A new procedure of initial boundary value problems using homotopy perturbation method

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Abstract

In this paper, , a new procedure is applied to treatment of initial boundary value problems by mixed initial and boundary conditions together to obtain a new initial solution at every iteration using homotopy perturbation method. The structure of a new successive initial solutions can give a more accurate solution. Some examples (linear and nonlinear, homogeneous and nonhomogeneous problems) are given in this paper to illustrate the effectiveness and convenience of a new procedure.

Keywords: initial boundary value problems, homotopy perturbation method.

1. Introduction

The initial boundary value problems were discussed by using either initial or one of the boundary conditions only by many researchers such as Wazwaz [1,2,3], Momani [4], Yulita Molliq et.al [5], Noor and Mohyud-Din [6], Biazar and Aminikhah [7], Shidfar and Molabahrami [8], Niu. and Wang [9], Chun and Sakthivel [10] and Wu. and Lee [11]. The researchers discussed those problems in a different methods and in a special case, Chun and Sakthivel [10] researched the homotopy perturbation technique for solving two-point boundary value problems by using one point only. The purpose of this paper is to employ a new procedure for treatment initial and boundary value problems by mixed initial and boundary conditions together to obtain a new initial solution at every iterations using homotopy perturbation method which is explain in the next section. These procedure to construct a new successive initial solutions can give a more accurate solution, some examples are given in this paper to illustrate the effectiveness and convenience of this technique. Recently, the treatment of the initial boundary value problems was found by Ali E. J. [12] by using a different method which is a variational iteration method.

2-Homotopy perturbation method [14,15]

Consider the following nonlinear differential equation;

$$A(u) = f(r) \quad r \in \Omega, \qquad (2.1)$$

with boundary conditions:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \qquad (2.2)$$

Where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of the domain Ω,

The operator A can be generally divided in two parts L and N, where L is linear, and N is nonlinear, therefore equation.(2.1) can be written as,

$$L(u) + N(u) = f(r), \qquad (2.3)$$

by using homotopy technique, one can construct a homotopy $v(r,p): \Omega \times [0,1] \to R$ which satisfies :

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0,\ p \in [0,1], \quad (2.4)$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \quad (2.5)$$

where $r \in \Omega$ and $p \in [0,1]$ is an embedding parameter, and u_0 is the initial approximation of equation.(2.3)which satisfies the boundary conditions. Hence, obviously we have

$$H(v,0) = L(v) - L(u_0) = 0, \quad (2.6)$$

$$H(v,1) = A(v) - f(r) = 0, \quad (2.7)$$

and the changing process of p from 0 to 1 is the same as changing $H(v,p)$ from $L(v) - L(u_0)$, to $A(v) - f(r)$. In topology, this is called deformation, $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic in topology. If, the embedding parameter $p; (0 \le p \le 1)$ is considered as a small parameter, applying the classical perturbation technique, we can assume that the solution of equation.(2.7)can be given as a power series in p, i.e.

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \quad (2.8)$$

and setting $p = 1$ results in the approximate solution of equation.(2.3) as ;

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots \quad (2.9)$$

3. A new procedure of initial boundary value problems.

To convey the basic idea for treatment of initial and boundary conditions for solving initial boundary value problems, we consider the following one dimensional differential equation

$$F\big(u(x,t)\big) = g(x,t), \quad 0 < x < 1, t > 0, \quad (3.1)$$

the initial conditions associated with (3.1) are of the form

$$u(x,0) = f_0(x),\ \frac{\partial u(x,0)}{\partial t} = f_1(x),$$

$$0 \le x \le 1, \quad (3.2)$$

and the boundary conditions are given by

$$u(0,t) = g_0(t), u(1,t) = g_1(t), t > 0, \quad (3.3)$$

where $f_0(x),\ f_1(x), g_0(t)\ and\ g_1(t)$ are given functions. The initial solution can be written as

$$u_0(x,t) = f_0(x) + tf_1(x). \quad (3.4)$$

We construct a new successive initial solutions u_n^* at every iteration for Eq. (3.1) by applying a new technique

$$u_n^*(x,t) = u_n(x,t) + (1-x)\,[g_0(t) - u_n(0,t)] + x[g_1(t) - u_n(1,t)],$$

$$n = 0,1,2,\ldots \quad (3.5)$$

Clearly that the new successive initial solutions u_n^* of Eq. (3.1) satisfying the initial and boundary conditions together as follows

$$if\ \ t = 0\ \ then\ \ u_n^*(x,0) = u_n(x,0),$$
$$if\ \ x = 0\ \ then\ \ u_n^*(0,t) = g_0(t),$$
$$if\ \ x = 1\ \ then\ \ u_n^*(1,t) = g_1(t). \quad (3.6)$$

generally, sometimes a new procedure of formula (3.5) can be applying for higher dimensional initial boundary value problems by mixed initial and boundary conditions together to have a new successive initial solutions u_n^* as follows

Consider the k dimensional equation

$$F\big(u(x_1,x_2,\ldots,x_k,t)\big) = g(x_1,x_2,\ldots,x_k,t),$$

$$0 < x_1,x_2,\ldots,x_k < 1, t > 0, \quad (3.7)$$

Then we have

$$u_n^*(x_1,x_2,\ldots,x_k,t) = u_n(x_1,x_2,\ldots,x_k,t) + (1 - x_1)[g_{01}(x_2x_3,\ldots,x_k,t) - u_n(0,x_2,\ldots,x_k,t)]$$

$$+x_1[g_{11}(x_2,x_3,\dots,x_k,t)-u_n(1,x_2,\dots,x_k,t)]$$

$$+\cdots(1-x_k)[g_{0k}(x_1,x_2,\dots,x_{k-1},t)-u_n(x_1$$

$$x_2,\dots,x_{k-1},0,t)]+x_k[g_{1k}(x_1,x_2,\dots,x_{k-1},t)$$

$$-u_n(x_1,x_2,\dots,x_{k-1},1,t)], \qquad (3.8)$$

where $n=0,1,2,\dots$, and the boundary conditions associated with Eq. (3.7) are of the form

$$u(0,x_2,\dots,x_k,t)=g_{01}(x_2,x_3,\dots,x_k,t),$$

$$u(1,x_2,\dots,x_k,t)=g_{11}(x_2,x_3,\dots,x_k,t),$$

$$u(x_1,0,\dots,x_k,t)=g_{02}(x_1,x_3,\dots,x_k,t),$$

$$u(x_1,1,\dots,x_k,t)=g_{12}(x_1,x_3,\dots,x_k,t),$$

$$\vdots$$

$$u(x_1,\dots,x_{k-1},0,t)=g_{0k}(x_1,\dots,x_{k-1},t),$$

$$u(x_1,\dots,x_{k-1},1,t)=g_{1k}(x_1,\dots,x_{k-1},t), \qquad (3.9)$$

and the initial conditions are given by

$$u(x_1,x_2,\dots,x_k,0)=f_0(x),$$

$$\frac{\partial u(x_1,x_2,\dots,x_k,0)}{\partial t}=f_1(x). \qquad (3.10)$$

Such as treatment of formula (3.8) is a very effective when $k=2$ as shown in this paper.

4. Applications and results

Example 1[4]: Consider the following one-dimensional heat-like problem:

$$\frac{\partial u}{\partial t}-\frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}=0,$$

$$0<x<1, t>0, \qquad (4.1)$$

subject to the initial conditions

$$u(x,0)=x^2, \qquad 0<x<1, \qquad (4.2)$$

and the boundary conditions

$$u(0,t)=0, \quad u(1,t)=\mathrm{e}^{\mathrm{t}}, \quad t>0. \qquad (4.3)$$

By applying a new approximations u_n^* in Eq. (3.5) we obtain

$$u_n^*(x,t)=u_n(x,t)+(1-x)[0-u_n(0,t)]+ x[\mathrm{e}^{\mathrm{t}}-u_n(1,t)], \quad n=0,1,2,\dots. \qquad (4.4)$$

Now, we begin with a new initial approximation u_0^* $(when\ n=0)$

$$u_0^*(x,t)=x^2+x\,(\mathrm{e}^{\mathrm{t}}-1).$$

we construct the following homotopy

$$(1-p)\left(\frac{\partial u}{\partial t}-\frac{\partial u_0}{\partial t}\right)+p\left(\frac{\partial u}{\partial t}-\frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}\right)=0$$

where the initial approximation $u_0(x,0)=x^2$,

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$$\frac{\partial u}{\partial t}+p\left(-\frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}\right)=0, \qquad (4.5)$$

we consider u as

$$u=u_0+pu_1+p^2u_3+\cdots \qquad (4.6)$$

substituting (4.6) in (4.5) and equating the coefficients of like powers of p, we get following set of differential equations with solving the systems accordingly, thus we obtain,

$$p^0: \frac{\partial u}{\partial t}=0, u_0(x,0)=x^2 \Rightarrow u_0(x,t)=x^2$$

$$p^1: \frac{\partial u_1}{\partial t}=\frac{1}{2}x^2\frac{\partial^2 u_0^*}{\partial x^2} \Rightarrow u_1=x^2 t$$

$$p^2: \frac{\partial u_2}{\partial t}=\frac{1}{2}x^2\frac{\partial^2 u_1^*}{\partial x^2} \Rightarrow u_2=x^2\frac{t^2}{2!}$$

$$p^3: \frac{\partial u_3}{\partial t}=\frac{1}{2}x^2\frac{\partial^2 u_2^*}{\partial x^2} \Rightarrow u_3=x^2\frac{t^3}{3!}$$

$$\vdots$$

$$p^n: \frac{\partial u_n}{\partial t}=\frac{1}{2}x^2\frac{\partial^2 u_{n-1}^*}{\partial x^2} \Rightarrow u_n=x^2\frac{t^n}{n!},$$

by setting $p=1$ in equation.(4.6), the solution of (4.1) can be obtain as

$$u=u_0+u_1+u_2+\cdots,$$

the series solution is given by

$$u(x,t) = x^2\left(1 + t + \frac{t^{2,}}{2!} + \frac{t^3}{3!} + \cdots\right), \quad (4.7)$$

and in a closed form by

$$u(x,t) = x^2 e^t. \quad (4.8)$$

Which is an exact solution.

Example 2[4]: We next consider the one-dimensional wave-like equation:

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2} = 0, 0 < x < 1, t > 0, (4.9)$$

subject to the initial conditions

$$u(x,0) = x, \frac{\partial u(x,0)}{\partial t} = x^2, \quad (4.10)$$

and the boundary conditions

$$u(0,t) = 0 \ , u(1,t) = 1 + \sinh t, \quad (4.11)$$

By applying a new approximations u_n^* in Eq.

(3.5) we obtain

$$u_n^*(x,t) = u_n(x,t) + (1-x)[0 - u_n(0,t)] +$$

$$x[1 + \sinh t - u_n(1,t)], \quad (4.12)$$

where $n = 0,1,2, \dots$.

Now, we begin with a new initial

approximation u_0^* $(when\ n = 0)$

$$u_0^*(x,t) = x + x^2 t + x\,(\sinh t - t). \quad (4.13)$$

we construct the following homotopy

$$(1-p)\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2}\right) + p\left(\frac{\partial^2 u}{\partial t^2} - \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}\right) = 0,$$
(4.14)

where the initial approximation is $u_0(x,0) = x$,

$$\frac{\partial^2 u}{\partial t^2} + p\left(-\frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}\right) = 0, \quad (4.15)$$

we consider u as

$$u = u_0 + pu_1 + p^2 u_2 + \cdots, \quad (4.16)$$

substituting (4.16) in (4.15) and equating the coefficients of like powers of p, we get following set of differential equations with solving the systems accordingly, thus we obtain,

$$p^0: \frac{\partial^2 u}{\partial t^2} = 0, u_0(x,0) = x$$

$$\Rightarrow u_0(x,t) = x + x^2\, t$$

$$p^1: \frac{\partial^2 u_1}{\partial t^2} = \frac{1}{2}x^2\frac{\partial^2 u_0^*}{\partial x^2} \Rightarrow u_1 = x^2\frac{t^3}{3!}$$

$$p^2: \frac{\partial^2 u_2}{\partial t^2} = \frac{1}{2}x^2\frac{\partial^2 u_1^*}{\partial x^2} \Rightarrow u_2 = x^2\frac{t^5}{5!}$$

$$p^3: \frac{\partial^2 u_3}{\partial t^2} = \frac{1}{2}x^2\frac{\partial^2 u_2^*}{\partial x^2} \Rightarrow u_3 = x^2\frac{t^7}{7!}$$

$\vdots$

$$p^n: \frac{\partial^2 u_n}{\partial t^2} = \frac{1}{2}x^2\frac{\partial^2 u_{n-1}^*}{\partial x^2} \Rightarrow u_n = x^2\frac{t^{2n+1}}{(2n+1)!},$$

by setting $p = 1$ in equation.(4.16), the solution of (4.9) can be obtain as

$$u = u_0 + u_1 + u_2 + \cdots,$$

the solution in a series form is given by

$$u(x,t) = x + x^2\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right), \quad (4.17)$$

and in a closed form by

$$u(x,t) = x + x^2 \sinh t. \quad (4.18)$$

Which is an exact solution.

Example 3[6]: Consider the two-dimensional initial boundary value problem :

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2}y^2\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}x^2\frac{\partial^2 u}{\partial y^2},$$

$$0 < x, y < 1, t > 0, \quad (4.19)$$

and the initial conditions

$$u(x,y,0) = x^2 + y^2,$$

$$\frac{\partial u(x,y,0)}{\partial t} = -(x^2+y^2), \qquad (4.20)$$

with boundary conditions

$u(0,y,t) = y^2 e^{-t}$,

$u(1,y,t) = (1+y^2)e^{-t}$,

$u(x,0,t) = x^2 e^{-t}$,

$u(x,1,t) = (1+x^2)e^{-t}$. (4.21)

By applying a new approximations u_n^* in Eq. (3.8) we obtain

$u_n^*(x,y,t) = u_n(x,y,t) + (1-x)[y^2e^{-t} -$

$u_n(0,y,t)] + x[(1+y^2)e^{-t} - u_n(1,y,t)] +$

$(1-y)[x^2e^{-t} - u_n(x,0,t)]$ +y[(1+x^2)

$e^{-t} - u_n(x,1,t)]$, (4.22)

where $n = 0,1,2,\dots$.

Now, we begin with a new initial

approximation u_0^* $(when\, n = 0)$

$u_0^*(x,y,t) = x^2 + y^2 - (x^2+y^2)t + (1-x)$

$[y^2e^{-t} + y^2t] + x[(1+y^2)e^{-t} - (1+y^2) +$

$1 + y^2\,)t] + (1-y)[x^2e^{-t} - x^2 + x^2t]$

$+y[(1+x^2)e^{-t} - (x^2+1) + (x^2+1)t]$

Then, we get

$u_0^*(x,y,t) = y^2 - xy^2 - (x+y) + (x+y)t +$

$(x^2+y^2)e^{-t} + (x+y)e^t$, (4.23)

we construct the following homotopy

$(1-p)(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2})$ + p $(\frac{\partial^2 u}{\partial t^2} - \frac{1}{2}y^2\frac{\partial^2 u}{\partial x^2}$

$-\frac{1}{2}x^2\frac{\partial^2 u}{\partial y^2}) = 0$ (4.24)

where the initial approximation is $u(x,y,0) = x^2 + y^2$,

$\frac{\partial^2 u}{\partial t^2} + p\left(-\frac{1}{2}y^2\frac{\partial^2 u}{\partial x^2} - \frac{1}{2}x^2\frac{\partial^2 u}{\partial y^2}\right) = 0$,
(4.25)

we consider u as

$u = u_0 + pu_1 + p^2u_3 + \cdots$,
(4.26)

substituting (4.26) in (4.25) and equating the coefficients of like powers of p, we get following set of differential equations with solving the systems accordingly, thus we obtain,

$p^0\!: \frac{\partial^2 u_0}{\partial t^2} = 0, u_0 = (x,y,0) = x^2 + y^2 \Rightarrow$

$u_0(x,y,t) = x^2 + y^2 - (x^2+y^2)t$,

$p^1\!: \frac{\partial^2 u_1}{\partial t^2} = \frac{1}{2}y^2\frac{\partial^2 u_0^*}{\partial x^2} + \frac{1}{2}x^2\frac{\partial^2 u_0^*}{\partial y^2} \Rightarrow$

$u_1 = -x^2 - y^2 + (x^2+y^2)t + (x^2+y^2)e^{-t}$,

by equation (4.23), we get

$u_1(x,y,t) = -x^2 - y^2 + (x^2+y^2)t$

$(x^2+y^2)e^{-t}$,

through, the solution of two steps iteration and by setting $p = 1$ in equation.(4.26), the solution of (4.19) can be obtain as

$u(x,y,t) = u_0(x,y,t) + u_1(x,y,t)$,

$u(x,y,t) = x^2 + y^2 - (x^2+y^2)t - x^2 - y^2$

$+(x^2+y^2)t + (x^2+y^2)e^{-t}$. (4.27)

We can readily check

$u(x,y,t) = (x^2+y^2)e^{-t}$. (4.28)

Which yields an exact solution of Eq. (4.19).

Example 4[6]: Consider the two-dimensional nonlinear inhomogeneous initial boundary value problem :

$$\frac{\partial^2 u}{\partial t^2} = \frac{15}{2}\left(x\,u_{xx}^2 + y\,u_{yy}^2\right) + 2x^2 + 2y^2,$$

$0 < x, y < 1, t > 0,$ (4.29)

and the initial conditions

$$u(x,y,0) = 0, \quad \frac{\partial u(x,y,0)}{\partial t} = 0, \qquad (4.30)$$

with boundary conditions

$u(0,y,t) = y^2t^2 + yt^6,$

$u(1,y,t) = (1+y^2)t^2 + (1+y)t^6,$

$u(x,0,t) = x^2t^2 + xt^6,$

$u(x,1,t) = (1+x^2)t^2 + (1+x)t^6.$ (4.31)

By applying a new approximations u_n^* in Eq. (3.8) we obtain

$$u_n^*(x,y,t) = u_n(x,y,t) + (1-x)[y^2t^2 + yt^6 - u_n(0,y,t)] + x[(1+y^2)t^2 + (1+y)t^6 - u_n(1,y,t)] + (1-y)[x^2t^2 + xt^6 - u_n(x,0,t)] + y[(1+x^2)t^2 + (1+x)t^6 - u_n(x,1,t)], \qquad (4.32)$$

where n=0,1,2,…,

now, we begin with a new initial

approximation u_0^* ($when$ n=0),

$$u_0^*(x,y,t) = (x^2+y^2)t^2 + (1-x)yt^6 + x(1+y)t^6 + (1-y)xt^6 + y(1+x)t^6.$$

Then, we get

$$u_0^* = (x^2+y^2)t^2 + 2(x+y)t^6, \qquad (4.33)$$

we construct the following homotopy

$$(1-p)\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2}\right) + p\left(\frac{\partial^2 u}{\partial t^2} - \frac{15}{2}x\left(\frac{\partial^2 u}{\partial x^2}\right)^2 - \frac{15}{2}y\left(\frac{\partial^2 u}{\partial y^2}\right)^2 - 2x^2 - 2y^2\right) = 0, \qquad (4.34)$$

where the initial approximation is $u(x,y,0) = 0,$

$$\frac{\partial^2 u}{\partial t^2} + p\left(\frac{\partial^2 u}{\partial t^2} - \frac{15}{2}x\left(\frac{\partial^2 u}{\partial x^2}\right)^2 - \frac{15}{2}y\left(\frac{\partial^2 u}{\partial y^2}\right)^2 - 2x^2 - 2y^2\right) = 0, \qquad (4.35)$$

we consider u as

$$u = u_0 + pu_1 + p^2u_3 + \cdots \qquad (4.36)$$

substituting (4.36) in (4.35) and equating the coefficients of like powers of p, we get following set of differential equations with solving the systems accordingly, thus we obtain,

$$p^0: \frac{\partial^2 u_0}{\partial t^2} = 0, u_0(x,y,0) = 0 \Rightarrow u_0(x,y,t) = 0$$

$$p^1: \frac{\partial^2 u_1}{\partial t^2} = \frac{15}{2}x\left(\frac{\partial^2 u_0^*}{\partial t^2}\right)^2 + \frac{15}{2}y\left(\frac{\partial^2 u_0^*}{\partial t^2}\right)^2 + 2x^2 + 2y^2,$$

$$\Rightarrow u_1 = (x^2+y^2)t^2 + (x+y)t^6,$$

by equation(4.33),we get

$$u_1(x,y,t) = (x^2+y^2)t^2 + (x+y)t^6.$$

through, the solution of two steps iteration and by setting $p = 1$ in equation.(4.36), the solution of (4.29) can be obtain as

$$u(x,y,t) = u_0(x,y,t) + u_1(x,y,t),$$

$$u(x,y,t) = (x^2+y^2)t^2 + (x+y)t^6. \qquad (4.37)$$

Which yields an exact solution of Eq. (4.29).

Example 5 [13]: Consider the two-dimensional initial boundary value problem:

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}y^2\frac{\partial^2 u}{\partial y^2},$$

$0 < x, y < 1, t > 0,$ (4.39)

and the initial conditions

$u(x,y,0) = x^2 + y^2,$

$$\frac{\partial u(x,y,0)}{\partial t} = y^2 - x^2, \qquad (4.40)$$

with boundary conditions

$u(0,y,t) = y^2e^t,$

$u(1,y,t) = e^{-t} + y^2e^t,$

$(x,0,t) = x^2e^{-t},$

$u(x,1,t) = e^t + x^2e^{-t}. \qquad (4.41)$

By applying a new approximations u_n^* in Eq. (3.8) we obtain

$u_n^*(x,y,t) = u_n(x,y,t) + (1-x)[y^2e^t -$

$u_n(0,y,t)] + x[e^{-t} + y^2e^t - u_n(1,y,t)]$

$(1- y)[x^2e^{-t} - u_n(x,0,t)] + y[e^t + x^2e^{-t}$

$-u_n(x,1,t)], \qquad (4\ .42)$

where $n = 0,1,2, \ldots$.

Now, we begin with a new initial

approximation u_0^* $(when\ n = 0)$

$u_0^*(x,y,t) = x^2 + y^2 + (y^2 - x^2)t +$

$(1-x)\ [\,y^2e^t - y^2 - y^2t\,] + x$

$[e^{-t} + y^2e^t - 1 - y^2 - (y^2-1)t\,] +$

$(1-y)\,[\,x^2e^{-t} - x^2 + x^2t\,] + y$

$[e^t + x^2e^{-t} - x^2 - 1 - (1-x^2)t].$

Then, we get

$u_0^*(x,y,t) = -(x+y) + (x-y)t + xe^{-t} +$

$ye^t + x^2e^{-t} + y^2e^t, \qquad (4.43)$

we construct the following homotopy

$(1-p)\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2}\right) + p(\frac{\partial^2 u}{\partial t^2} - \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2} -$

$\frac{1}{2}y^2\frac{\partial^2 u}{\partial y^2}) = 0, \qquad (4.44)$

where the initial approximation is

$u(x,y,0) = x^2 + y^2,$

$\frac{\partial^2 u}{\partial t^2} + p\left(-\frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2} - \frac{1}{2}y^2\frac{\partial^2 u}{\partial y^2}\right) = 0, \qquad (4.45)$

we consider u as

$u = u_0 + pu_1 + p^2u_3 + \cdots \qquad (4.46)$

substituting (4.46) in (4.45) and equating the coefficients of like powers of p, we get following set of differential equations with solving the systems accordingly, thus we obtain,

$p^0 : \frac{\partial^2 u_0}{\partial t^2} = 0, u_0 = (x,y,0) = x^2 + y^2$

$\Rightarrow u_0(x,y,t) = x^2 + y^2 + (y^2 - x^2)t,$

$p^1 : \frac{\partial^2 u_1}{\partial t^2} = \frac{1}{2}y^2\frac{\partial^2 u_0^*}{\partial x^2} + \frac{1}{2}x^2\frac{\partial^2 u_0^*}{\partial y^2} \Rightarrow$

$u_1 = -x^2 - y^2 - (y^2 - x^2)t + x^2e^{-t} + y^2e^t,$

by equation(4.43) , we get

$u_1(x,y,t) = -x^2 - y^2 - (y^2 - x^2)t +$

$x^2e^{-t} + y^2e^t,$

through, the solution of two steps iteration and by setting $p = 1$ in equation.(4.46), the solution of (4.39) can be obtain as

$u(x,y,t) = u_0(x,y,t) + u_1(x,y,t),$

$u(x,y,t) = x^2 + y^2 + (y^2 - x^2)t - x^2 - y^2$

$-(y^2 - x^2)t + x^2e^{-t} + y^2e^t. \qquad (4.47)$

We can readily check

$u(x,y,t) = x^2e^{-t} + y^2e^t. \qquad (4.48)$

Which yields an exact solution of Eq. (4.39).

Example 6 [13]: Consider the two-dimensional initial boundary value problem:

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}y^2\frac{\partial^2 u}{\partial y^2},$$

$$0 < x, y < 1, t > 0, \quad (4.49)$$

and the initial conditions

$$u(x,y,0) = y^2, \quad \frac{\partial u(x,y,0)}{\partial t} = x^2, \quad (4.50)$$

with boundary conditions

$u(0,y,t) = y^2 \cosh t,$

$u(1,y,t) = \sinh t + y^2 \cosh t,$

$u(x,0,t) = x^2 \sinh t,$

$u(x,1,t) = x^2 \sinh t + \cosh t. \quad (4.51)$

By applying a new approximations u_n^* in Eq. (3.8) we obtain

$u_n^*(x,y,t) = u_n(x,y,t) + (1-x)[y^2 \cosh t$ -

$u_n(0,y,t)] + x[\sinh t + y^2 \cosh t - u_n(1,y,t)]$

$+(1-y)[x^2 \sinh t - u_n(x,0,t)] + y[x^2 \sinh t$

$+ \cosh t - u_n(x,1,t)], \quad (4.52)$

where $n = 0,1,2,\dots$.

Now, we begin with a new initial

approximation u_0^* $(when\ n = 0)$

$u_0^*(x,y,t) = y^2 + x^2 t + (1-x)[y^2 \cosh t$

$-y^2] + x[\sinh t + y^2 \cosh t - y^2 - t] +$

$(1-y)[x^2 \sinh t - x^2 t] + y$

$[x^2 \sinh t + \cosh t - 1 - x^2 t],$

Then, we get

$u_0^*(x,y,t) = -y - xt + x sinh(t) +$

$x^2 \sinh(t) + y \cosh(t) + y^2 \cosh(t), \quad (4.53)$

we construct the following homotopy

$$(1-p)\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2}\right) + p\left(\frac{\partial^2 u}{\partial t^2} - \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}\right.$$

$$\left. - \frac{1}{2}y^2\frac{\partial^2 u}{\partial y^2}\right) = 0,$$

where the initial approximation is

$u(x,y,0) = y^2,$

$$\frac{\partial^2 u}{\partial t^2} + p\left(-\frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2} - \frac{1}{2}y^2\frac{\partial^2 u}{\partial y^2}\right) = 0, \quad (4.54)$$

we consider u as

$$u = u_0 + pu_1 + p^2 u_3 + \cdots \quad (4.55)$$

substituting (4.55) in (4.54) and equating the coefficients of like powers of p , we get following set of differential equations with solving the systems accordingly, thus we obtain,

$$p^0: \frac{\partial^2 u_0}{\partial t^2} = 0, u_0 = (x,y,0) = y^2$$

$\Rightarrow u_0(x,y,t) = y^2 + x^2 t,$

$$p^1: \frac{\partial^2 u_1}{\partial t^2} = \frac{1}{2}x^2\frac{\partial^2 u_0^*}{\partial x^2} + \frac{1}{2}y^2\frac{\partial^2 u_0^*}{\partial y^2}$$

$\Rightarrow u_1 = -y^2 - x^2 t + x^2 \sinh(t) + y^2 \cosh(t),$

by equation(4.53),we get

$u_1 = -y^2 - x^2 t + x^2 \sinh(t) + y^2 \cosh(t).$

Through, the solution of two steps iteration and by setting $p = 1$ in equation.(4.55), the solution of (4.49) can be obtain as

$u(x,y,t) = u_0(x,y,t) + u_1(x,y,t),$

$u(x,y,t) = y^2 + x^2 t - y^2 - x^2 t +$

$x^2 sinht + y^2 cosht \quad (4.56)$

We can readily check

$$u(x,y,t) = x^2 \sinh t + y^2 \cosh t. \quad (4.57)$$

Which yields an exact solution of Eq. (4.49).

5. Conclusions

In this paper, a very effective to construct a new initial successive solutions u_n^* by mixed initial and boundary conditions together which explained in formula (3.5) and (3.8) when $k = 2$ which is used to find successive approximations u_n of the solution u by homotopy perturbation method to solve initial boundary value problems. These technique to construct of a new successive initial solutions can give a more accurate solution. It is important and obvious that the exact solutions have found directly from a first iteration for the last four examples but if we use initial conditions only we will have exact solution by calculating infinite successive solutions u_n which closed form by Eq. (2.9).

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