A suggested method for finding the general solution of the beloved equation: $u''(x)+3u(x)u'(x)+u^3(x)=0$.

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Abstract

Our aim in this paper is to find the general solution of the beloved equation which its general form is given by $u''(x)+3u(x)u'(x)+u^3(x)=0$ by using suitable substitution .

1-INTRODUCTION .

The beloved equation arises in many areas. Some of these are the analysis of the fusion of pellets, the theory of univalent functions, the stability of gaseous spheres, operator Yang–Baxter equations, motion of a free particle in a space of constant curvature, the stationary reduction of the second member of the Burgers hierarchy [1] , [6] .

The beloved equation is linearizable by a point transformation [6] , and by a generalized Sund-man transformation [6] , into the equation $y''=0$ and $y''+3y'+2y=0$, respectively.

The beloved equation Possesses both Left Painleve Series (LPS) and Right Painleve Series (RPS) [4] .

In this paper , we shall find the general solution of the beloved equation by using the reduction of order and the suitable substitution $u^2(x)=tp$, where $p=\dfrac{du}{dx}$.

Deffinitions and Basic concepts .

Definition [4] : The general form of the Right Painleve Series (RPS) is given by : $u=\sum_{i=o}^{\infty}\alpha_i x^{-p+i}$.

And the Left Painleve Series (LPS) is given by : $u=\sum_{i=o}^{\infty}\alpha_i x^{-p-i}$.

Definition [4]: the general form of Riccati transformation is given by

$$u=\alpha\frac{w'}{w}$$

where u and w are function of x .

2- The general solution of the beloved equation by using (RPS) and (LPS) [4] :

The beloved equation is given by : $u''(x)+3u(x)u'(x)+u^3(x)=0$

To find the general solution of it , get the leading order behaviour

$$u=\alpha\, x^{-p} \qquad \ldots\ldots\ldots\ldots.(1)$$

$$\Rightarrow \alpha(-p)(-p-1)x^{-p-2}+3\alpha^2(-p)x^{-2p-1}+\alpha^3x^{-3p}=0 \qquad \ldots\ldots\ldots\ldots(2)$$

If p=1 $\Rightarrow$ α is given as $\alpha^2-3\alpha+2=0 \Leftrightarrow \alpha=1,2$ …(3)

Resonances

$u=\alpha x^{-1}+\beta x^{r-1}$ …(4)

$r^2+3(\alpha-1)r+3\alpha^2-6\alpha+2=0$ by [4] .

$\Leftrightarrow r^2+3(\alpha-1)r+\alpha(2\alpha-3)=0$ …(5)

When (3) is taken into account. Hence

$r=-\alpha$, $3-2\beta=-1,1;-2,-1$ by [4] …(6)

So , the general solution of beloved equation is given by

…(7)

$$u=\frac{2A_1x+2B_1}{A_1x^2+2B_1x+C_1}$$

where A_1, $B_1$and C_1 are arbitrary constants .

Note : In fact the above solution yields under especial conditions for the and by getting the first term of the (R P S) . β and α values of p ,

3- Solution of beloved equation by Riccati transformation [6] :

Consider the beloved equation : **u″(x)+3u(x)u′(x)+u^3(x)**=0 _

By using Riccati transformation [2],[3],

$$u=\alpha\frac{w'}{w}$$

…(8)

We get

…(9)

$$\Leftrightarrow\frac{w'''}{w}+3(\alpha-1)\frac{w'w''}{w^2}+(\alpha^2-3\alpha+2)\frac{w'^3}{w^3}=0$$

…(10)

If $\alpha=1\ \Rightarrow$

$$w'''=0$$

…(11)

The coefficient of $\dfrac{w'^3}{w^3}$ is the same as for the Painleve analysis. $\alpha=1$ leads the RPS and $\alpha=2$ to the LP S

$ww'''+3w'w''=0$ ***…(12)***

w^2 integrating factor

$\Rightarrow w''=\dfrac{K}{w3}$ …(13)

Equation (13) Ermakov-Pinney equation [5]

$$\alpha\left(\frac{w'''}{w}-\frac{3w'w''}{w^2}+\frac{2w'^3}{w^3}\right)+3\alpha^2\frac{w'}{w}\left(\frac{w''}{w}-\frac{w'^2}{w^2}\right)+\alpha^3\frac{w'^3}{w^3}=0$$

The beloved equation possesses eight Lie point symmetries and is linearisable to $U''=0$ by a point transformation [4] .

Note : In fact the Riccati transformation transform the beloved equation to the nonlinear third order ordinary differential equation and its solution yields $.\alpha$ under some conditions of values of

4-Solution of beloved equation by using a nonlocal symmetry [1] :

To solve the beloved equation **u″(x)+3u(x)u′(x)+u^3(x)=0** ,

- We illustrate the procedure of reduction of order on the first member of the system (the beloved equation) using the nonlocal symmetry Σ_1. The associated Lagrange's system is

$$\frac{dx}{0}=\frac{du}{u}=\frac{du'}{u'-u^2}$$
...(14)

after one has removed the common factor of exp[-∫xdt]. The characteristics are t and

$$w=\frac{u'}{u}+u$$
…(15)

The reduced equation, an elementary Riccati equation, is

$$\frac{dw}{dx}=-w^2$$

which is easily solved to give

$$w=\frac{1}{K+x}\ .$$

When we combine this with (15) we obtain the linear first-order differential equation

$$\frac{d}{dx}\left(\frac{1}{u}\right)+\frac{1}{K+x}\left(\frac{1}{u}\right)=1$$

which has the solution

$$u(x)=\frac{(x+K)}{c+\frac{1}{2}(x+K)^2}$$

Where K and c are arbitrary constants .

<u>Note :</u> we saw at the three above methods for finding the solution of the beloved equation , there are some conditions to find the solution . Now we introduce method to solve the beloved equation without using any conditions and we will transform it to first order ordinary differential equation .

5- The general solution of the beloved equation: $u''(x)+3u(x)u'(x)+u^3(x)=0$

To solve the beloved equation :

$$u''(x)+3u(x)u'(x)+u^3(x)=0$$

Let $u'(x)=p(x)$ so

$$u''(x)=p\frac{dp}{du}\Rightarrow p\frac{dp}{du}+3u(x)p+u^3(x)=0$$

Suppose $u^2(x)=pt \Rightarrow 2u(x)du = pdt + tdp \Rightarrow$

$$du=\frac{pdt+tdp}{2t^{\frac{1}{2}}p^{\frac{1}{2}}}\Rightarrow pdp+(3p^{\frac{3}{2}}t^{\frac{1}{2}}+p^{\frac{3}{2}}t^{\frac{3}{2}})\frac{tdp+pdt}{2t^{\frac{1}{2}}p^{\frac{1}{2}}}=0$$

$$(2+3t+t^2)dp+(3+t)pdt=0\Rightarrow\frac{dp}{p}+\frac{(3+t)dt}{2+3t+t^2}=0\Rightarrow$$

$$\int\frac{dp}{p}+\int\frac{(3+t)dt}{(t+1)(t+2)}=\int 0dt\Rightarrow \ln p+\ln(t+1)^2-\ln(t+2)=\ln c_1$$

$$\Rightarrow \ln\frac{p(t+1)^2}{t+2}=\ln c_1\Rightarrow p(t^2+2t+1)=c_1(t+2)$$

Since , $t=\frac{u^2(x)}{p}$

$$\Rightarrow p\left(\frac{u^4(x)}{p^2}+\frac{2u^2(x)}{p}+1\right)-c_1\frac{u^2(x)}{p}-2c_1=0$$

$,then$

$$u^4(x)+2u^2(x)p+p^2-c_1u^2(x)-2c_1p=0$$

$,then$

$$P^2+(2u^2(x)-2c_1)p+u^4(x)-c_1u^2(x)=0$$

, then

$$p=c_1-u^2(x)\pm\sqrt{c_1^2-c_1u^2(x)}$$

, then

Now;let

$$u(x)=\sqrt{c_1}\sin\sigma\Rightarrow\frac{du}{dx}=p=\sqrt{c_1}\cos\sigma\frac{d\sigma}{dx}\Rightarrow$$

$$\sqrt{c_1}\cos\sigma\frac{d\sigma}{dx}+c_1\sin^2\sigma=c_1\pm\sqrt{c_1^2-c_1^2\sin^2\sigma}\Rightarrow$$

$$\frac{1}{\sqrt{c_1}}\left(\frac{\cos\sigma}{1-\sin^2\sigma+\cos\sigma}\right)d\sigma=dx\Rightarrow\frac{1}{\sqrt{c_1}}\int\frac{d\sigma}{1+\cos\sigma}=\int dx\Rightarrow$$

$$\frac{1}{\sqrt{c_1}}(\csc\sigma-\cot\sigma)=x+c_2\Rightarrow\frac{1}{\sqrt{c_1}}\left(\frac{1-\cos\sigma}{\sin\sigma}\right)=x+c_2$$

$$\Rightarrow\frac{1}{\sqrt{c_1}}\left(\frac{1-\sqrt{1-\sin^2\sigma}}{\sin\sigma}\right)=x+c_2$$

Since :

$$\sin\sigma=\frac{u(x)}{\sqrt{c_1}}\Rightarrow\frac{1}{\sqrt{c_1}}\left(\frac{1-\sqrt{1-\frac{u^2(x)}{c_1}}}{\frac{u(x)}{\sqrt{c_1}}}\right)=x+c_2\Rightarrow$$

$$1-\sqrt{1-\frac{u^2(x)}{c_1}}=(x+c_2)u(x)\Rightarrow1-(x+c_2)u(x)=\sqrt{1-\frac{u^2(x)}{c_1}}\Rightarrow$$

$$1-2(x+c_2)u(x)+(x+c_2)^2u^2(x)=1-\frac{u^2(x)}{c_1}\Rightarrow$$

$$(x+c_2)^2u^2(x)-2(x+c_2)u(x)+\frac{u^2(x)}{c_1}=0\Rightarrow$$

$$u(x)[(x+c_2)^2u(x)-2(x+c_2)+\frac{u(x)}{c_1}]=0$$

Either u(x)=0 is trivial solution

Or

$$(x+c_2)^2u(x)-2(x+c_2)+\frac{u(x)}{c_1}=0\Rightarrow$$

$$[(x+c_2)^2+\frac{1}{c_1}]u(x)=2(x+c_2)\Rightarrow$$

$$u(x)=\frac{2(x+c_2)}{(x+c_2)^2+\frac{1}{c_1}};\quad c_1\neq0$$

, where c_1 and c_2 are arbitrary constants . .

المستخلص

هدفنا في هذا البحث أيجاد الحل العام للمعادلة المحبوبة التي صيغتها العامة :-

$$u''(x)+3u(x)u'(x)+u^3(x)=0$$

بأستخدام تعويض مناسب .

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