Construction Of Complete (K,N)-Arcs In PG (2,8) FOR M < N

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Abstract

In this work, we construct complete (k,n)-arcs and we find some of them are maximum for some n, 2 < n < 8. if n = 2, where every arc which constructed by the equation of the conic called conic arc, and from it we constructs complete arcs and we prove it's maximum by taking the union of two ,three and six conics , respectively. and then we show By adding the points of index zero for the $(k_6,6)$-arc, $(k_7,7)$-arc, respectively, we get a maximum complete $(k_8,8)$-arc.

1. Introduction //

Hassan,(2001),(1) showed the classification and construction of (k,3)-arcs in PG(2,4), PG(2,8) and PG(2,16).Mohammed,(1988),(2)showed the classification and construction of(k,3)-arcs in PG(2,5).Abdul-Hussain,(1997), (3) showed the classification and construction of (k,4)-arcs in PG(2,5). Faiyad,(2000), (4) constructed and classified the(k,3)-arcs in PG(2,7) and kareem,(2000), (5) constructed and classified the (k,3)-arcs in PG(2,9),all of them used the algebraic method. Kadhum , (2001), (6) constructed the (k, n)-arcs from the (k, m)-arcs for m < n in PG(2,5)and PG(2,7).

In this work we find the maximum (k_n, n)-arcs in PG(2,8),$3<n\leq 8$ from the maximum (k,2)-arcs (conics) :

2. Preliminaries //

2.1 Definition (1)

The projective plane PG(2,8) over Galois Field GF(8) contains 73 points , 73 lines , every line contains 9 points and every point is on 9 line . Let Pi and L_i , i = 1,2,…,73 , be the points and lines of PG(2,8) , respectively . Let i stands for the point Pi and all the points and lines of PG(2,8) are given in the table :

I	P_i			L_i								
1	1	0	0	2	10	18	26	34	42	50	58	66
2	0	1	0	1	10	11	12	13	14	15	16	17
3	1	1	0	3	10	19	28	37	46	55	64	73
4	2	1	0	8	10	24	27	41	44	54	61	71
5	3	1	0	6	10	22	31	35	49	53	60	72
6	4	1	0	5	10	21	32	39	43	52	65	70
7	5	1	0	9	10	25	29	38	48	51	63	68
8	6	1	0	4	10	20	30	40	47	57	59	69
9	7	1	0	7	10	23	33	36	45	56	62	67
10	0	0	1	1	2	3	4	5	6	7	8	9
11	1	0	1	2	11	19	27	35	43	51	59	67
12	2	0	1	2	16	24	32	40	48	56	64	72
13	3	0	1	2	14	22	30	38	46	54	62	70
14	4	0	1	2	13	21	29	37	45	53	61	69
15	5	0	1	2	17	25	33	41	49	57	65	73
16	6	0	1	2	12	20	28	36	44	52	60	68
17	7	0	1	2	15	23	31	39	47	55	63	71
18	0	1	1	1	18	19	20	21	22	23	24	25
19	1	1	1	3	11	18	29	36	47	54	65	72
20	2	1	1	8	16	18	33	35	46	52	63	69
21	3	1	1	6	14	18	27	39	45	57	64	68
22	4	1	1	5	13	18	31	40	44	51	62	73
23	5	1	1	9	17	18	30	37	43	56	60	71
24	6	1	1	4	12	18	32	38	49	55	61	67
25	7	1	1	7	15	18	28	41	48	53	59	70
26	0	2	1	1	58	59	60	61	62	63	64	65
27	1	2	1	4	11	21	31	41	46	56	58	68
28	2	2	1	3	16	25	30	39	44	53	58	67
29	3	2	1	7	14	19	29	40	49	52	58	71

I	P_i			L_i								
30	4	2	1	8	13	23	28	38	43	57	58	72
31	5	2	1	5	17	22	27	36	48	55	58	69
32	6	2	1	6	12	24	33	37	47	51	58	70
33	7	2	1	9	15	20	32	35	45	54	58	73
34	0	3	1	1	42	43	44	45	46	47	48	49
35	1	3	1	5	11	20	33	38	42	53	64	71
36	2	3	1	9	16	19	31	36	42	57	61	70
37	3	3	1	3	14	23	32	41	42	51	60	69
38	4	3	1	7	13	24	30	35	42	55	65	68
39	5	3	1	6	17	21	28	40	42	54	63	67
40	6	3	1	8	12	22	29	39	42	56	59	73
41	7	3	1	4	15	25	27	37	42	52	62	72
42	0	4	1	1	34	35	36	37	38	39	40	41
43	1	4	1	6	11	23	30	34	48	52	61	73
44	2	4	1	4	16	22	28	34	45	51	65	71
45	3	4	1	9	14	21	33	34	44	55	59	72
46	4	4	1	3	13	20	27	34	49	56	63	70
47	5	4	1	8	17	19	32	34	47	53	62	68
48	6	4	1	7	12	25	31	34	43	54	64	69
49	7	4	1	5	15	24	29	34	46	57	60	67
50	0	5	1	1	66	67	68	69	70	71	72	73
51	1	5	1	7	11	22	32	37	44	57	63	66
52	2	5	1	6	16	20	29	41	43	55	62	66
53	3	5	1	5	14	25	28	35	47	56	61	66
54	4	5	1	4	13	19	33	39	48	54	60	66
55	5	5	1	3	17	24	31	38	45	52	59	66
56	6	5	1	9	12	23	27	40	46	53	65	66
57	7	5	1	8	15	21	30	36	49	51	64	66
58	0	6	1	1	26	27	28	29	30	31	32	33
59	1	6	1	8	11	25	26	40	45	55	60	70
60	2	6	1	5	16	23	26	37	49	54	59	68
61	3	6	1	4	14	24	26	36	43	53	63	73
62	4	6	1	9	13	22	26	41	47	52	64	67
63	5	6	1	7	17	20	26	39	46	51	61	72
64	6	6	1	3	12	21	26	35	48	57	62	71
65	7	6	1	6	15	19	26	38	44	56	65	69
66	0	7	1	1	50	51	52	53	54	55	56	57
67	1	7	1	9	11	24	28	39	49	50	62	69
68	2	7	1	7	16	21	27	38	47	50	60	73
69	3	7	1	8	14	20	31	37	48	50	65	67
70	4	7	1	6	13	25	32	36	46	50	59	71
71	5	7	1	4	17	23	29	35	44	50	64	70
72	6	7	1	5	12	19	30	41	45	50	63	72
73	7	7	1	3	15	22	33	40	43	50	61	68

2.2 Definition (2)

A (k,n)-arc in PG(2,p^n) is a set of k points no n+1 of them are collinear. A(k,2)-arc is called k-arc which is a set of k points no three of them are collinear. A(k,n)-arc is complete if it is not contained in a(k+1,n)-arc .The maximum number of points that a(k,2)-arc can have is m(2,p) , and this arc is an oval .

2.3 Theorem (3)

$$m(2,p) = \begin{cases} p+1 & \text{for } p \text{ odd} \\ p+2 & \text{for } p \text{ even} \end{cases}$$

2.4 Definition (4)

A line L in PG(2,p) is an i-secant of a(k,n)-arc if $|L \cap K| = i$

A 2- secant is called a bisecant line .

2.5 Definition (1)

A variety V(F) is a subset of PG(2,p) s.t. V(F)={P(A) $\in$ PG (2,p) | F(A)=0}

2.6 Theorem

In PG(2,q) with q even , the nucleus of the conic :
$V(a_{11}x_1^2+a_{22}x_2^2+a_{33}x_3^2+a_{12}x_1x_2+a_{13}x_1x_3+a_{23}x_2x_3)$ is $P(a_{23}, a_{13}, a_{12})$.

2.7 Definition

Let Q(2,p) be the set of quadrics in PG(2,p) , that is the varieties V(F) , where :
$F= a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{23}x_2x_3$

If $A=\begin{bmatrix} a_{11} & \frac{a_{12}}{2} & \frac{a_{13}}{2} \\ \frac{a_{12}}{2} & a_{22} & \frac{a_{23}}{2} \\ \frac{a_{13}}{2} & \frac{a_{23}}{2} & a_{33} \end{bmatrix}$ is a non-singular,

then the quadric is a conic .

2.8 Definition (5)

A point N which is not on a(k,n) – arc has index i if there are exactly i(n-secants) of the arc through N , we denote the number of points N of index i by N_i

2.9 Remark (5)

The (k,n)-arc is complete iff N_0=0.Thus the arc is complete iff every point of PG(2,p^n) lies on some n - secant of the arc .

3.The Construction of (k,2)-Arcs in PG (2,8)

Let A={1,2,10,19}be the set of unit and reference points , where : 1(1,0,0) , 2 (0 ,1 ,0) , 10 (0 , 0 ,1) and 19 (1,1,1) . A is a(4 ,2)-arc since no three points of A are collinear .

3.1 The Conics in PG (2,8) Through the Unit and Reference Points (1)

The general equation of the conic is: $a_1x_1^2+a_2x_2^2+a_3x_3^2+a_4x_1x_2+a_5x_1x_3+a_6x_2x_3=0$ ..(1)

By substituting the points of A in (1) we get $a_1= a_2 = a_3= 0$ and $a_4+a_5+a_6=1$

So (1) becomes: $a_4x_1x_2+a_5x_1x_3+a_6x_2x_3=0$..(2)

if $a_4=0$, then the conic is degenerated therefore , $a_4 \neq 0$, similarly $a_5 \neq 0$ and $a_6 \neq 0$

Dividing equation (2) by a_4 , we get :

$x_1x_2+\alpha x_1x_3+\beta x_2x_3=0$..(3) $\alpha = \frac{a_5}{a_4}, \beta = \frac{a_6}{a_4}$

then β =-(1+α)=1+α [mod 2] and (3) can be wrilten as: $x_1x_2+\alpha x_1x_2+(1+\alpha)x_2x_3=0$..(4)

where $\alpha \neq 0$ and $\alpha \neq 1$, for if $\alpha = 0$ or α =1, we get a degenerated conic,i . e . , α =2,3,4,5,6,7,[mod 8]

3.2.The Equations of the Conics in PG(2,8) Through the Unit and Reference Points(1)

For any valne for α there is aunique conic contains nine points

1. If α =2,then the equation of the conic C_1 is $x_1x_2+2x_1x_3+3x_2x_3=0$, its nucleus is P_1 (3,2,1) which is the point 29 .C_1= {1,2,10,19,39,44,57,62,72} $\bigcup$ {29}
2. If α =3 , then the equation of the conic C_2 is $x_1x_2+3x_1x_3+2x_2x_3=0$, its nucleus is P_2(2,3,1) which is the point 36 C_2 = {1,2,10,19,30,48,53,65,71} $\bigcup$ {36}
3. If α = 4 ,then the equation of the conice C_3 is $x_1x_2+4x_1x_3+5x_2x_3 = 0$, its nuleus is P_3(5,4,1) which is the point 47 C_3 = {1,2,10,19,29,41,56,60,70} $\bigcup$ {47}
4. If α =5 , then the equation of the conic C_4 is $x_1x_2+5x_1x_3+4x_2x_3=0$, its nucleus is P_4(4,5,1) Which is the point 54 C_4 = {1,2,10,19,32,36,49,63,69} $\bigcup$ {54}
5. If α = 6 , then the equation of the conic C_5 is $x_1x_2+6x_1x_3+7x_2x_3$ =0 , its nucleus is P_5(7,6,1) Which is the point 65 C_5 = {1,2,10,19,31,40,45,54,68} $\bigcup$ {65}
6. If α =7 , then the equation of the conic C_6 is : $x_1x_2+7x_1x_3+6x_2x_3 = 0$, its nucleus is P_6(6,7,1)Which is the point 72 C_6 = {1,2,10,19,33,38,47,52,61} $\bigcup$ {72}

3.3 The Construction of Maximum Complete (k,4)-Arcs from the Conics in PG(2,8)

We have constructed six distinct conics .We choose two conics having no points incommon except the unit and reference point , so we take the two conics C_3 and C_5 . Let A=$C_3 \bigcup C_5$,then we find A is a (k,4)-arc , which is incomplete , since there exist points of index zero for A we add seven of these points to A , which are {5,6,13,14,20,23,64} , we obtain a maximum complete (23,4) – arc which is . A={1,2,3,10,19,29,41,56,60,70,47,31,40,45,54,68,65,5,6,13,14,20,23,64} .

3.4 The Construction of Maximum Complete (k,5)-Arcs

We take the union of three conics which are C_3 , C_5 and C_1 let B=$C_3 \bigcup C_5 \bigcup C_1$, we obtain a(k,5)-arc which is in complete since there are points of index zero for B .We add eight of these points to B which are {5,6,20,23,8,37,42,58}.We obtain a maximum complete(29,5)-arc which is B={1,2,10,19,29,41,56,60,70,47,31,40,45,54,68, 65,39,44, 57,62,72,5,6,20,23,8,37,42,58}.

3.5 The Constuction of Maximum Complete (k,6)-Arcs

We take the union of four conics which are C_3 ,C_5 ,C_1 , and C_4 , we obtain a(k,6)-arc which is incomplete since there are points of index zero for this arc. We add the points of the conics C_2 and C_6 , so this arc is the union of all the six conics , let C=$C_3 \bigcup C_5 \bigcup C_1 \bigcup C_4 \bigcup C_2 \bigcup C_6$. There are no points of index zero to C , so this arc is a maximum complete (34,6)-arc which is C={1,2,10,19,29,41,56,60,70,47,31,40,45,54,68, 65,39,44,57,62,72,32,36,49,63,69,30,48,53,71,33,38,52,61}

3.6 The Construction of Maximum Complete (k,7)-Arcs

The (k,6)-arc is incomplete (k,7)-arc since there are some points of index zero for it. We add eleven of these point , which are {3,4,5,6,7,16,17,27,37,46,51} . We obtain a maximum complete (45,7)-arc which is D={1,2,10,19,29,41,56,60, 70,47,31,40,45, 54,68,65,39,44,57,62,72,32,36,49,63,69,30,48,53,71,33,38,52,61,3,4,5,6,7,16,17,27,37,46,51}

3.7 The Construction of Maximum Complete (k,8)-Arcs

The (k,7)-arc is incomplete (k,8)-arc since there are points of index zero for (k,7)-arc. We add twelve of these points which are {8,11,12,13,14,20,23,25,35,43,55,64}.We obtain a maximum complete (57,8)-arc which is

E={1,2,10,19,29,41,56,60,70,47,31,40, 45,68,65,39,44,57,62,72,32,36,49,63,69, 30,48,53,71,33,38,52,61,3,4,5,6,7,16,17, 27,37,46,51,8,11,12,13,14,20,23,25,35,43,55,64}

4. The Results and Discussion

1. We constructed maximum complete (k,n)-arcs from the maximum complete (k,2)-arcs where $4 \leq n \leq 8$
2. We cannot construct a (k,3)-arc from the union of two conics . having no points incommon except the unit and reference points . But we can construct a (k,3)-arc by taking union of two conics having points incommon other than the unit and reference points , say C_2 and C_4
3. We constructed the maximum complete (k,6)-arc by taking the union of all the conics without adding the points of index zero since the number of the points of index zero is zero.

will be like Exhaustive Search which can take up n step in the worst case.

References //

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