Near Prime Spectrum

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Abstract:

Let R be a commutative ring with identity . It is well known that a topology was defined for $Spec(R) = \{I: I \text{ is a prime ideal of } R\}$ called the Zariski topology (prime spectrum) . In this paper we will generalize this idea for near prime ideal . If N be a commutative near-ring with identity , P_N be a near prime ideal of N and define $Spec(N) = \{P_N: P_N \text{ is a near prime ideal }\}$. Then $Spec(N)$ can be endowed with a topology similar to the Zariski topology which is called near Zariski topology (near prime spectrum) . we studies and discuss some of properties of such topology .

Keywords : prime ideal ;near prime ideal ;prime spectrum ; near prime spectrum .

1-Introduction .

Let R be a ring with identity . The theory of the prime spectrum of R where $Spec(R) = \{I: I \text{ is a prime ideal of } R\}$ has been form 1930 . The modern theory was developed by Jacobson and Zariski mainly [3] . A topology was defined on $Spec(R)$ as follows : For each ideal E of R if $V(E) = \{P \in Spec(R): E \subseteq P\}$, then the collection $\xi = \{V(I): I \text{ is an ideal of } R\}$ satisfies the axioms of closed subset of a topology for $X(R) = Spec(R)$, called the Zariski topology for $Spec(R)$ [11 p. 100-101] . In 1992 this idea generalized for modules by P. A. Hummadi [12] . She proved that , if M is a multiplication R-module (R is a commutative ring with identity) or more generally a locally cyclic module , then $Spec(M) = \{P: P \text{ is a prime submodule of } M\}$ can be endowed with a topology similar to the Zariski topology . later , in 1994 [13] , the prime spectrum for modules was studied , some new results were given . Chin-pi L. (1995)[9] , Chin-pi L. (1999)[10] , have studied Zariski topology on the prime spectrum of a module .

In this work we generalize this idea for near prime ideals . Let N be a commutative near ring with identity .If $Spec(N)$ is the set of all near prime ideals of N , for a subset E of Nwe defined$V(E)$as the set of all near prime ideal containing E . Some properties of the sets $V(E)$ are given . it is shown that $Spec(N)$ can be endowed with a topology similar to the Zariski topology which is called near prime spectrum topology . Some properties of the space $Spec(N)$ are given . It is shown , how some algebraic properties of N are reflected on the topological properties of $Spec(N)$. We are show that $Spec(N)$ is compact space , T_0 space and any near-ring homomorphism $\varphi: N_1 \to N_2$ induces a continuous map $\varphi^*: Spec(N_2) \to Spec(N_1)$ define by $\varphi^*(P_N) = \varphi^{-1}(P_N)$ where P_N is a near prime ideal of N_2 .

2-Some definitions and construction of the Near Prime Spectrum .

Definition 2.1. [14]

Let N be a nonempty set with two binary operations $(+)$, $(.)$. $(N,+,.)$ is called near-ring if and only if :

1. $(N,+)$ is a group (not necessarily commutative) .
2. $(N,.)$ is a semi group , that is , $(.)$ is closed on N and satisfies the associativety .
3. For all $n_1,n_2,n_3 \in N$; $(n_1+n_2).n_3 = n_1.n_3 + n_2.n_3$ (right distributive law)

This near-ring will be termed as right near-ring . If $n_1.(n_2+n_3) = n_1.n_2 + n_1.n_3$ instead of condition (3) the set N satisfies , then we call N a left near-ring .

If $1.n = n$ $(n.1 = n)$ then N has a left identity(right identity) . If $(N,+)$ is abelian , we call N an abelian near-ring . If $(N,.)$ is commutative we call N itself a commutative near-ring [14] . Clearly if N is commutative near-ring then left and right distributive law is satisfied and $1.n = n.1 = n$, N is called unital commutative near-ring .

In this paper the word (N be a near-ring) shall mean a unital commutative near-ring .

Definition 2.2. [14]

Let $(N,+,.)$ and $\left(N_1,+',.'\right)$ be two near-rings then $h: N \to N_1$ is called a near-ring homomorphism if for each $m,n \in N$, $h(m+n) = h(m)+'h(n)$ and $h(m.n) = h(m).'h(n)$, and h is said to be unital if h is near-ring homomorphism and $h(1_N) = 1_{N_1}$.

Definition 2.3. [7]

A nonempty subset A of N is called N-subgroup of N , if $(A,+)$ is a sub group of $(N,+)$ and $NA \subseteq A$.

Definition2.4. [14]

Let N be a near-ring , $n \in N$, we say that n is a nilpotent if $n^m = 0$ for some $m > 0$

Definition2.5.

Let N be a near-ring with identity , for $x \in N$, x is called a unit if it has inverse for $(.)$, i.e. , there exists $x^{-1} \in N$ such that $x.x^{-1} = x^{-1}.x = 1$.

Definition2.6. [14]

Let N be a near-ring and I_N is a nonempty subset of N . $(I_N,+,.)$ is called an ideal of N if and only if :

1- $(I_N,+)$ is normal subgroup of $(N,+)$.

2- $I_N N \subseteq I_N$ and for all $n,n_1 \in N$ and for all $i \in I_N, n(n_1+i) - nn_1 \in I_N$.

If $(I_N,+)$ is normal subgroup of $(N,+)$ and $I_N N \subseteq I_N$ then $(I_N,+,.)$is called right ideals of N while $(I_N,+)$ is normal subgroup of $(N,+)$ with for all $n,n_1 \in N$ and for all

$i \in I_N, n(n_1 + i) - nn_1 \in I_N$ are called left ideals .

Definition2.7. [5]

Let N be a near-ring and I_N be a subset of N . We write $\sqrt{I_N} = \{x \in N : x^k \in I_N \ for\ some\ positive\ integer\ k\}$. Then $\sqrt{I_N}$ is called radical set . If N is an abelian near-ring then $(\sqrt{I_N}, +, .)$ is a near ideal .

Definition2.8.

Let N be a near-ring and S_N is a proper near ideal of N , then S_N is called a near primary ideal if $a.b \in S_N$ with $a \notin S_N$ implies $b^n \in S_N$ for some positive integer$n$.

Definition2.9. [14]

Let$P_N$be a proper near ideal of $N.P_N$is called a near prime ideal if for each near ideal $P_{1N}, P_{2N}$of N and $P_{1N.}.P_{2N} \subseteq P_N$ implies $P_{1N} \subset P_N$ or $P_{2N} \subset P_N$.

Remark 2.10.

1. Every near prime ideal is a near primary ideal .
2. If N is a abelian near-ring and I_N is a primary near ideal of N , then the near ideal $(\sqrt{I}, +, .)$ is a near prime .
3. A proper near ideal P_N of the near-ring N is a near prime ideal if for all $a, b \in N, a.b \in P_N$ implies either $a \in P_N$ or $b \in P_N$.

Definition 2.11. [14]

Let N be a near-ring , N is a near integral domain if N has no non-zero divisors of zero .i.e. N be a near-ring and there exist element $a$has this properties . Let$a \in N$, $a \neq 0$, there exist $b \in N$, $b \neq 0$ and $a \cdot b = 0$.

Definition 2.12. [1]

Let N be a near-ring , $I_N$be a near ideal of N . Let $N/I_N = \{n + I_N, n \in N\}$ be the set of cosets of $I_N$in N . then $(N/I_N, +, .)$ is called the quotient near-ring of N or over I_N , where + and . are defined by$(n_1 + I_N) + (n_2 + I_N) = (n_1 + n_2) + I_N$ and $(n_1 + I_N) . (n_2 + I_N) = (n_1 . n_2) + I_N$ for all n_1 , $n_2$in N.

Definition 2.13.

Let N be a near-ring and I_N be a near ideal of N , then the function $nat_{I_N}: N \to N/I_N$ is called natural near-ring homomorphism defined as $nat_{I_N}(x) = x + I_N$ for each $x \in N$. It is clear that the function nat_{I_N} is a near-ring homomorphism , denoted by π_{I_N} .

Proposition 2.14. [1]

Let N be a near-ring and $P_N$be a proper near ideal in N . Then , P_N is a prime near ideal if and only if N/P_N is a near integral domain .

Definition 2.15.

A near ideal M_N of the near-ring N is called a near maximal ideal provided that $M_N \neq N$ and whenever J_N is a near ideal of N with $M_N \subset J_N \subseteq N$, then $J_N = N$.

Definition 2.16 [14]

Let $(F_N, +, .)$ be a near-ring we say that $(F_N, +, .)$ be a near-field if $(F_N - \{0\}, .)$ is a group .

Definition 2.17. [14]

Let I_N be a near ideal of N ; the prime radical of I_N denoted by $P(I_N) = \bigcap_{I_N \subseteq P_N} P_N$.

Theorem2.18. [1]

Let M_N be a near ideal in a near-ring N , then N/M_N is a near field if and only if M_N is a near maximal ideal .

Proposition 2.19. [1]

In near-ring every near maximal ideal is a near prime ideal .

Proposition 2.20.

Let $spec\ (N)$ be the set of all near prime ideal , let $V(E) = \{P_N \in Spec(N) : E \subset P_N\}$ then ,

1. $V(0) = Spec(N)$, $V(1) = \emptyset$.
2. If $(E_i)_{i \in I}$ is any family subset of N . Then $V(\bigcup_{i \in I} E_i) = \bigcap_{i \in I} V(E_i)$
3. $V(I_N \cap J_N) = V(I_N J_N) = V(I_N) \cup V(J_N)$. For any I_N, J_N is a near ideal of N .

<u>**Proof**</u> **:-**

1. Since $\{0\} \subset P_N$ for every near prime ideal P_N , then $V(0) = Spec(N)$, let $V(1) \neq \emptyset$ so there exist P_N is a near prime ideal such that $P_N \in V(1)$ and $\{1\} \subset P_N$. So $1 \in P_N$, this is a contradiction because P_N is a near prime ideal then $1 \notin P_N$, so $V(1) = \emptyset$.■

2. $P_N \in V(\bigcup_{i \in I} E_i) \Leftrightarrow \bigcup_{i \in I} E_i \subseteq P_N \Leftrightarrow E_i \subseteq P_N\ , for\ each\ i \in I \Leftrightarrow P_N \in V(E_i)\ , for\ each\ i \in I \Leftrightarrow P_N \in \bigcap_{i \in I} V(E_i)$. ■

3. If P_N is a near prime ideal of N , and if $P_N \notin V(I_N)$ and $P_N \notin V(J_N)$ then the sets $I_N \setminus P_N$ and $J_N \setminus P_N$ are nonempty . Let $x \in I_N \setminus P_N$ and $y \in J_N \setminus P_N$, then $xy \in I_N J_N \setminus P_N$, and therefore $P_N \notin V(I_N J_N)$. It follows from this that $V(I_N J_N) \subset V(I_N) \cap V(J_N)$. But also $I_N J_N \subset I_N \cap J_N$, $I_N \cap J_N \subset I_N$ and $I_N \cap J_N \subset J_N$, and therefore $V(I_N \cap J_N) \subset V(I_N J_N)$, $V(I_N) \subset V(I_N \cap J_N)$ and $V(J_N) \subset V(I_N \cap J_N)$. Thus $V(I_N) \cup V(J_N) \subset V(I_N \cap J_N) \subset V(I_N J_N) \subset V(I_N) \cap V(J_N)$. And therefore $V(I_N) \cup V(J_N) = V(I_N \cap J_N) = V(I_N J_N)$. ■

Definition 2.21.

Let N be a near-ring , $spec(N)$ be the set of all near prime ideal of N is called near prime spectrum . And let E be a subset of N : define $V(E)$ as the set of all near prime ideal containing E . Then collection of all $V(E)$ satisfies topological space is called a near Zariski topology on $Spec\,(N)$ and denoted by $\mathcal{T}$. In $N = \{0,1,2,3,4,5,6,7\}$, define $+$ and $\cdot$ as follows :-

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	0	5	6	7	4
2	2	3	0	1	6	7	4	5
3	3	0	1	2	7	4	5	6
4	4	7	6	5	0	3	2	1
5	5	4	7	6	1	0	3	2
6	6	5	4	7	2	1	0	3
7	7	6	5	4	3	2	1	0

·	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	0	1	0	1	1	0
2	0	2	0	2	0	2	2	0
3	0	3	0	3	0	3	3	0
4	4	4	4	4	4	4	4	4
5	4	5	4	5	4	5	5	4
6	4	6	4	6	4	6	6	4
7	4	7	4	7	4	7	7	4

Let $P_1 = (0), P_2 = (3), P_3 = (5), P_4 = (7)$ are a near prime ideal and let $Spes(N) = \{P_1, P_2, P_3, P_4\}$, $V(E) = \{P \in Spec(N), E \subset P\}$. Then $(Spec(N), V(E))$ is a near Zariski Topology .

Note 2.22.

1. If $(E_i)_{i\in\Lambda}$ is a family of near ideals then $\bigcap_{i\in\Lambda} V(E_i) = V(\sum_{i\in\Lambda} E_i)$.
2. Let N be a unital commutative near-ring . Each element f of N determines an open subset $H(f)$ of $Spec(N)$, where $H(f) = \{P_N \in Spec\,N : f \notin P_N\}$, i.e. , this open set is the complement of closed set consisting of all near prime ideal of N that contain the near ideal (f) generated by the element f of N . Note that $H(f) \cap H(g) = H(fg)$ for all elements f and g of N . Indeed let P_N be any near prime ideal of N . Then $fg \notin P_N$ if and only if $f \notin P_N$ and $g \notin P_N$. Thus $P_N \in H(fg)$ if and only if $P_N \in H(f)$ and $P_N \in H(g)$.
3. Let $I_N$be a near ideal of the near-ring N . Then $Spec(N) \setminus V(I_N) = \{P_N \in Spec(N): I_N \not\subset P_N\} = \bigcup_{f\in I_N}\{P_N \in Spec(N) : f \notin P_N\} = \bigcup_{f\in I_N} H(f)$. It follows that the collection of subset of $Spec(N)$ that are of the form $H(f)$ for some $f \in N$ is a basis for the topology of $Spec\,N$, since each open subset of $Spec(N)$ is a union of open set of this form .
4. Given any collection $\{I_\lambda : \lambda \in \Lambda\}$of near ideals of N , we can form their sum $\sum I_\lambda$such that $\lambda \in \Lambda$, which the near ideal consisting of all elements of$N$that can be expressed as a finite sum of the form $x_1 + x_2 + \cdots + x_r$where each

summand x_i is an element of some near ideal I_{λ_i} belonging to the collection .

5. Give any two near ideals I_{1N} and I_{2N} of N , we can form their product $I_{1N}I_{2N}$. This near ideal $I_{1N}I_{2N}$ is the near ideal of N consisting of all elements of N that can be expressed as a finite sum of the form $x_1y_1 + x_2y_2 + \cdots + x_ry_k$ with $x_i \in I_{1N}$ and $y_i \in I_{2N}$ for $i = 1,2,\dots,r$.
6. Since $R \subseteq N$ clearly that $Spec(R) \subseteq Spec(N)$.

Lemma2.23.

Let N be a near-ring , I_N, J_N are a near ideals then :

1. $V(I_N) \subseteq V(J_N)$ if $J_N \subseteq I_N$.
2. $V(I_N) \subseteq V(J_N)$ if and only if $P(J_N) \subseteq P(I_N)$.

Lemma2.24.

Let $Spec(N)$ be a near Zariski topology and (0) be a prime ideal then ;

1. $H(f) = \emptyset$ if and only if f is near nilpotent .
2. $H(f) = Spec(N)$ if and only if f is a near unit .
3. Let I_N be a near ideal of N then $P(I_N) = \{f \in N : H(f) \cap V(I_N) = \emptyset\}$.

<u>Proof</u> :-

1-Let $H(f) = \emptyset$, then $V(f) = V(0)$, by lemma (2.23), $P(f) = P(0)$. So $f^n = 0$. Thus , f is a near nilpotent .

Conversely ; let $f^n = 0$ for some $n > 0 \Longrightarrow f = 0 \Longrightarrow f \in P(0)$ then $f \in P_N$ for all P_N is near prime ideal . Then $H(f) = \emptyset$. ■

2-Let $H(f) = Spec(N) \Longrightarrow f \notin P_N$ for all $P_N \in Spec(N) \Longrightarrow 1 \in (f)$, (since every $I_N \subsetneq N$ there exist a maximal near ideal M_N such that $I_N \subseteq M_N$). Thus f is near unit .

Conversely ; let f is a near unit $\Longrightarrow f.f^{-1} = 1 \Longrightarrow f.1/f = 1$ by note (2.22)$H(f) \cap H(1/f) = Spec(N)$. ■

3-It follows from definition (2.17) that $P(I_N)$ is the intersection of all near prime ideals P_N of N , $I_N \subseteq P_N$, thus an element f of N belongs to $P(I_N)$ if and only if $f \in P_N$ for all $P_N \in V(I_N)$, and thus if and only if $H(f) \cap V(I_N) = \emptyset$. ■

Definition2.25.

Let N be a near-ring , the near nil radical of N is the set of all nilpotent elements of the near-ring , denoted by H.

Note2.26.

If r and s are elements of a commutative near-ring and if $r^m = 0$ and $s^n = 0$ then $(r+s)^{m+n} = 0$. Also $(-r)^m = 0$, and $(tr)^m = 0$ for all $t \in N$.

It follows that the near nil radical of a near-ring is a near ideal of that N.

Proposition2.27.

The near nil radical of N is the intersection of all the near prime ideals of N .

Proposition2.28

Let N be a near-ring , $Spec(N)$ be a near prime spectrum space and $P_1, P_2 \in Spec(N)$. Then

1. $\overline{\{P_1\}} = V(P_1)$.
2. $P_2 \in \overline{\{P_1\}}$ if and only if $P_1 \subseteq P_2$.
3. The set $\{P_1\}$ is a closed in $Spec(N)$ if and only if P_1 is near maximal ideal .

Proof :-

1-Let $V(I_N)$ be a closed sub set of X containing P_1 , that is $P_1 \in V(I_N)$. Clearly $V(P_1)$ is a closed set containing P_1 . Hence by lemma (2.23) (1) , $V(P_1) \subseteq V(I_N)$, therefore $V(P_1)$ is the smallest closed set containing $\{P_1\}$, i.e. $\overline{\{P_1\}} = V(P_1)$. ■

2-$P_2 \in \overline{\{P_1\}} \Leftrightarrow P_2 \in V(P_1) \Leftrightarrow P_1 \subset P_2$.

3-Is clear .

Definition2.29.[4]

Let (X, T) be a topological space we say that X is hyper connected (irreducible) if X satisfies one of the following equivalent two conditions ;

1. every pair of nonempty open sets in X intersect .
2. every nonempty open set in X is dense .

Recall that a topological space (X, T) is said to be disconnected if X can be expressed as the union of two disjoint nonempty open subsets of X , otherwise X is said to be connected . [8]

Remark2.30.

Every irreducible topological space is connected

Definition 2.31.

Let N be a unital commutative near-ring . A subset$V$ of N is said to be multiplicative subset if $1 \notin V$and$a.b \in V$for all $a \in V$ and$b \in V$.

Proposition 2.32.

Let N near-ring , let I_N be a near ideal of N , then $V(I_N)$ is an irreducible topological space if and only if the $(P(I_N), +, .)$ is a near prime ideal of N .

Proof :-

Suppose that $V(I_N)$ is an irreducible topological space . Let n_1 and n_2 be elements of $N \setminus P(I_N)$. Then $H(n_1) \cap V(I_N)$ and $H(n_2) \cap V(I_N)$ are nonempty by note (2.22) (2) . Now $H(n_1 n_2) = H(n_1) \cap H(n_2)$, and therefore $H(n_1 n_2) \cap V(I_N)$ is the intersection of the nonempty open subset $H(n_1) \cap V(I_N)$ and $H(n_2) \cap V(I_N)$ of $V(I_N)$. It follows from the irreducibility of $V(I_N)$ that $H(n_1 n_2) \cap V(I_N)$ is itself nonempty , and therefore $n_1 n_2 \in N \setminus P(I_N)$. Thus if $V(I_N)$ is an irreducible topological space then the complement $N \setminus P(I_N)$ of $P(I_N)$ is a multiplicative subset of N , and therefore $P(I_N)$ is a near prime ideal of N .

Conversely ; suppose that $P(I_N)$ is a near prime ideal of N. W_1 and W_2 be nonempty subsets of $V(I_N)$. Any open subset of $V(I_N)$ is a union of subsets of $V(I_N)$ each of which is of the form $H(n) \cap V(I_N)$ for some $n \in N$. Therefore there exist elements n_1 and n_2 of N such that $H(n_1) \cap V(I_N)$ and $H(n_2) \cap V(I_N)$ are nonempty , $H(n_1) \cap V(I_N) \subset W_1$ and $H(n_2) \cap V(I_N) \subset W_2$. Then $n_1 \notin P(I_N)$ and $n_2 \notin P(I_N)$. But then $n_1 n_2 \notin P(I_N)$, because the complement of a near prime ideal is a multiplicative subset of N . It follows that $H(n_1 n_2) \cap V(I_N)$ is nonempty . Thus $W_1 \cap W_2$is nonempty . We have $V(I_N)$ is irreducible . ■

3-Some Properties of the space $Spec(N)$.

Theorem3.1.

The near prime spectrum $Spec(N)$ of any commutative near-ring N is a compact topological space .

Proof :-

Let $\{U_\lambda : \lambda \in \Lambda\}$ be any open cover of $Spec(N)$. Then there exists a collection $\{I_\lambda : \lambda \in \Lambda\}$ of near ideals of N such that $spec(N) \setminus U_\lambda = V(I_\lambda) = \{P \in Spec(N) : I_\lambda \subset P\}$ for each open set U_λ in the given collection . Let the near ideal I_N be the sum $\sum_{\lambda \in \Lambda} I_\lambda$ of all the near ideals I_λ in this collection . Then $V(I_N) = \bigcap_{\lambda \in \Lambda} V(I_\lambda) = \bigcap_{\lambda \in \Lambda} (Spec(N) \setminus U_\lambda) = Spec(N) \setminus \bigcup_{\lambda \in \Lambda} U_\lambda = \emptyset$

Thus there is no near prime ideal P_N of N with $I_N \subset P_N$. But any proper near ideal of N is contained in some near maximal ideal ([6] theorem 3-30) and moreover every near maximal ideal is a near prime ideal by proposition (2.19) . We conclude that $I_N = N$, and therefore every element of the near-ring N may be expressed as a finite sum where each of the summands belongs to one of the near ideals I_λ. In particular there exists elements $x_1, x_2, \ldots, x_k$ of N and near ideals $I_{\lambda_1}, I_{\lambda_2}, \ldots, I_{\lambda_k}$ in the collection $\{I_\lambda : \lambda \in \Lambda\}$, such that $x_i \in I_{\lambda_i}$ for $i = 1,2,3, \ldots, k$ and $x_1 + x_2 + \cdots + x_k = 1$. But then $\sum_{i=1}^{k} I_{\lambda_i} = N$ and there fore $Spec(N) \setminus \bigcup_{i=1}^{k} U_{\lambda_i} = \bigcap_{i=1}^{k} V(I_\lambda) = V\left(\sum_{i=1}^{k} I_{\lambda_i}\right) = V(N) = \emptyset$ and therefore $\{U_{\lambda_i}, i = 1,2, \ldots, k\}$ is an open cover of $Spec(N)$. Thus every open cover of $Spec(N)$ has a finite sub cover . We conclude that $Spec(N)$ is a compact topological space . ■

Corollary3.2.

Let I_N be a near ideal of near-ring N . Then the closed subset $V(I_N)$ of $Spec(N)$ is a compact set .

Lemma 3.3.

Let $\varphi: N_1 \to N_2$ be a near-ring unital homomorphism between near-rings N_1 and N_2 . Then $\varphi: N_1 \to N_2$ induces a continuous map $\varphi^*: Spec(N_2) \to Spec(N_1)$ from $Spec(N_2)$ to $Spec(N_1)$, where $\varphi^*(P_N) = \varphi^{-1}(P_N)$ for every near prime ideal P_N of N_2 .

Proof :-

Let P_{N2} be a near prime ideal of N_2. Now $1_{N2} \notin P_{N2}$, because P_{N2} is a proper near ideal of N_2 , then $1_{N1} \notin \varphi^{-1}(P_{N2})$, since $\varphi(1_{N1}) = 1_{N2}$. It follows that $\varphi^{-1}(P_{N2})$ is a proper near ideal of N_1 .

Let x and y be elements of N_1 . Suppose that $xy \in \varphi^{-1}(P_{N2})$. Then $\varphi(x)\varphi(y) = \varphi(xy)$ and therefore $\varphi(x)\varphi(y) \in P_{N2}$. But P_{N2} is a near prime ideal of N_2 , and therefore either $\varphi(x) \in P_{N2}$ or $\varphi(y) \in P_{N2}$. Thus either $x \in \varphi^{-1}(P_{N2})$ or $y \in \varphi^{-1}(P_{N2})$. This shows that $\varphi^{-1}(P_{N2})$ is a near prime ideal of N_1 . We conclude that there is a well-defined function $\varphi^*: Spec(N_2) \to Spec(N_1)$ such that $\varphi^*(P_{N2}) = \varphi^{-1}(P_{N2})$ for all near prime ideal P_{N2} of N_2 .

Now we prove that φ^* is a continuous function , let I_{N1} be a near ideal of N_1

$\varphi^{*-1}\big(V(I_{N1})\big) = \{P_{N2} \in Spec(N_2): \varphi^*(P_{N2}) \in V(I_{N1})\}$. since $\varphi^*(P_{N2}) = \varphi^{-1}(P_{N2})$

$=\{P_{N2} \in Spec(N_2): \varphi^{-1}(P_{N2}) \in V(I_{N1})\}$

$=\{P_{N2} \in Spec(N_2) : I_{N1} \subset \varphi^{-1}(P_{N2})\}$

$=\{P_{N2} \in Spec(N_2): \varphi(I_{N1}) \subset P_{N2}\} = V(\varphi(I_{N1}))$

Thus , $\varphi^*: Spec(N_2) \to Spec(N_1)$ is a continuous function .■

Recall that a continuous open (closed) bijective map $f: X \to Y$, where X and Y are topological spaces , is called a homeomorphism and is denoted by $f: X \cong Y$. Two spaces X, Y are homeomorphic , written $X \cong Y$, if there is a homeomorphism $f: X \cong Y$ [8] .

Proposition3.4.

Let N be a near-ring , let I_N be a proper ideal of N , and let $\pi_I: N \to N/I_N$ be the corresponding quotient near-ring homomorphism onto the quotient near-ring N/I_N . Then the induced map ${\pi_I}^*: Spec(N/I_N) \to Spec\ (N)$ maps $Spec(N\ /\ I_N)$ homeomorphically onto the closed set $V(I_N)$.

Proof :-

1-To prove ${\pi_I}^*: Spec(N/I_N) \to Spec(N)$ is onto function . Let q_N be a near prime ideal of N/I_N . Then $I_N \subset {\pi_I}^{-1}(q_N)$ and therefore ${\pi_I}^*(q_N) \subset V(I_N)$. We conclude that ${\pi_I}^*(Spec(N/I_N)) \subset V(I_N)$. For prove $V(I_N) \subset {\pi_I}^*(Spec(N/I_N))$, let P_N be a near prime ideal of N belonging to $V(I_N)$ then $I_N \subset P_N$, and let $q_N = \pi_I(P_N)$. Now since $I_N \subset P_N$ and therefore ${\pi_I}^{-1}(q_N) = I + P_N = P_N$. It follows from that the q_N must be a proper near ideal of N/I_N . Let x, y by elements of N with the property that

$(x+I_N)(y+I_N) \in q_N$. then $\pi_I(xy) \in q_N$, $\pi_I(x)\pi_I(y) \in q_N$, and therefore $xy \in P_N$. But then either $x \in P_N$ or $y \in P_N$, and thus either $x+I_N \in q_N$ or $y+I \in q_N$ .This shows that q_N is a near prime ideal of N / I_N . Moreover $P_N = \pi_I{}^*(q_N)$. We conclude that $\pi_I{}^*(Spec\,(N/I_N)) = V(I_N)$.

2-To prove $\pi_I{}^*: Spec(N/I_N) \to Spec(N)$ is a one-to-one function . If q_{1N}, q_{2N} are near prime ideals of N / I_N , and $\pi_I{}^*(q_{N1}) = \pi_I{}^*(q_{N2})$ by lemma (3.3) . Thus , $\pi_I{}^{-1}(q_{1N}) = \pi_I{}^{-1}(q_{2N})$, and therefore . $q_{1N} = \pi_I(\pi_I{}^*(q_{1N})) = \pi_I(\pi_I{}^*(q_{2N})) = q_{2N}$, so $q_{1N} = q_{2N}$. then from (1,2) we get $\pi_I{}^*: Spec(N/I_N) \to Spec(N)$ maps the near spectrum $Spec(N/I_N)$ bjectively onto the closed sub set $V(I_N)$ of the near spectrum $Spec(N)$ of N .

Let q_N is a near ideal of N/I_N and p_N be a near prime ideal . Then $\pi_I\left(\pi_I^{-1}(q_N)\right) = q_N$ and $\pi_I\left(\pi_I^{-1}(p_N)\right) = p_N$. It follows that $\pi_I^{-1}(q_N) \subseteq \pi_I^{-1}(p_N)$ if and only if $q_N \subset p_N$. But then $V\left(\pi_I^{-1}(q_N)\right) \cap V(I_N) = \{P \in V(I_N): \pi_I^{-1}(q_N) \subset P\} = \pi_I^*\{p_N \in Spec(N/I_N):\ \pi_I^{-1}(q_N) \subset \pi_I^{-1}(p_N)\} = \pi_I^*\{p_N \in Spec(N/I_N): q_N \subset p_N\} = \pi_I^*(V(q_N))$.

Thus the continuous function $\pi_I{}^*: Spec(N/I_N) \to Spec(N)$ maps closed subsets of $Spec(N/I_N)$ onto closed subsets of $V(I_N)$. But any continuous closed bijection between two topological spaces is a homeomorphism . We conclude therefore that the function $\pi_I{}^*$ maps $Spec(N/I_N)$ homeomrphically onto $V(I_N)$. ■

Proposition3.5.

Let $\varphi: N_1 \to N_2$ be a near-ring homomorphism , and $\varphi^*: Spec(N_2) \to Spec(N_1)$ be the induced map . Let $\pi: N_2 \to N_3$ be another near-ring homomorphism . Then $(\pi \circ \varphi)^* = \varphi^* \circ \pi^*$.

<u>Proof</u> :-

Let $I_N \in Spec(N_3)$. Then $(\pi \circ \varphi)^*(I_N) = (\pi \circ \varphi)^{-1}(I_N) = \varphi^{-1}(\pi^{-1}(I_N)) = \varphi^{-1}(\pi^*(I_N)) = \varphi^*(\pi^*(I_N)) = (\varphi^* \circ \pi^*)(I_N)$ therefore $(\pi \circ \varphi)^* = (\varphi^* \circ \pi^*)$. ■

Proposition3.6.

Let $\varphi: N_1 \to N_2$ be a near-ring homomorphism between near-rings N_1 and N_2 . If I_N is a near ideal of N_1 . Then

1. $\varphi^{*-1}(V(I_N)) = V(\varphi(I_N))$.
2. If φ is surjective , then φ^* is a homeomorphism of $Spec(N_2)$ onto the closed subset $V(ker\varphi)$ of $Spec(N_1)$.

<u>Proof</u> :-

1.$P_N \in {\varphi^*}^{-1}(V(I_N)) \Leftrightarrow \varphi^*(P_N) \in V(I_N) \Leftrightarrow I_N \subseteq \varphi^*(P_N) \Leftrightarrow I_N \subseteq \varphi^{-1}(P_N) \Leftrightarrow \varphi(I_N) \subseteq P_N \Leftrightarrow P_N \in V(\varphi(I_N))$. ■

2.It is well known that , if φ is surjective , then there is a one-to -one correspondence between near prime ideal of

N_2 and near prime ideal of N_1 containing $ker\varphi$, this means that $\varphi^*: Spec(N_2) \to V(ker\varphi)$ is bijective , by lemma (3.3) φ^* is continuous . It remains to show that $\varphi^*(H(f))$ is open in $V(ker\varphi)$ for each $f \in N_2$. Now $\varphi^*(H(f)) = \{\varphi^*(I_N) : I_N \in Spec(N_2)\ and\ f \notin I_N\} = \{P_N \in V(ker\varphi) : \varphi^{-1}(f) \notin P_N\} = H(\varphi^{-1}(f)) \cap V(ker\varphi)$. Hence $\varphi^*(H(f))$ is open in $V(ker\varphi)$, thus . φ^* is an open map . Hence φ^* is a homeomorphism from $Spec(N_2)$ onto $V(ker\varphi)$. ■

Proposition 3.7.

Let N be a near-ring , and let H be the near nilradical of N . Then the near quotient homomorphism $\nu: N \to N / H$ induces a homeomorphism $\nu^*: Spec(N/H) \to Spec(N)$ between the spectra of N / H and N .

Proof :-

Let H be the near nil radical we starts this proof from $V(H) = Spec(N)$. $V(H) = \{P_N \in Spec(N) : H \subseteq P_N\} = \left\{P_N \setminus \bigcap_{q_N \in Spec\ N} q_N \subseteq P_N\right\} = Spec(N)$. But for any near ideal I_N of N , the quotient homomorphism from N to N/I_Na induces a homomorphism between $Spec(N/I_N)$ and $V(I_N)$. It follows that the quotient homomorphism $\nu: N \to N / H$ induces a homomorphism between $Spec(N)$ and $Spec(N/I_N)$. ■

Theorem 3.8.

If N be a near-ring and (0) is near prime ideal then $Spec(N)$ is an irreducible space .

Proof :-

Let $H(f)$, $H(g)$ be any two nonempty basic open subset of $Spec(N)$. Since (0) is a near prime ideal of N , (0) belongs to every nonempty basic open subset $H(f)$ and $H(g)$ of $Spec(N)$, consequently $Spec(N)$ is irreducible .

Corollary 3.9.

If N be a near-ring and $P(0)$ is near prime , then $Spec(N)$ is a connected space .

Corollary 3.10.

Let N be a commutative near-ring with identity , and let H be the near nilradical of N . Suppose that the spectrum $Spec(N)$ of N is an irreducible topological space . Then N / H is a near integral domain .

Proof :-

Since $Spec(N)$ is irreducible then H is a near prime ideal of N and therefore N / H is a near integral domain by using proposition (2.14) . ■

Definition 2.11.

A near-ring N is said to be near noetherian if every near ideal is finitely generated . Recall that a topological space (X,T) is noetherian if every ascending chain of open subsets of X is finite . Since closed subsets are complements of open subset , it comes to the same thing to say that the closed subset of X satisfy the descending chain condition , i.e. , every descending chain of closed subsets of X is finite [2] .

Proposition 3.12.

If N is a near noetherian near-ring . Then $Spec(N)$ is a noetherian space .

<u>Proof</u> :-

Let $V(I_1) \supseteq V(I_2) \supseteq \cdots$ be an arbitrary descending chain of closed subsets of $Spec(N)$ where $I_1, I_2, \ldots.$ are near ideal of N . Then by lemma (2.23.) (2) $P(I_1) \subseteq P(I_2) \subseteq \cdots$. But N is near Noetherian , so there exists $n \in N$ such that $P(I_1) \subseteq P(I_2) \subseteq \cdots \subseteq P(I_n) = P(I_{n+1})$ hence $V(I_1) \supseteq V(I_2) \supseteq \cdots \supseteq V(I_n) = V(I_{n+1})$ that is $H(I_n) = H(I_{n+1})$. Therefore $Spec(N)$ is a Noetherian space . ■

Proposition 3.13.

Let N be a near-ring . Then the space $Spec(N)$ is a T_0 space .

<u>Proof</u> :-

Let $P_1, P_2 \in Spec(N)$ and $P_1 \neq P_2$ then $P_1 \nsubseteq P_2$ or $P_2 \nsubseteq P_1$, let $H(f) = \{P_1 \in Spec(N): f \notin P_1\}$ and $P_1 \nsubseteq P_2$. Then We get $P_1 \in H(f), P_2 \notin H(f)$. Thus $Spec(N)$ is a T_0 space . ■

Proposition 3.14.

Let N be a near-ring . Then the space $Spec(N)$ is T_1 if and only if $Spec(N) = Max(N)$ is the set of all near maximal ideal of N .

<u>Proof</u> :-

Assume that $Spec(N)$ is a T_1-space . Hence every singleton is closed . But $\{P\}$ is closed by using proposition (2.28) . Hence every near prime ideal is near maximal . Equivalently , $Spec(N) = Max(N)$. Conversely ; Let $Spec(N) = Max(N)$. Since $V(P) = \{P\}$for all $P \in Spec(N)$. Then $Spec(N)$ is T_1 . ■

Proposition 3.15.

If N is a near integral domain , then the following statements are equivalent :

1. $Spec(N)$ indiscrete topological space .
2. N is a near field .

<u>Proof</u> :-

(1) → (2) . Assume that of $Spec(N)$ is indiscrete space . Then for $0 \neq f \in N$ either $H(f) = \emptyset$ or $H(f) = X$. If $H(f) = \emptyset$, then f is near nilpotent by lemma (2.24) (1) . So $f = 0$, which this is a contradiction . So

$H(f) = X$, and by lemma (2.24) (2) f is a near unit . Hence N is a near filed .

(2) → (1) . Let N be a near filed . Then the only near prime ideal of N is (0) , i.e. , $X = \{(0)\}$. Hence $Spec(N)$ is the indiscrete space . ■

Proposition 3.16.

Let (0) is a near prime ideal . If N is a near-ring which is not a near field , then the space $Spec(N)$ can not be T_1 .

Proof :-

Assume a $Spec(N)$ is T_1 ; therefore , by proposition (3.14) , $Spec\ N = Max(N)$. That is (0) is a near maximal ideal ; consequently N is a near field , which is a contradiction . ■

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