3) $f(f^{-1}(B_1)) = B_1$.

(4) If $A_1 \subseteq A_2$, then $f(A_1) \subseteq f(A_2)$.

(5) If $B_1 \subseteq B_2$, then $f^{-1}(B_1) \subseteq f^{-1}(B_2)$.

Proposition 1.12[7] :

Let f be a function from a set M into a set N. If B_1 and B_2 are fuzzy subsets of N,

then $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

2-Fuzzy Modules and Fuzzy Submodules

In this section, we recall the definitions of fuzzy modules and fuzzy submodules with some fundamental results about them.

Definition 2.1 [8], [5] : Let M be an R-module. A fuzzy set X of M is called fuzzy module of an R-module M if :

(1) $X(x-y) \geq \min\{X(x), X(y)\}$, for all $x, y \in M$
(2) $X(rx) \geq X(x)$, for all $x \in M$ and $r \in R$.
(3) $X(0) = 1$.

Definition 2.2 [3] : Let X and A be two fuzzy modules of an R-module M. A is called a fuzzy submodule of X if $A \subseteq X$.

Proposition 2.3 [6] : Let A be a fuzzy set of an R-module M. Then the level subset A_t, $t \in [0, 1]$ is a submodule of M if and only if A is a fuzzy submodule of X where X is a fuzzy module of an R-module M.

Now, we shall give some properties of fuzzy submodules, which will be used in the next section.

Definition 2.4 [4] : If A is a fuzzy module of an R-module M, then the submodule A_t of M is called the level submodule of M where $t \in [0, 1]$.

Proposition 2.5 [7]:

Let X be a fuzzy module of an R-module M. Let $\{A_\alpha, \alpha \in \Lambda\}$ be a family of fuzzy submodules of X, then

(1) $\bigcap_{\alpha \in \Lambda} A_\alpha$ is a fuzzy submodule of X.

(2) If $\{A_\alpha, \alpha \in \Lambda\}$ is a chain, then $\bigcup_{\alpha \in \wedge} A_\alpha$ is a fuzzy submodule of X.

Definition 2.6 [10] :

Let A and B be two fuzzy subset of an R-module M. Then $(A + B)(x) = \sup\{\min\{A(a), B(b), x = a + b\}, a, b \in M$, for all $x \in M$. A + B is a fuzzy subset of M.

Proposition 2.7 [10] :

Let A and B be two fuzzy submodules of a fuzzy module X, then A + B is a fuzzy submodule of X.

Definition 2.8 [3] : Let X and Y be two fuzzy modules of R-modules M_1 and M_2 respectively. $f: X \to Y$ is called a fuzzy homomorphism if $f : M_1 \to M_2$ is homomorphism and $Y(f(x)) = X(x)$, for each $x \in M_1$.

Proposition 2.9[7] : Let X and Y be two fuzzy modules of R-modules M_1 and M_2 respectively. Let $f : X \to Y$ be a fuzzy homomorphism.

If A and B are two fuzzy submodules of X and Y respectively, then :

(1) f (A) is a fuzzy submodule of Y. [13]
(2) $f^{-1}(B)$ is a fuzzy submodule of X. [13]
(3) $f(A \cap A_1) = f(A) \cap f(A_1)$ is a fuzzy submodule of Y, A_1 is a fuzzy submodule of X. [13]

Definition 2.10 [4] :

Let X be a fuzzy module. A proper fuzzy submodule A of X is called a prime fuzzy submodule whenever$(r_t a_k) \subseteq A$ for fuzzy singleton r_t of R and $a_k \subseteq X$ we have either $r_t \subseteq (A : X)$ or $a_k \subseteq A$ where: $(A \underset{R}{:} X) = \{r_t : r_t X \subseteq A, r_t$ fuzzy singleton of R$\}$.

3-A fuzzy semi-essential submodule of a fuzzy module

In this section we give the definition of a fuzzy semi-essential submodule of a fuzzy module and we study its properties about the image or inverse image of any fuzzy semi-essential submodule and we study anther properties about it .

Definition 3.1:-[1] A non-zero R-submodule N of an R-module M is called a semi-essential submodule of M if $N \cap P=(0)$ for all a prime submodule P of M then P=(0) .

Definition 3.2 Let A be a fuzzy module of an R-module M if B a fuzzy submodule of A then B is called a fuzzy semi-essential submodule of A if for all fuzzy prime submodule P of A and $B \cap P=0_1$ then $P=0_1$.

Example 3.3 Let A be a fuzzy module of a Z-module Z^2 with scalar multiplication $(a,b)r=(ar,br)$ defined as:

$$A((a,b))=\begin{cases} 1 & \text{if } a+b \in Z_e \\ 1/2 & \text{if } a+b \in Z_o \end{cases}$$

$$B((a,b))=\begin{cases} 1 & \text{if } a,b \in Ze \\ 1/3 & \text{if } a,b \in Zo \\ 1/4 & \text{otherwise} \end{cases}$$

To prove B is a fuzzy semi-essential submodule of A , Let P be a fuzzy prime submodule of A such that $B \cap P=((0,0))_1$

It is means that

$$\min \{B(a,b),\ P\{(a,b)\}= \begin{cases} 1 & \text{if } (a,b)=(0,0) \\ 0 & \text{if } (a,b)\neq(0,0) \end{cases}$$

for all $(a,b) \in Z^2$

Now if $(a,b)=(0,0) \Longrightarrow P((a,b))=1$ (since P is a fuzzy submodule)

If $(a,b) \neq (0,0) \Longrightarrow P(a,b)=0$

(since $B((a,b)) \neq 0$ for all $(a,b) \in Z^2$)

Therefore $P=(0,0)_1$.

Hence B is a fuzzy semi-essential submodule of A Now we recall the definition of a semi-essential submodule .

Proposition 3.4 Let A be a fuzzy module of an R-module M . A fuzzy submodule B of A is a fuzzy semi-essential submodule if B_t is semi-essential submodule of A for all $t \in (0,1]$.

Proof:-

B is a fuzzy submodule of A (by proposition 2.3)

suppose U is a prime fuzzy submodule of A and $B \cap U=O_1$ this implies

$(B \cap U)_t=B_t \cap U_t =(0_1)_t =\{0\}$ for all $t \in (o,1]$ (by proposition 1.8)

$U_t=(0_1)_t$(since B_t is a semi-essential submodule of A)

then $U=O_1$ (by proposition 1.8)

Therefore, B is a fuzzy semi-essential submodule of A.

Proposition 3.5 Let X be a fuzzy module of an R-module M if A and B are two fuzzy semi-essential submodule of X then $A\cup B$ is a fuzzy semi-essential submodule of X whenever $A\subseteq B$ or $B\subseteq A$.

Proof: If $B\subseteq A$,

$A\cup B$ is fuzzy submodule of X(by proposition 2.5)

If $(A\cup B)\cap U=0_1$
Where U is a prime fuzzy submodule of X

$(A\cup B)\cap U=(A\cap U)$ (since $B\subseteq A$).

$=0_1$

(since A is fuzzy semi-essential submodule of X) .

This implies $U=O_1$,hence $A\cup B$ is a fuzzy semi-essential submodule of X .

Similarly if $A\subseteq B$.

Proposition 3.6

Let A be a fuzzy submodule of a module M and B be a fuzzy semi-essential submodule of A then the set $B^*=\{X\in M :B(x)> 0\}$ is a semi-essential submodule of M.

Proof: $B^*=\{X\in M :B(x)> 0\} =\bigcup_t Bt$ where $t \in (o,1]$(by proposition 3.4, 3.5).

B^* is semi-essential submodule of M .

Proposition3.7

Let $\mathcal{F}$ be a bijective function from a fuzzy module A into a fuzzy module B ($\mathcal{F}$:A $\to$B) . If N be a fuzzy semi-essential submodule of A then $\mathcal{F}$(N) is fuzzy semi-essential submodule of B .

Proof :

$\mathcal{F}$(N) is submodule of B (by proposition 2.9).

Now, Suppose that U be a prime fuzzy submodule of B such that $\mathcal{F}(N)\cap U=0_1$

$\mathcal{F}^{-1}(\mathcal{F}(N)\cap U)=\mathcal{F}^{-1}(0_1)$

$\mathcal{F}^{-1}(\mathcal{F}(N))\cap\mathcal{F}^{-1}(U)= O_1$ (by proposition 1.11 and $\mathcal{F}$ is a bijective).

$N\cap\mathcal{F}^{-1}(U)=O_1$ (since $\mathcal{F}$ is a bijective)

$\mathcal{F}^{-1}(U)=O_1$ (since N is a fuzzy semi-essential submodule and $\mathcal{F}^{-1}(U)$ is submodule of A)

$\mathcal{F}(\mathcal{F}^{-1}(U))=\mathcal{F}(O_1)$ (by proposition 1.11 and $\mathcal{F}$ is a bijective). Thus $U=O_1$

Then $\mathcal{F}$(N) is fuzzy semi-essential submodule of B .

Proposition3.8

Let $\mathcal{F}$ be a bijective function from a fuzzy module A into a fuzzy module B ($\mathcal{F}$:A $\to$B) if M be a fuzzy semi-essential submodule of B then $\mathcal{F}^{-1}$(M) is fuzzy semi-essential submodule of A .

Proof:

$\mathcal{F}^{-1}$(M) is a fuzzy submodule of A (by proposition 2.9).

Suppose Ube a prime fuzzy submodule of A such that $\mathcal{F}^{-1}(M)\cap U=O_1$

$$\mathcal{F}(\mathcal{F}^{-1}(M)\cap U)=\mathcal{F}(O_1)$$

$\mathcal{F}(\mathcal{F}^{-1}(M))\cap \mathcal{F}(U)=\mathcal{F}(0_1)$ (by proposition 1.11).

$M \cap \mathcal{F}(U) = O_1$ (since M is fuzzy semi-essential of A).

$\mathcal{F}(U) = O_1$

$\mathcal{F}^{-1}(\mathcal{F}(U)) = \mathcal{F}^{-1}(O_1)$

$U=O_1$ (by proposition 1.11 and $\mathcal{F}$ is a bijective).

Then $\mathcal{F}^{-1}(M)$ is fuzzy semi-essential submodule of A .

Proposition 3.9

Let A and B be two fuzzy semi-essential submodule of a fuzzy module X of an R-module M then A+B is also fuzzy semi-essential submodule of X .

Proof :

A+B is a fuzzy submodule of X (by proposition 2.7).

Since A and B are fuzzy semi-essential submodule of X that mean (i.e.)

if $A \cap U=O_1 \Longrightarrow U=O_1$

if $B \cap U=O_1 \Longrightarrow U=O_1$, For all U is a prime fuzzy submodule of X

Now , if $(A+B) \cap U=O_1$ for all a prime fuzzy submodule U of X

Then $(A \cap U)+(B \cap U)=O_1$ (Distribution law).

Therefore $(A \cap U) =O_1$ and $(B \cap U)=O_1$ implies $U=O_1$ (since both A and B are semi-essential submodule of X), Yields A+B is a fuzzy semi-essential submodule of X .

Proposition 3.10 Let A,B,M and N are fuzzy module of R-module X such that $A \subseteq B \subseteq M \subseteq N$ and A is fuzzy semi-essential submodule of N then B is fuzzy semi-essential submodule of M .

Proof : Let U be a fuzzy submodule of M and $B \cap U=O_1$

then $A \cap U=O_1$ (Since $A \subseteq B$).

Since A is fuzzy semi-essential submodule of N and $(U \subseteq M \subseteq N)$ Then $U=O_1$

Therefore,

B is fuzzy semi-essential submodule of M.

Proposition3.11

Let X be a fuzzy module of an R-module M and let N_1 and N_2 are two fuzzy submodules of X such that $N_1$is a fuzzy submodule of N_2 then N_1 is a fuzzy semi-essential submodule of X then N_2 is a fuzzy semi-essential submodule of X.

Proof : Suppose that U be a fuzzy prime submodule of X ,

such that $U \cap N_2=O_1$

$U \cap N_1=O_1$ (since $N_1 \subseteq N_2$).

$U=O_1$ (N_1 is a fuzzy semi-essential submodule of X).

Then N_2 is a fuzzy semi-essential submodule of X.

REFERENCES

[1] Al-daban. N K,"semi-essential *submodules and semi- uniform module*" , M.Sc . Thesis, University of Tikrit,(2005).

[2] Kumar R., "*Fuzzy Semiprimary Ideals of Rings*", Fuzzy Sets and Systems, Vol. 42, PP. 263-272, (1991).

[3] Majumdar S., "*Theory of Fuzzy Modules*", Eull Col. Math. Sce., Vol. 82, PP. 395-399, (1990).

[4] Martinez L., "Fuzzy Modules Over Fuzzy Rings in Connection with Fuzzy Ideal of Ring", J. Fuzzy Math. Vol. 4, PP. 843-857, (1996).

[5] Mashinchi M. and Zahedi M. M., "*On L-Fuzzy Primary Submodules*", Fuzzy Sets and Systems, Vol. 49, PP. 231-236, (1992).

[6] Mukhejee T.K., Sen M. K. and Roy D., "*On Submodules and their Radicals*", J. Fuzzy Math. Vol. 4, PP. 549-558, (1996).

[7] Rabi H. J., "*Prime Fuzzy Submodule and Prime Fuzzy Modules*", M. Sc. Thesis, University of Baghdad, (2001).

[8] Rosenfeld .A., "Fuzzy groups," J. Math. Anal. Appl., vol. 35, pp. 512-517, 1971

[9] Zadeh L. A., "Fuzzy Sets, Information and control" , Vol. 8, PP. 338-353, 1965.

[10] Zahedi M. M., "On L-Fuzzy Residual Quotient Modules and P. Primary Submodules", Fuzzy Sets and Systems, Vol. 51, PP. 33-344, (1992).

[11] Zahedi M. M., " A characterization of L-Fuzzy Prime Ideals", Fuzzy Sets and Systems, Vol. 44, 147-160, (1991).

[12] Zhao Jiandi, Shi K. Yue M., "*Fuzzy Modules over Fuzzy Rings*", The J. of Fuzzy Math. Vol. 3, PP. 531-540, (1993).

[13] Zahedi M. M.., "*Some Results on L- Fuzzy Modules* ", Fuzzy Sets and Systems, Vol. 55, PP. 355-363, (1993).

الموديول الجزئي الضبابي شبه-الجوهري من الموديول الضبابي

أ.م.حسين هادي النعماني

شكر نعمة مبرد العياشي

جامعة الكوفة/كلية التربية للبنات/قسم الرياضيات

الخلاصة

في هذا البحث ,قدمنا مفهوم الموديول الجزئي الضبابي شبه-الجوهري من الموديول الضبابي ودرسنا الشرط الكافي والضروري الذي يجعل الموديول الجزئي الضبابي من الموديول الضبابي موديولاً شبه-جوهري كذلك درسنا الصورة الضبابية والصورة العكسية الضبابية للموديول الجزئي شبه الجوهري وخصائص اخرى