Solution of Fractional Delay Linear Integro-differential Equations Using Variational Approach and Approximate method

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Abstract:-

In this paper fractional linear integro-differential equations with and without delays is studied, and solving such equations by using the variational approach and singularity methods.

Keyword:- Integral equations, Delay integral equations, Fractional differential equations, Calculus of variation ,Collocation method.

1-Introduction:-

The importance of delay or functional integral equations is of its ability to give information about the solution not only at the present time ,but also for an interval. Many numerical and approximate methods may be used to solve such type of delay integral equations, and among such methods is the variational approach which is of minimizing certain functional,[1],[2].

Fractional calculus is commonly called generalized differ-integration which means arbitrary orders (real or complex) derivatives and integrals. Many questions arise concerning differintegration,in which the early beginning of fractional calculus was in 1695 when G.W.Leibnize wrote a letter from Hanover,Germany,September 30,1695 to G.A.L.Hopital said that $d^{1/2}\ x = x\sqrt{\frac{dx}{x}}$ which is an apparent paradox,[4].

The first application of fractional calculus is due to Abel in 1823 in solving an integral equation which arises in the tautochrone proplem.This problem sometimes called "Isochrones problem" is that of finding the shape of a frictionless wire lying in a vertical plane such that the time of slide of a bead placed on the wire slides to the lowest point of the wire in the same time regardless of where the bead is placed,[3],[4].

In addition, an integral equation is said to be singular when:

1- Either or both of the limits of the integration become infinite.
2- If the kernel becomes infinite for one more points of the interval under discussion.
3- The range of the integration is infinite or the kernel k(x,y) is discontinuous.

Such type of equations may be solved easily using numerical and approximate methods,[5],[6],[7].

2.Preliminaries:

2.1-Fractional Linear Integro-differational equation without delay:-

The linear Integro-differational equation without delay in unknown function f(x) is defined by:

$$\frac{df}{dx} = g(x) + \int_a^x k(x,y)f(y)df \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where g and k are given functions, a is constant and $x \in [a,b]$.

The given function k(x,y) which depends on the current variable x as well as the auxiliary variables y is known as the kernel or nucleus of the Integro-differational equation,g(x) is called the driving term,[8],[9],[10],[11].

The fractional linear Integro-differational equation of order α is defined by:

$$D^{\alpha}f(x) = g(x) + \int_a^x k(x,y)f(y)dy. \ (2)$$

where α may be real or complex.

and by Riemann-Liouville definition of fractional derivatives of orderα,[9],[12]

Is defined as:

$D^{\alpha}f(x) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_a^x \frac{f(y)}{(x-y)^{\alpha-m+1}}dy \ldots (3)$

Where m is the integral defined by m-1$< \alpha < m$.

Then,We get the more general form for this type of equations which is:

$\frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_a^x \frac{f(y)}{(x-y)^{\alpha-m+1}}dy = g(x) + \int_a^x k(x,y)f(y)dy \ldots (4)$

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$g(x) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_a^x \frac{f(y)}{(x-y)^{\alpha-m+1}}dy - \int_a^x k(x,y)f(y)dy \ldots. (5)$

This equation is the non-homogenous Volterra integral equation without delay of the first kind.

Where The order α is either assumed to be $\alpha > 1$ or $\alpha < 1$.

2.2Fractional Linear Integro- differational equation with delay:-

The linear Integro- differational equation with delay and unknown function f(x) may be given as:

$\frac{df}{dx}g(x) + \int_a^x k(x,y)f(y-\tau)dy \ldots\ldots (6)$

where g,k are given functions and τ,a are constant with τ is positive , k(x,y) is the kerenel of this integral equation.

The fractional linear Integro- differational equation of order α is:

$D^{\alpha}f$=g(x)+$\int_a^x k(x,y)f(y-\tau)dy \ldots\ldots (7)$

and by Riemann-Liouville fractional derivatives of order α,yields:

$\frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_a^x \frac{f(y)}{(x-y)^{\alpha-m+1}}dy = g(x) + \int_a^x k(x,y)f(y-\tau)dy \ldots\ldots (8)$

Then:-

$g(x) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_a^x \frac{f(y)}{(x-y)^{\alpha-m+1}}dy - \int_a^x k(x,y)f(y-\tau)dy \ldots\ldots\ldots\ldots. (9)$

Equation (9) is a first kind non-homogenous Volterra integral equation with delay.

The fractional order α is either $\alpha > 1$ or $\alpha < 1$ and we will consider two cases:

3-The Main Results and Comparisions:

3.1- Fractional Linear Integro-differential equation without delay:

Consider two cases for fractional derivatives :

<u>Case-1- (if $1 < \alpha \leq 2$):</u>

For example taking $\alpha = 1.5$ then m=2 and the equation (5) becomes:

$$g(x) = \frac{1}{\Gamma(0.5)}\frac{d^2}{dx^2}\int_a^x \frac{f(y)}{(x-y)^{0.5}}dy - \int_a^x k(x,y)f(y)dy \ldots\ldots\ldots (10)$$

This equation may be solved by using the Variational approach as follows:

The linear operator related to this equation is given by:

$$L = \frac{1}{\Gamma(0.5)}\frac{d^2}{dx^2}\int_a^x \frac{dy}{(x-y)^{0.5}} - \int_a^x k(x,y)dy \ldots. (11)$$

This operator satisfy Lf=g and it is easily checked that it is linear and symmetric. Therefore the variational formulation of the fractional Integro-differational equation without delay is given by:

$F(f) = \frac{1}{2} < Lf, Lf > - < g, Lf >$

$= \frac{1}{2}\int_0^T Lf(x)Lf(x)dx - \int_0^T g(x)Lf(x)dx$

$=\int_0^T \left[\frac{1}{2}\left[\frac{d^2}{dx^2}\int_a^x \frac{1}{\sqrt{\pi}}(x-y)^{-0.5}f(y)dy - \int_a^x k(x,y)f(y)dy\right]^2 - \left[g(x)\left[\frac{d^2}{dx^2}\int_a^x \frac{1}{\sqrt{\pi}}(x-y)^{-0.5}f(y)\int_a^x k(x,y)f(y)dy\right]\right]\right]$…….(12)

For complete illustration ,consider the following example:-

<u>Exampal(1):</u>

Consider the non- homogenous fractional Volterra integral equation without delay of the first kind:-

$g(x) = \frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_0^x (y+1)(x-y)^{-1/2}dy - \int_0^x (x^2-y^2)(y+1)\,dy \ldots\ldots\ldots\ldots (13)$

Where the kernel is given by:

k(x,y)=$x^2 - y^2$

and the exact solution for this problem is given by:

f(x)=x+1 , where f(0)=1

We write the function f as a linear combination of some basis functions,as:

f(x)=$1 + a_1x + a_2x^2 + a_3x^3$

and the linear operator is taken to be:

$L = \frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_0^x (x-y)^{-1/2}dy - \int_0^x (x^2-y^2)\,dy \ldots\ldots\ldots (14)$

Then the Variational formulation releted to equation (13) is:

F(f)=

$$\int_{0.01}^{1}\left[\frac{1}{2}\left[\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_0^x (xy)^{-0.5}\ f(y,a_1,a_2,a_3)dy - \int_0^x (x^2-y^2)f(y,a_1,a_2,a_3)dy\right]^2 - \left[g(x)\left[\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_0^x (x-y)^{-0.5} f(y,a_1,a_2,a_3)dy - \int_0^x (x^2-y^2)f(y)dy\right]\right]\right] \ldots..(15)$$

and hence the problem is reduced now to minimize the functional (15).

The obtained numerical results are:

$a_1 = 1.00051$, $a_2 = 5.2713\times 10^{-7}$, $a_3 = -2.926\times 10^{-8}$

We get the following results presented in table (1) with its comparison with the exact solution.

Table(1)

x	Exact solution	Approximate solution	error
0	1.0	1.0	0
0.1	1.1	1.100051	0.000051
0.2	1.2	1.200102	0.000102
0.3	1.3	1.300153	0.000153
0.4	1.4	1.400204	0.000204
0.5	1.5	1.500255	0.000255
0.6	1.6	1.600306	0.000306
0.7	1.7	1.700357	0.000357
0.8	1.8	1.800408	0.000408
0.9	1.9	1.900459	0.000459
1	2	2.00051	0.00051

Case-2- (if $0 < \alpha < 1$)

In this case suppose that for simplicity $0 < \alpha < 1$ then $\alpha = \frac{1}{2}$,m=1 and equation (5) becomes singular integral equation therefore, this equation may be solved by using numerical and approximation methods such as quadrature methods and collocation method,[5].

We will solve this type of equation (6) by using the collocation method as in the following example.

Exampal-2-:

Consider the singular integral equation:-

$$\frac{2}{\sqrt{\pi}}x^{1/2} - \frac{5x^3}{6} = \frac{1}{\sqrt{\pi}}\frac{d}{dx}\int_0^x (x-y)^{-1/2} f(y)dy - \int_0^x (x+y)f(y)dy \ldots\ldots\ldots\ldots(16)$$

Where $x\in[0,1]$

The exact solution is f(x)=x and the kernel function is k(x,y)=x+y

Hence using collocation method,we let:

$f(x)=a_1+a_2x$…………………(17)

and substituting (17) in (16) yields:-

$$\frac{2}{\sqrt{\pi}}x^{1/2} - \frac{5x^3}{6} = \frac{1}{\sqrt{\pi}}\frac{d}{dx}\int_0^x (x-y)^{-1/2}(a_1 + a_2\, y)dy - \int_0^x (x+y)(a_1+a_2y)dy \ldots..(18)$$

Therefore,we have two unknowns a_1 and a_2 to be evaluated and by evaluating equation (18) and using the minimization technique an a_1 and a_2 of the functional (18) we get the following results for a_1 and a_2 :

$a_{1=} -1.526 \times 10^{-6}$, $a_{2=}1.02421$

Therefore:

f(x)= -1.526×10^{-6} +1.02421x

A comparison between the exact and approximate results are given in table (2)

Table(2)

x	**Exact solution**	**Approximate solution**	**Error**
0	0	-1.526$\times 10^{-6}$	0.000001526
0.1	0.1	0.102419474	-0.002419474
0.2	0.2	0.204840474	-0.004840474
0.3	0.3	0.307261474	-0.007261474
0.4	0.4	0.4096827474	-0.009682474
0.5	0.5	0.512103474	-0.012103474
0.6	0.6	0.614524474	-0.014524474
0.7	0.7	0.716945474	-0.016945474
0.8	0.8	0.819366474	-0.019366474
0.9	0.9	0.921787474	-0.021787474
1	1	1.024208474	-0.024208474

3.2-Fractional Linear Integro-differational equation with delay:

Consider two cases for fractional derivative and with time delay.

Cases-1- (If $1 < \alpha \leq 2$)

Suppose that $\alpha = 1.5$,then m=2 and equation (9) becomes:

$$g(x) = \frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x (x-y)^{-1/2} f(y)\,dy - \int_a^x k(x,y)f(y-\tau)dy \ldots (19)$$

Where $\tau > 0$, and g is given function.

This equation may be solved by using the Variational approach as follows**:**

The linear operator related equation (19) is given by :

$$L=\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x (x-y)^{-1/2}\,dy - \int_a^x k(x,y)D\,dy$$

Where D is the shift operator defined by:

Dy(x)=y(x-τ)

and the integral equation may be re written as:

$$\left(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x (x-y)^{-1/2}\,dy - \int_a^x k(x,y)D\,dy\right)f(y) = g(x)$$

$$\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x (x-y)^{-1/2} f(y)dy - \int_a^x k(x,y)Df(y)dy = g(x)$$

$$\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x (x-y)^{-1/2} f(y)dy - \int_a^x k(x,y)f(y-\tau)dy = g(x)$$

Where the operator L is linear since :

$L(c_1u_1 + c_2u_2)=$

$\left(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x (x-y)^{-1/2}\,dy - \int_a^x k(x,y)D\,dy\right)(c_1u_1(x) + c_2u_2(x)$

$$=\left(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x (x-y)^{-1/2}\,dy - \int_a^x k(x,y)D\,dy\right)c_1u_1(x) + \left(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x (x-y)^{-1/2}\,dy - \int_a^x k(x,y)D\,dy\right)c_2u_{2(x)}$$

$$=\left(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x (x-y)^{-1/2}c_1u_1(y)\,dy - \int_a^x k(x,y)Dc_1u_1(y)dy\right) + \left(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x (x-y)^{-1/2}c_2u_2(y)\,dy - \int_a^x k(x,y)Dc_2u_2(y)dy\right)$$

$$=\left(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x (x-y)^{-1/2}c_1u_1(y)\,dy - \int_a^x k(x,y)c_1u_1(y-\tau)dy\right) + \left(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x (x-y)^{-1/2}c_2u_2(y) - \int_a^x k(x,y)c_2u_2(y-\tau)dy\right)$$

$$= c_1\left(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x (x-y)^{-1/2}u_1(y)\,dy - \int_a^x k(x,y)u_1(y-\tau)dy\right) + c_2\left(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x (x-y)^{-1/2}u_2(y)\,dy - \int_a^x k(x,y)u_2(y-\tau)dy\right)$$

$$=c_1\left(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x (x-y)^{-1/2}u_1(y)\,dy - \int_a^x k(x,y)Du_1(y)dy\right) + c_2\left(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x (x-y)^{-1/2}u_2(y)\,dy - \int_a^x k(x,y)Du_2(y)dy\right)$$

$$=c_1\left(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x (x-y)^{-1/2}\,dy - \int_a^x k(x,y)D\,dy\right)u_1(x) + c_2\left(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x (x-y)^{-1/2}\,dy - \int_a^x k(x,y)D\,dy\right)u_{2(x)}$$

$$= c_1Lu_1(x) + c_2Lu_2(x)$$
$$= c_1L(u_1) + c_2L(u_2)$$

Also,the linear operator L is symmetric with respect to the chosen bilinear form:-

$(u,v) =< u, Lv >$

Where $< u,v >= \int_0^T u(x)v(x)dx$

And since :

$$(Lu_1,u_2) =< Lu_1,Lu_2 > = \int_0^T Lu_1(x)Lu_2(x)dx$$

$$=\int_0^T\left(\left(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x (x-y)^{-1/2}\,dy - \int_a^x k(x,y)D\,dy\right)u_1(x)\left(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x (x-y)^{-1/2}\,dy - \int_a^x k(x,y)D\,dy\right)u_{2(x)}\right)dx$$

$$=\int_0^T\left(\left(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x (x-y)^{-1/2}u_1(y)\,dy - \int_a^x k(x,y)Du_1(y)dy\right)\left(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x (x-y)^{-1/2}c_2u_2(y)\,dy - \int_a^x k(x,y)Dc_2u_2(y)dy\right)\right)dx$$

$$=\int_0^T\left(\left(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x (x-y)^{-1/2}u_1(y)\,dy - \int_a^x k(x,y)u_1(y-\tau)dy\right)\left(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x (x-y)^{-1/2}u_2(y)\,dy - \int_a^x k(x,y)u_2(y-\tau)dy\right)\right)dx$$

$=\int_0^T\Big(\Big(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x(x-y)^{-1/2}u_2(y)\,dy-\int_a^x k(x,y)u_2(y-\tau)dy\Big)$

$$\left(\begin{array}{c}\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x(x-y)^{-1/2}u_1(y)\,dy\\ -\int_a^x k(x,y)u_1(y-\tau)dy\end{array}\right)\Big)dx$$

$=\int_0^T\Big(\Big(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x(x-y)^{-1/2}u_2(y)\,dy-\int_a^x k(x,y)Du_2(y)dy\Big)$

$\Big(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x(x-y)^{-1/2}u_1(y)\,dy-\int_a^x k(x,y)Du_1(y)dy\Big)\Big)dx$

$=\int_0^T\Big(\Big(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x(x-y)^{-1/2}\,dy-\int_a^x k(x,y)Ddy\Big)u_2(x)$

$\Big(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x(x-y)^{-1/2}\,dy-\int_a^x k(x,y)Ddy\Big)u_1(x)\Big)dx$

$= \int_0^T Lu_2\,(x)Lu_1dx$

$=< Lu_2, Lu_1 >$

$= (Lu_2, u_1)$

Thus, the Variational formulation of equation (18) is given by:

$$F(f)=\frac{1}{2}< Lf,Lf > -< g,Lf >$$

$$=\frac{1}{2}\int_0^T Lf(x)Lf(x) - \int_0^T g(x)Lf(x)dx$$

$=\int_0^T\left[\frac{1}{2}Lf(x)Lf(x)-g(x)Lf(x)\right]$

$=\int_0^T[\frac{1}{2}(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x(x-y)^{-1/2}\,dy-\int_a^x k(x,y)Ddy)f(x))^2-g(x)(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x(x-y)^{-1/2}\,dy-\int_a^x k(x,y)Ddy)f(x))]dx$

$=\int_0^T[\frac{1}{2}(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x(x-y)^{-1/2}f(y)\,dy-\int_a^x k(x,y)Df(y)dy)^2-g(x)(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x(x-y)^{-1/2}f(y)dy-\int_a^x k(x,y)Df(y)dy)]dx$

$=\int_0^T[\frac{1}{2}(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x(x-y)^{-1/2}f(y)\,dy-\int_a^x k(x,y)f(y-\tau)dy)^2-g(x)\Big(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_a^x(x-y)^{-1/2}f(y)dy-\int_a^x k(x,y)f(y-\tau)dy)\Big]dx \ldots\ldots(20)$

Now,consider the following illustrative example:

Example -3-

Consider the following non-homogenous Volterra integral equation with constant delay of the first kind:

$$\frac{5}{2\sqrt{\pi}}x^{-1/2}+\frac{4}{3}x^4-x^3=\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_0^x(x-y)^{-1/2}f(y)\,dy-\int_0^x k(x,y)f(y-1)dy \ldots\ldots(21)$$

Where k(x,y)=2xy, and the exact solution of this problem is given by:

$f(x)=2x+1$

The linear operator related to (21) is taken to be:-

$$L=\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_0^x(x-y)^{-1/2}\,dy-\int_0^x 2xy\,Ddy$$

Where Df=f(y-τ) , $\tau=1$

The functional f may be approximated as:

$f(x)=1+a_1x+a_2x^2+a_3x^3$

Hence ,the variational formulation related to equation (21) is:

$$F(f)=\int_{0.01}^1[\frac{1}{2}(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_0^x(x-y)^{-1/2}f(y,a_1,a_2,a_3)\,dy-\int_0^x 2xyf(y-1,a_1,a_2,a_3)dy)^2-g(x)\Big(\frac{1}{\sqrt{\pi}}\frac{d^2}{dx^2}\int_0^x(x-y)^{-1/2}f(y,a_1,a_2,a_3)\,dy-\int_0^x 2xyf(y-1,a_1,a_2,a_3)dy)\Big]dx \ldots.(22)$$

The problem is reduced to minimize the functional (22) which will give the following results:

$a_1=1.9852$, $a_2=6.5\times10^{-4}$, $a_3=-1.295\times10^{-9}$

A comparison between the exact and approximate results are given in table (3).

Table(3)

x	Exact solution	Approximate solution	error
0	1	1	0
0.1	1.2	1.1985264999	0.001473501
0.2	1.4	1.397065999	0.002934001
0.3	1.6	1.5956184999	0.004381501
0.4	1.8	1.794183999	0.005816001
0.5	2	1.99276244999	0.007237501
0.6	2.2	2.19135399	0.008646001
0.7	2.4	2.389958499	0.010041501
0.8	2.6	2.588575995	0.011424005
0.9	2.8	2.7872064928	0.012793508
1	3	2.98584999	0.01415001

Case -2- if $0 < \alpha < 1$

In this case suppose for simplicity that $\alpha = \frac{1}{2}$ and hence m=1 which will becomes singular delay integral equation ,as it is illustrated in the following example:

Example -4-:

Consider the singular integral equation:-

$$\frac{1}{\sqrt{\pi}}\left(\frac{-8}{3}x^{3/2} + 2x^{1/2}\right) + \frac{x^7}{5} - \frac{x^6}{4} + \frac{x^5}{3} = \frac{1}{\sqrt{\pi}}\frac{d}{dx}\int_0^x (x-y)^{-1/2} f(y)\,dy - \int_0^x k(x,y) f(y-1)dy \ldots\ldots\ldots.(23)$$

Where $f(x) = x^2 + x$ is the exact solution and k(x,y)=x^2y^2 *is the* kerenel of the integral quation.

Hence,using the approximate solution:

$f(x)$=a_1+a_2 x+$a_3 x^2$

$f(y-1) = a_1 + a_2(y-1) + a_3(y-1)^2$..(24)

and substituting (24) in(23) yields,to:

$$\frac{1}{\sqrt{\pi}}\left(\frac{-8}{3}x^{3/2} + 2x^{1/2}\right) + \frac{x^7}{5} - \frac{x^6}{4} + \frac{x^5}{3} = \frac{1}{\sqrt{\pi}}\frac{d}{dx}\int_0^x \binom{x}{-y}^{-1/2} f(y, a_1, a_2, a_3)\,dy - \int_0^x x^2y^2 f(y-1, a_1, a_2, a_3) \ldots (25)$$

Functional (2) may be minimized for a_1, a_2 and a_3 which will give the following results:

a_1=1.1×10^{-5} , $a_2 = 0.9561$, $a_3 = 0.97005$

Hence: f(x)=1.1×$10^{-5} + 0.9561$x +0.97005x^2

The obtained approximate results are given in table (4) and its comparison with the exact solution.

Table(4)

x	Exact solution	Approximate solution	Error
0	0	0.000011	-0.000011
0.1	0.11	0.9648115	-0.8548115
0.2	0.24	0.230033	0.009967
0.3	0.39	0.3741455	0.0158545
0.4	0.56	0.537659	0.022351
0.5	0.75	0.7205735	0.0294265
0.6	0.96	0.922889	0.037111
0.7	1.19	1.1446055	0.0453945
0.8	1.44	1.385723	0.054277
0.9	1.71	1.6462415	0.0637585
1	2	1.926161	0.073839

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