On A Class Of P - Valent Functions Defined By Integral Operator

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Abstract : In this paper , we study a new class $\Re M^*$ of p-valent functions with negative coefficient defined by integral operator in the unit disk . We obtain some results of this class .

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1. introduction :

Let $\Re M^*$ denote of a class of multivalent analytic functions of the form :

$$f(z) = mz^p + \sum_{n=p+1}^{\infty} a_n z^n \quad , \quad p \in IN \, , m > 0$$

(1)

which are analytic and multivalent functions in the unit disk

$$U = \{z : z \in C \text{ and } |z| < 1 \}.$$

In the next we defined a new integral operator .

Definition (1) : Let the function f defined by (2) in the class. Then

$$WR_\beta^\mu(f(z)) = \frac{1}{\Gamma(\mu+1)} \int_0^1 t^\beta \left(\log\frac{1}{t}\right)^\mu f(\frac{z}{t^\beta})dt$$

$$= mz^p - \sum_{n=2}^{\infty} \left(\frac{1}{1-\beta(n-1)}\right)^{\mu+1} a_n z^n$$

$$= mz^p - \sum_{n=2}^{\infty} \Psi(\beta,\mu,n) a_n z^n$$

where $\Psi(\beta,\mu,n) = \left(\frac{1}{1-\beta(n-1)}\right)^{\mu+1}$

and $\beta \le 0 \ , \ 0 \le \mu \le 1$.

Definition(2) : A function f defined by (1) belonging to the class $\Re M^*$ is in the class $\Re M^{*p}(\alpha,\lambda,\mu,\beta)$ if it satisfies the condition:

$$\left|\frac{z^2(1-\alpha)\left(WR_\beta^\mu(f(z))\right)'' - z(p-1)\left(WR_\beta^\mu(f(z))\right)'}{\alpha z^2\left(WR_\beta^\mu(f(z))\right)'' + z(1-\lambda)WR_\beta^\mu(f(z))'}\right| < 1$$

, (3)

where $\beta \le 0$, $0<\alpha<1$, $0\le\lambda<1$, $0\le\mu\le1$.

For a given real number z_0 $(0<z_0<1)$. Let $\Re^{pi}(i=0,1)$ be a subclass of $\Re_p^*$ satisfies the condition $z_0^{-p}f(z_0)\le1$ and $p^{-1}z_0^{1-p}f(z_0)\le1$ respectively .

Let

$$\Re M^{*pi}(\alpha,\lambda,\mu,\beta,z_0) = \Re M^{*p}(\alpha,\lambda,\mu,\beta)\cap\Re^{pi}(i=0,1)$$

(4)

Some another classes studied by W.G. Atshan and S.R. Kulkarni [2], S.R. Kulkarni and Mrs.S.S. Joshi[3],consisting of multivalent and meromorphic univalent functions respectively .

2. Coefficient Estimates :

In the next theorem , we obtain a necessary and sufficient condition for function to be in the class $\Re M^{*p}(\alpha,\lambda,\mu,\beta)$.

Theorem (1): A function f defined (2) be in the class $\Re M^{*p}(\alpha,\lambda,\mu,\beta)$ if and only if

$$\sum_{n=p+1}^{\infty} n[1+(n-p)-\lambda\theta]\Psi(\beta,\mu,n)a_n \le mp(1-\lambda) \qquad (5)$$

Under the parametric restraints given (3) , the result is sharp .

Proof : Let the inequality (5) holds true . For $|z|=1$, we have

$$\left|z^2(1-\alpha)\left(WR_\beta^\mu(f(z))\right)'' - z(p-1)\left(WR_\beta^\mu(f(z))\right)'\right|$$

$$-\left|\alpha z^2\left(WR_\beta^\mu(f(z))\right)'' + z(1-\lambda)WR_\beta^\mu(f(z))'\right|$$

$$=\left|\begin{array}{l} m(1-\alpha)p(p-1)z^p \\ -(1-\alpha)\sum_{n=p+1}^{\infty}n(n-1)\Psi(\beta,\mu,n)a_nz^n \\ -mp(p-1)z^p+(p-1)\sum_{n=p+1}^{\infty}n\Psi(\beta,\mu,n)a_nz^n \end{array}\right|$$

$$-\left|mp(p-1)\alpha z^p - \alpha\sum_{n=p+1}^{\infty}n(n-1)\Psi(\beta,\mu,n)a_nz^n\right.$$

$$\left.+mp(1-\lambda)z^p-(1-\lambda\theta)\sum_{n=p+1}^{\infty}n\Psi(\beta,\mu,n)a_nz^n\right|$$

$$=\left|\begin{array}{l} m\alpha p(p-1)z^p + \\ \sum_{n=p+1}^{\infty}n[(1-\alpha)(n-1)-(p-1)]\Psi(\beta,\mu,n)a_nz^n \end{array}\right|$$

$$-\left|\begin{array}{l} mp[(p-1)\alpha+(1-\lambda)]z^p- \\ \sum_{n=p+1}^{\infty} n[(\alpha(n-1)+(1-\lambda\theta)]\Psi(\beta,\mu,n)a_n z^n \end{array}\right|$$

$$\leq m\alpha p(p-1)|z|^p+$$
$$\sum_{n=p+1}^{\infty} n[(1-\alpha)(n-1)-(p-1)]\Psi(\beta,\mu,n)a_n|z|^n$$

$$-[mp[(p-1)\alpha+(1-\lambda)]]|z|^p+$$
$$\sum_{n=p+1}^{\infty} n[(\alpha(n-1)+(1-\lambda\theta)]\Psi(\beta,\mu,n)a_n|z|^n$$

$$=\sum_{n=p+1}^{\infty} n[(1+(n-p)-\lambda]\Psi(\beta,\mu,n)a_n$$
$$-mp(1-\lambda\theta).$$

Hence by principle of the maximum modulus, $f(z)\in\Re M^{*p}(\alpha,\lambda,\mu,\beta)$.

Conversely, assume that $f(z)\in\Re M^{*p}(\alpha,\lambda,\mu,\beta)$. Then

$$\left|\frac{z^2(1-\alpha)\left(WR_\beta^\mu(f(z))\right)''-z(p-1)\left(WR_\beta^\mu(f(z))\right)'}{\alpha z^2\left(WR_\beta^\mu(f(z))\right)''+z(1-\lambda)WR_\beta^\mu(f(z))'}\right|$$
$$\left[\left|\frac{m\alpha p(p-1)z^p+\sum_{n=p+1}^{\infty} n[(1-\alpha)(n-1)-(p-1)]\Psi(\beta,\mu,n)a_n z^n}{mp[(p-1)\alpha+(1-\lambda)]z^p-\sum_{n=p+1}^{\infty} n[(\alpha(n-1)+(1-\lambda\theta)]\Psi(\beta,\mu,n)a_n z^n}\right|<1\right]$$

,

for all $z\in U$. Using the fact $\operatorname{Re}(z)\leq|z|$ for z it follows that

$$\operatorname{Re}\left\{\frac{m\alpha p(p-1)z^p+\sum_{n=p+1}^{\infty} n[(1-\alpha)(n-1)-(p-1)]\Psi(\beta,\mu,n)a_n z^n}{mp[(p-1)\alpha+(1-\lambda\theta)]z^p-\sum_{n=p+1}^{\infty} n[(\alpha(n-1)+(1-\lambda)]\Psi(\beta,\mu,n)a_n z^n}\right\}<1$$

. (6)

Now choose the values of z on the real axis so that $\dfrac{\left(WR_\beta^\mu(f(z))\right)''}{\left(WR_\beta^\mu(f(z))\right)'}$ is real . Upon clearing the dominator in (6) and letting $z\to 1^-$ through positive values , we obtain

$$\sum_{n=p+1}^{\infty} n[1+(n-p)-\lambda\theta]\Psi(\beta,\mu,n)a_n\leq mp(1-\lambda)$$

.

The result is sharp.

The proof is completes .

<u>Corollary (1):</u> Let $f(z)\in\Re M^{*p}(\alpha,\lambda,\mu,\beta)$. Then

$$a_n\leq\frac{mp(1-\lambda)}{n[1+(n-p)-\lambda\theta]\Psi(\beta,\mu,n)} \quad .$$

(7)

Theorem (2): Let $f(z)\in \Re M^{*p}(\alpha,\lambda,\mu,\beta)$ .Then $f(z)\in \Re M^{*p0}(\alpha,\lambda,\mu,\beta,z_0)$ if and only if

$$\sum_{n=p+1}^{\infty}\left(\frac{n\left[1+(n-p)-\lambda\theta\right]\Psi(\beta,\mu,n)}{p(1-\lambda)}-z_0^{n-p}\right)a_n\le 1 \quad . \qquad (8)$$

Proof : Suppose that $f(z)\in \Re M^{*p0}(\alpha,\lambda,\mu,\beta,z_0)$, we have

$$z_o^{-p}f(z_0)=m-\sum_{n=p+1}^{\infty}a_n z_o^{n-p} \quad ,$$

which gives

$$m\le 1+\sum_{n=p+1}^{\infty}a_n z_o^{n-p} \quad .$$

Put m in (5) , we get

$$\sum_{n=p+1}^{\infty}n\left[1+(n-p)-\lambda\theta\right]\Psi(\beta,\mu,n)a_n$$
$$\le p(1-\lambda)\left[1+\sum_{n=p+1}^{\infty}a_n z_o^{n-p}\right]$$

which is equivalent to

$$\sum_{n=p+1}^{\infty}\left(\frac{n\left[1+(n-p)-\lambda\theta\right]\Psi(\beta,\mu,n)}{p(1-\lambda)}-z_0^{n-p}\right)a_n\le 1$$
.

Conversely , let the inequality (8) holds true , then we have

$$\sum_{n=p+1}^{\infty}\left(\frac{n\left[1+(n-p)-\lambda\theta\right]\Psi(\beta,\mu,n)}{p(1-\lambda)}-z_0^{n-p}\right)a_n$$

$$=m\sum_{n=p+1}^{\infty}\left(\frac{n\left[1+(n-p)-\lambda\theta\right]\Psi(\beta,\mu,n)}{mp(1-\lambda)}a_n\right)$$
$$-\sum_{n=p+1}^{\infty}a_n z_0^{n-p}$$

$$\le m-\sum_{n=p+1}^{\infty}a_n z_o^{n-p}=z_o^{-p}f(z_0)\le 1.$$

Then $f(z)\in \Re M^{*p0}(\alpha,\lambda,\mu,\beta,z_0)$. (9)

Corollary (2): Let $f(z)\in \Re M^{*p0}(\alpha,\lambda,\theta,\beta,z_0)$. Then

$$a_n\le \frac{p(1-\lambda)}{n\left[1+(n-p)-\lambda\right]\Psi(\beta,\mu,n)-p(1-\lambda)z_o^{n-p}} \quad . \qquad (10)$$

Theorem (3): Let $f(z)\in \Re^{*p}(\alpha,\lambda,\mu,\theta,\beta)$ .Then $f(z)\in \Re^{*p1}(\alpha,\lambda,\mu,\theta,\beta,z_0)$ if and only if

$$\sum_{n=p+1}^{\infty}\left(\frac{n\left[1+(n-p)-\lambda\theta\right]\Psi(\beta,\mu,n)}{p(1-\lambda)}-(np^{-p})z_0^{n-p}\right)a_n\le 1 \quad . \qquad (11)$$

Proof : Since $f(z)\in \Re M^{*p1}(\alpha,\lambda,\mu,\beta,z_0)$, we have

$$p^{-1}z_o^{1-p}f'(z_0)=m-\sum_{n=p+1}^{\infty}np^{-1}a_n z_o^{n-p} \quad ,$$
(12)

which gives

$$m \le 1 + \sum_{n=p+1}^{\infty} np^{-1} a_n z_o^{n-p}$$
(13)

Substituting this value of m (given (13)) in Theorem 1 we get desire assertion .

Corollary (3): Let $f(z) \in \Re M^{*p0}(\alpha, \lambda, \mu, \beta, z_0)$. Then

$$a_n \le \frac{p(1-\lambda)}{n[1+(n-p)-\lambda\theta]\Psi(\beta,\mu,n) - n(1-\lambda\theta)z_o^{n-p}} .$$ (14)

Definition (3) : Let $f(z)$ defined by the form

$$f(z) = kz + \sum_{n=p+1}^{\infty} a_n z^n \; (a_n \ge 0, k > 0, p \in IN) .$$

be in the class $\Re M^*$. Then $f(z)$ is said $k-$close – to – convex of order $\alpha, (0 \le \alpha < 1)$ if and only if

$$\left| \frac{f'(z)}{z^{p-1}} - kp \right| \le 1 - \alpha \; . \qquad (15)$$

4. Radius of k - close – to – convex

In the next theorem , we obtain the radius of $k-$close – to – convex for $f(z) \in \Re M^{*p}(\alpha, \lambda, \mu, \beta)$.

Theorem (4) : If $f(z) \in \Re M^{*p}(\alpha, \lambda, \mu, \beta)$, then $f(z)$ is $k-$closed – to – convex in the disk $|z| < r_{p^1}(\alpha, \lambda, \mu, \beta)$,where

$$r_{p^1}(\alpha, \lambda, \mu, \beta) = \inf_n \left\{ \frac{(1-\alpha)[1+(n-p)-\lambda\theta]\Psi(\beta,\mu,n)}{mp(1-\lambda)} \right\}^{\frac{1}{n-1}} \qquad (16)$$

Proof: For $|z| < r_{p^1}(\alpha, \lambda, \mu, \beta)$ we have

$$\left| \frac{f'(z)}{z^{p-1}} - mp \right| \le \sum_{n=p+1}^{\infty} n a_n |z|^{n-p}$$

Thus $\left| \frac{f'(z)}{z^{p-1}} - mp \right| \le 1 - \alpha$ if

$$\sum_{n=p+1}^{\infty} \left(\frac{n}{1-\alpha} \right) a_n |z|^{n-p} \le 1. \qquad (17)$$

According to Theorem (1) , we have

$$\sum_{n=p+1}^{\infty} \frac{n[1+(n-p)-\lambda]\Psi(\beta,\mu,n)}{mp(1-\lambda)} a_n \le 1.$$
(18)

Hence (17) will be true if

$$\frac{n|z|^{n-p}}{1-\alpha} \le \frac{n[1+(n-p)-\lambda]\Psi(\beta,\mu,n)}{mp(1-\lambda)}$$

equivalently if

$$|z| \le \left\{ \frac{(1-\alpha)[1+(n-p)-\lambda]\Psi(\beta,\mu,n)}{mp(1-\lambda)} \right\}^{\frac{1}{n-1}}$$

$n \ge 2$. (19)

The proof is completes .

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