Some Properties Of N-Co probabilistic Normed Space And Co-probabilistic Dual Space Of N-Co probabilistic Normed Space

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Abstract:

The primary purpose of this paper is to introduce the, N-coprobabilistic normed space, coprobabilistic dual space of N-coprobabilistic normed space and give some facts that are related of them.

1. Introduction:-

In [1], Menger replaced the nonnegative real numbers as values of the metric by a distribution function and defined probabilistic metric spaces. The concept of random normed spaces were introduced by Serstnev (2). In (3), Jebril and Hatamleh introduced the concept of random n-normed linear space as a generalization of n-normed space which introduced by Gunawan and Mashadi (4). In this paper we introduced a N-coprobabilistic normed linear space depending on the idea of random n-normed linear space was introduced by Jebril and Hatamled (3) and investigate their important properties. Then we shall introduce the definition of N-coprobabilistic bounded linear functional. Thereafter we prove the set of all N-coprobabilistic bounded linear functional is a coprobabilistic norm linear space.

2. Some Results Of N-Coprobabilistic Normed Space:-

In this section we give the definition of N-coprobabilistic normed space and prove some basic concepts of it.

We start this section by giving a definition of N-normed linear space.

Definition (2.1), (4):-

Let X to be a linear space over R (field of real numbers) of dimension greater than and equal to n where n>2 . A function $\|.,\ldots,\|: X\times X\times\ldots\times X \longrightarrow R$ satisfy the following axioms:

(N_1) $\|x_1,x_2,...,x_n\|=0$ *if and only if* $x_1,x_2,...,x_n$ *are* linearly dependent.

(N_2) $\|x_1,x_2,\ldots,x_n\|$ is invariant under any permutation of $x_1,x_2,\ldots,x_n$.

(N_3) $\|x_1,x_2,...,cx_n\|=|c|\|x_1,x_2,...,x_n\|$ *for any* $c\in R$.

(N_4) $\|x_1,x_2,...,x_n+x'_n\|\le\|x_1,x_2,...,x_n\| + \|x_1,x_2,...,x'_n\|$

is said to be an N-norm on X and the pair $(X,\|.,\ldots,\|)$ is called an N-normed space.

Definition (2.2), (4):-

Let $(X,\|.,\ldots,\|)$ be a N-normed linear space, a sequence $\{x_k\}$ in X is said to be convergent to a point $x\in X$ in case $\lim\limits_{k\to\infty}\|x_1,x_2,\ldots,x_{n-1},x_k-x\|=0$ for each $x_1,x_2,\ldots,x_{n-1}\in X$. In this case, x is said to be the limit of the sequence $\{x_k\}$ and we denote it by $\lim x_k$. Otherwise, the sequence is divergent.

Definition (2.3), (4):-

Let $(X,\|.,\ldots,\|)$ be a N-normed linear space, a sequence $\{x_k\}$ in X is said to be Cauchy sequence in case

$\lim_{k\to\infty}\|x_1,x_2,\ldots,x_{n-1},x_{k+p}-x_k\|=0$ for each $x_1,x_2,\ldots,x_{n-1}\in X$ and P=1,2,….

Definition (2.4), (4):-

A N-normed linear space in which every Cauchy sequence is convergent is said to be complete.

Definition (2.5), (1):-

A nonascending probability distribution function η on R that is a left continuous nonascending real function on R with $\eta(x)=1$ for all $x\le 0$ and $\eta(+\infty-)=0$

Notes (2.6), (1):-

1-The family of all a nonascending probability distribution functions will be denoted by ∇^+

2- By setting $F\le G$ whenever $F(x)\le G(x)$, for each $x\in R$, one introduces natural ordering in ∇^+.

3-If $a\in R$, then ε_a will be an element of ∇^+ defined by

$$\varepsilon_a(t)=\begin{cases}1 & \text{if } t\le a\\ 0 & \text{if } t>a\end{cases}$$

Definition (2.7), (1):-

Let X be a linear space over a field R, a mapping C* from X into ∇^+ is said to be a coprobabilistic norm on X in case the following conditions hold:

(c-N_1) $C_x^*=\varepsilon_0$ if and only if x=0

(c-N_2) If $0\ne c\in R$ then $C_{cx}^*(t)=C_x^*(\frac{t}{|c|})$

(c-N_3) $C_{x+x'}^*(s+t)\le \max\{C_x^*(s),C_{x'}^*(t)\}$, for each $s,t\in R$.

The pair (X,C^*) will be referred to a coprobabilistic normed linear space (briefly c-NLS).

Example (2.8), (1):-

Let $(X,\|.\|)$ be an normed space. For each $x\in X$ and $t\in R$, a function

$$C_x^*(t)=\begin{cases}1 & \text{if } t\le\|x\|\\ 0 & \text{if } t>\|x\|\end{cases}$$

………………………...(2.1)

is coprobabilistic normed linear space.

Now, we introduce the concept of N-coprobabilistic normed linear space as a development of coprobabilistic normed linear space.

Definition (2.9)-

Let X be a linear space of dimension greater than or equal to N and a mapping C from $X\times X\times\ldots\times X=X^n$ into ∇^+ is said to be a N-coprobabilistic norm on X in case the following conditions hold:

(N-c-N_1) $C_{(x_1,x_2,\ldots x_n)}=\varepsilon_0$ if and only if $x_1,x_2,\ldots,x_n$ are linearly dependent.

(N-c-N_2) $C_{(x_1,x_2,\ldots,x_n)}$ is invariant under any permutation of $x_1,x_2,\ldots,x_n$.

(N-c-N_3) If $0\ne c\in R$ *then* $C_{(x_1,x_2,\ldots,cx_n)}(t)=C_{(x_1,x_2,\ldots,x_n)}(\frac{t}{|c|})$

(N-c-N_4) $C_{(x_1,x_2,\ldots,x_n+x_n')}(s+t)\le\max\begin{Bmatrix}C_{(x_1,x_2,\ldots,x_n)}(s)\\ ,C_{(x_1,x_2,\ldots,x_n')}(t)\end{Bmatrix}$, for each $s,t\in R$.

The pair (X,C) will be referred to N-coprobabilistic normed linear space (briefly N-C-NLS).

In order to make definition (2.9) as clear as possible we will consider the following examples.

Example (2.10):-

Let $(X,\|.,\ldots,.\|)$ be an N-normed space. For each $(x_1,x_2,\ldots,x_n)\in X^n$ and $t\in R$. Defined

$$C_{(x_1,x_2,\ldots,x_n)}(t)=\begin{cases}1 & \text{if } t\le\|x_1,x_2,\ldots,x_n\|\\ 0 & \text{if } t>\|x_1,x_2,\ldots,x_n\|\end{cases}$$

…(2.2)

In order to prove that (X,C) is a N-C-NLS, we must verify the above four (N-C-N) conditions.

(N-C-N_1) It is easy to check that

$$C_{(x_1,x_2,\ldots,x_n)} = \varepsilon_0$$

(N-C-N_2) Since $\|x_1,x_2,\ldots,x_n\|$ is invariant under any permutation of $x_1,x_2,\ldots,x_n$ it follow that $C_{(x_1,x_2,\ldots,x_n)}$ is invariant under any permutation of $x_1,x_2,\ldots,x_n$

(N-C-N_3) If $0 \neq c \in R$ and t>0, then one can get the following two cases:-

(i) If $t \leq \|x_1,x_2,\ldots,cx_n\|$ then

$$C_{(x_1,x_2,\ldots,x_n)}(\frac{t}{|c|}) = 1.$$

(ii) If $t > \|x_1,x_2,\ldots,cx_n\|$ then

$$C_{(x_1,x_2,\ldots,x_n)}(\frac{t}{|c|}) = 0$$

(N-C-N_4) For each $s,t \in R$, we must prove

$$C_{(x_1,x_2,\ldots,x_n+x'_n)}(s+t) \leq \max\left\{C_{(x_1,x_2,\ldots,x_n)}(s), C_{(x_1,x_2,\ldots,x'_n)}(t)\right\}$$

…….(2.3)

To do this consider the following two cases

(1) If $s+t \leq \|x_1,x_2,\ldots,x_n+x'_n\|$ then

$$C_{(x_1,x_2,\ldots,x_n+x'_n)}(s+t) = 1$$

And

$$s+t \leq \|x_1,x_2,\ldots,x_n\| + \|x_1,x_2,\ldots,x'_n\|$$

In this case one can recognize the following two cases.

(i) If $s \leq \|x_1,x_2,\ldots,x_n\|$ then either $t \leq \|x_1,x_2,\ldots,x'_n\|$ or $t > \|x_1,x_2,\ldots,x'_n\|$

If $t \leq \|x_1,x_2,\ldots,x'_n\|$ the proof of ineq.(2.3) is trivial. On the other hand, if $t > \|x_1,x_2,\ldots,x'_n\|$ then

$C_{(x_1,x_2,\ldots,x'_n)}(t) = 0$. Hence, ineq.(2.3) holds in this case

(ii) If $s > \|x_1,x_2,\ldots,x_n\|$ then $t \leq \|x_1,x_2,\ldots,x'_n\|$ and hence.

$C_{(x_1,x_2,\ldots,x_n)}(s) = 0$ and

$C_{(x_1,x_2,\ldots,x'_n)}(t) = 1$. Thus ineq. (2.3) holds.

(2) If $s+t > \|x_1,x_2,\ldots,x_n+x'_n\|$ then

$C_{(x_1,x_2,\ldots,x_n+x'_n)}(t) = 0$. On the other hand, if

$$s+t \leq \|x_1,x_2,\ldots,x_n\| + \|x_1,x_2,\ldots,x'_n\|.$$

Thus ineq.(2.3) holds. If

$$s+t > \|x_1,x_2,\ldots,x_n\| + \|x_1,x_2,\ldots,x'_n\|$$

Then one can see the following two cases:-

(i) If $s \leq \|x_1,x_2,\ldots,x_n\|$ then $t > \|x_1,x_2,\ldots,x'_n\|$. Hence ineq.(2.3) holds.

(ii)If $s > \|x_1,x_2,\ldots,x_n\|$ then either $t \leq \|x_1,x_2,\ldots,x'_n\|$ or $t > \|x_1,x_2,\ldots,x'_n\|$

If $t \leq \|x_1,x_2,\ldots,x'_n\|$ then

$$C_{(x_1,x_2,\ldots,x'_n)}(t) = 1$$

and $C_{(x_1,x_2,\ldots,x_n)}(s) = 0$.

Thus ineq.(2.3) holds. If $t > \|x_1,x_2,\ldots,x'_n\|$ then

$$C_{(x_1,x_2,\ldots,x'_n)}(t) = 0$$

Hence ineq.(2.3) holds.

Definition (2.11):-

Let (X,C) be a N-C-NLS, a sequence $\{x_k\}$ in X is said to be convergent to a point $x \in X$ in case $\lim\limits_{k\to\infty} C_{(x_1,x_2,\ldots,x_{n-1},x_k-x)}(t) = 0$ for each t>0 and for each $x_1,x_2,\ldots,x_{n-1} \in X$ .In this case, x is said to be the limit of the sequence $\{x_k\}$ and we denote it by $\lim x_k$. Otherwise, the sequence is divergent.

Definition (2.12):-

Let (X,C) be a N-C-NLS, a sequence $\{x_k\}$ in X is said to be Cauchy sequence in case $\lim\limits_{k\to\infty} C_{(x_1,x_2,\ldots,x_{n-1},x_{k+p}-x_k)}(t) = 0$

for each t>0, $x_1, x_2, \ldots, x_{n-1} \in X$ and P=1,2,….

Definition (2.13):-

A N-coprobilistic normed linear space in which every Cauchy sequence is convergent is said to be complete.

3.Coprobabilistic Dual Space Of N-Coprobabilistic Normed Linear Space:-

In this section, we give a definitions of N-coprobabilistic bounded linear functional, uniformly N-coprobabilistic bounded and coprobabilistic dual space of N-coprababilistic normed linear space. Also some results related to them are discussed.

Definition (3.1):-

Let $T:(X,C) \longrightarrow (R,C^*)$ be a linear functional where (X,C) be a N-coprobabilistic normed linear space and C^* be the coprobabilistic norm defined ineq.(2.1). T is said to be N-coprobabilistic bounded on X^n in case there exists a positive number M such that for all $(x_1, x_2, \ldots, x_n) \in X^n$ and t>0,

$$C^*_{T(x_1,x_2,\ldots,x_n)}(t) \le C_{(x_1,x_2,\ldots,x_n)}\left(\frac{t}{M}\right)$$

Theorem (3.2):-

Let T be a N-coprobabilistic bounded linear functional on X^n. If $x_1, x_2, \ldots, x_n$ are linearly dependent then $T(x_1, x_2, \ldots, x_n) = 0$.

Proof:-

Suppose $x_1, x_2, \ldots, x_n$ are linearly dependent. Since T is a N coprobabilistic bounded linear functional. Then there exists M>0 such tha

$$C^*_{T(x_1,x_2,\ldots,x_n)}(t) \le C_{(x_1,x_2,\ldots,x_n)}\left(\frac{t}{M}\right),\ \forall (x_1, x_2, \ldots, x_n) \in X^n \text{ and } t > 0$$

.

$$\text{Then } C^*_{T(x_1,x_2,\ldots,x_n)}(t) \le 0. \text{ Hence } T(x_1, x_2, \ldots, x_n) = 0$$

.

In 2010, Dinda and et. al., [5] gave a relation between fuzzy antinormed linear space and normed space. Here we gave the same idea but for N-coprababilistic normed linear space and N-normed space.

Theorem (3.3):-

Let (X,C) be a N-coprobabilistic normed space satisfying the following two conditions

(N-C - N_5) for each t>0,

$C_{(x_1,x_2,\ldots,x_n)}(t) < 1$ implies $x_1, x_2, \ldots, x_n$ are Linearly dependent.

(N-C-N_6) For $x_1, x_2, \ldots, x_n$ are linearly independent, $C_{(x_1,x_2,\ldots,x_n)}(t)$ is a continuous of $t \in R$ and strictly decreasing in the subset $\{t : 0 < C_{(x_1,x_2,\ldots,x_n)} < 1\}$ of R. And let $\{\|.,\ldots,\| : \alpha \in (0,1)\}$ is a ascending family of N-norms of X. defined

$\|x_1, x_2, \ldots, x_n\|_\alpha = Inf\{\ t : C_{(x_1,x_2,\ldots,x_n)}(t) \le 1 - \alpha\},$
$where\ \alpha \in (0,1)$

.

Also, Let C' be a mapping from $X \times X \times \ldots \times X = X^n$ into ∇^+ defined by

$$C'_{(x_1,x_2,\ldots,x_n)}(t) = \begin{cases} Inf\{1-\alpha : \|x_1, x_2, \ldots, x_n\|_\alpha \le t\} & if\ x_1, x_2, \ldots, x_n\ are \\ & linearly\ indepent, t \ne 0 \\ 1 & otherwise \end{cases}$$

Then, C=C'

Proof:-

The prove directly follows from change the proof of the theorem in (2.4) in [5] to N-coprobabilistic normed space.

Definition (3.4):-

Let $T:(X,C) \longrightarrow (R,C^*)$ be a linear functional where (X,C) be a N-coprobabilistic normed linear space satisfying condition the (N-C - N_5) and (R,C*) be a coprobabilistic normed linear space where C* defined in eq.(2.1). T is said to be uniformly N-coprobabilistic bounded in case there exist M>0 such that

$|T(x_1, x_2, \ldots, x_n)| \ge M\|x_1, x_2, \ldots, x_n\|_\alpha,$
$\alpha \in (0,1)$

Where $\{\|.,\ldots,\| : \alpha \in (0,1)\}$ is a ascending family of N-norms.

Theorem (3.5):-

Let $T:(X,C) \longrightarrow (R,C^*)$ be a linear functional where (X,C) be a N-coprobabilistic normed linear space satisfying the condition (N-C-N_5) and (R,C*) be a coprobabilistic normed linear space where C* defined in eq.(2.1). If T is N-coprobabilistic bounded on X^n then it is uniformly N-coprobabilistic bounded.

<u>Proof:-</u>

Suppose T is N-coprobabilistic bounded. Then there exists M>0 suchthat

$$\forall (x_1,x_2,\ldots,x_n) \in X^n \text{ and } s>0, \quad C^*_{T(x_1,x_2,\ldots,x_n)}(s) \le C_{(x_1,x_2,\ldots,x_n)}\left(\frac{s}{M}\right)$$

Then

$$C^*_{T(x_1,x_2,\ldots,x_n)}(s) \le C_{(x_1,x_2,\ldots,Mx_n)}(s)$$

$$\|x_1,x_2,\ldots,Mx_n\|_\alpha > t \text{ then } \operatorname{Inf}\{s : C_{(x_1,x_2,\ldots,Mx_n)}(s) \le 1-\alpha\} > t$$

Hence,

$$\exists s_0 > t \text{ such that } C_{(x_1,x_2,\ldots,Mx_n)}(s_0) \le 1-\alpha$$

Therefore,

$$\exists s_0 > t \text{ such that } C^*_{T(x_1,x_2,\ldots,x_n)}(s_0) \le 1-\alpha$$

Then, $|T(x_1,x_2,\ldots,x_n)| \ge s_0 > t$

Hence,

$$|T(x_1,x_2,\ldots,x_n)| \ge \|x_1,x_2,\ldots,Mx_n\|_\alpha$$

Thus T is uniformly N-coprobabilistic bounded.

<u>Note (3.6):-</u>

Let (X,C) be a N-coprobabilistic normed linear space. We denote by $\beta(X^n,R)$ the set of all N-coprobabilistic bounded linear functionals on X^n and we call β, the coprobabilistic dual space of X^n.

<u>Theorem (3.7):-</u>

$\beta(X^n,R)$ is a linear space.

<u>Proof:</u>

Let $T_1, T_2 \in \beta(X^n,R)$, then there exist two positive numbers M_1 and M_2 such that for each $(x_1,x_2,\ldots,x_n) \in X^n$ and $t>0$

$$C^*_{T_1(x_1,x_2,\ldots,x_n)}(t) \le C_{(x_1,x_2,\ldots,x_n)}\left(\frac{t}{M_1}\right)$$

and

$$C^*_{T_2(x_1,x_2,\ldots,x_n)}(t) \le C_{(x_1,x_2,\ldots,x_n)}\left(\frac{t}{M_2}\right)$$

Thus for any two none scalars α and λ we have

$$C^*_{(\alpha T_1+\lambda T_2)(x_1,x_2,\ldots,x_n)}(t)$$

$$\le \max\{C^*_{(\alpha T_1)(x_1,x_2,\ldots,x_n)}\left(\frac{t}{2}\right),$$

$$C^*_{(\lambda T_2)(x_1,x_2,\ldots,x_n)}, \frac{t}{2})\}$$

$$\le \max\{C_{(x_1,x_2,\ldots,x_n)}\left(\frac{t}{2|\alpha|M_1}\right),$$

$$C_{(x_1,x_2,\ldots,x_n)}\left(\frac{t}{2|\lambda|M_2}\right)\}$$

Choose $M = \operatorname{Max}\{2|\alpha|M_1, 2|\lambda|M_2\} + 1.$

Thus $M \ge 2|\alpha|M_1$ and $M \ge 2|\lambda|M_2$

This implies that

$$\frac{t}{2|\alpha|M_1} \ge \frac{t}{M} \text{ and } \frac{t}{2|\lambda|M_2} \ge \frac{t}{M} \text{ for each } t>0$$

Hence,

$$C_{(x_1,x_2,\ldots,x_n)}\left(\frac{t}{2|\alpha|M_1}\right) \le C_{(x_1,x_2,\ldots,x_n)}\left(\frac{t}{M}\right) \text{ and}$$

$$C_{(x_1,x_2,\ldots,x_n)}\left(\frac{t}{2|\lambda|M_2}\right) \le C_{(x_1,x_2,\ldots,x_n)}\left(\frac{t}{M}\right).$$

Therefore,

$$C^*_{(\alpha T_1+\lambda T_2)(x_1,x_2,\ldots,x_n)}(t) \le C_{(x_1,x_2,\ldots,x_n)}\left(\frac{t}{M}\right) \text{ for}$$

each $t>0$. Then we have for each $(x_1,x_2,\ldots,x_n) \in X^n$ and $t>0$, there exists M>0 such that

$C^*_{(\alpha T_1+\lambda T_2)(x_1,x_2,\dots,x_n)}(t) \le C_{(x_1,x_2,\dots,x_n)}\left(\frac{t}{M}\right)$.

This implies that $\alpha T_1 + \lambda T_2 \in \beta(X^n, R), \forall \alpha, \lambda \in R$.

Hence $\beta(X^n, R)$ is a linear space.

Definition (3.13):-

Let (X,C) be a N-coprobabilistic normed space satisfying the following two conditions

(N-C $-N_5$) for each t>0, $C_{(x_1,x_2,\dots,x_n)}(t) < 1$ implies $x_1, x_2, \dots, x_n$ are Linearly dependent.

(N-C-N_6) For $x_1, x_2, \dots, x_n$ are linearly independent, $C_{(x_1,x_2,\dots,x_n)}(t)$ is a continuous of $t \in R$ and strictly decreasing in the subset $\{t : 0 < C_{(x_1,x_2,\dots,x_n)} < 1\}$ of R.

And let $\{\|.,\dots,\| : \alpha \in (0,1)\}$ is a ascending family of N-norms of X. defined. and $T \in \beta(X^n, R)$, we define

$$\|T\|^*_\alpha = Inf\left\{ \begin{array}{l} \dfrac{|T(x_1,x_2,\dots,x_n)|}{\|x_1,x_2,\dots,x_n\|_{1-\alpha}} : x_1,x_2,\dots,x_n \\ are\ linear\ independent \end{array} \right\},$$

$\forall \alpha \in (0,1)$

Again we define

$$C^{**}_T(s) = \begin{cases} Inf\{1-\alpha \in (0,1) : \|T\|^*_\alpha \le s\} \\ \quad for\ (T,s) \ne (0,0) \\ 1 \\ \quad for\ \ (T,s) = (0,0) \end{cases}$$

It is clear that C** is coprobabilistic norm on β

Reference:-

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بعض الخواص للفضاء المعياري N- المرافق الاحتمالي والفضاء المواجة المرافق الاحتمالي للفضاء المعياري N- المرافق الاحتمالي

الخلاصة :-

الغرض الرئيسي من هذا البحث هو تقديم الفضاء المعياري N- المرافق الاحتمالي, الفضاء المواجه المرافق الاحتمالي للفضاء المعياري N - المرافق الاحتمالي وإعطاء بعض الحقائق المتعلقة بهم .