

On Fuzzy Co-Pre-Hilbert Spaces

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Abstract

This paper introduces the concepts fuzzy pre-Hilbert spaces and fuzzy co-pre-Hilbert spaces and proves some theorems in this subject.

Keywords: fuzzy pre-Hilbert space, fuzzy co-pre-Hilbert space.

1. Introduction

The notion of fuzzy pre-Hilbert spaces can be considered as the generalization of that notion of pre-Hilbert spaces. The definition of these spaces has been introduced in [3, 4]. The study mentions the definition of fuzzy pre-Hilbert space and then it introduces a definition for fuzzy co-pre-Hilbert space and discusses some properties on this subject.

2. Preliminaries

Definition(2.1):[1] A binary operation $*$ $: [0,1] \times [0,1] \rightarrow [0,1]$ is a t- norm if $*$ is satisfies the following conditions:

- (i) $*$ Is commutative and associative;
- (ii) $a * 1 = a$ for all $a \in [0,1]$;
- (iii) $a * b \leq c * d$ whenever $a \leq c$ and $a \leq d$

$$a, b, c, d \in [0, 1].$$

Definition(2.2):[2,5] A binary operation $\circ : [0,1] \times [0,1] \rightarrow [0,1]$ is a t-conorm if $\circ$ is satisfies the following conditions:

- (i) $\circ$ Is commutative and associative;
- (ii) $a \circ 0 = a$ for all $a \in [0,1]$;
- (iii) $a \circ b \leq c \circ d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Definition (2.3): [4] Let X be a real vector space, $*$ be a continuous t-norm on $I = [0, 1]$. A function $F : X \times X \times R \rightarrow [0, 1]$ is called a fuzzy pre-Hilbert function if satisfies the following axioms for every $x, y, z \in X$ and $s, t, r \in R$

- (1) $F(x, x, 0) = 0$ and $F(x, x, t) > 0$ for all $t > 0$;
- (2) $F(x, x, t) \neq H(t)$ for some $t \in R \Leftrightarrow x \neq 0$;
- (3) $F(x, y, t) = F(y, x, t)$;
- (4) For any real number α

$$F(\alpha x, y, t) = \begin{cases} F\left(x, y, \frac{t}{\alpha}\right) & , \alpha > 0 \\ H(t) & , \alpha = 0 \\ 1 - F\left(x, y, \frac{t}{-\alpha}\right) & , \alpha < 0 \end{cases}$$

$$(5) F(x, x, t) * F(y, y, s) \leq F(x + y, x + y, t + s);$$

$$(6) \sup_{s+r=t} (F(x, z, s) * F(y, z, r)) = F(x + y, z, t);$$

$$(7) F(x, y, \cdot): R \rightarrow [0, 1] \text{ is continuous on } R/\{0\};$$

$$(8) \lim_{t \rightarrow \infty} F(x, y, t) = 1.$$

$(X, F, *)$ is a fuzzy pre-Hilbert space.

Note: $H(t) = \begin{cases} 1 & , t > 0 \\ 0 & , t \leq 0 \end{cases}$

Lemma (2.4): [3] If $(X, F, *)$ is a fuzzy pre-Hilbert space then $F(x, y, t)$ is non-decreasing with respect to t , for each $x, y \in X$.

Proof: Let $t < s$ for all $s, t \in R$,

$$F(x, y, t) = F(x, y, t) * 1 = F(x, y, t) *$$

$$F(0, 0, s - t) \leq F(x, y, s).$$

Lemma (2.5): If $(X, F, *)$ is a fuzzy pre-Hilbert space then:

$$(1) F(ax, ay, t) = F(x, y, \frac{t}{a^2}) \text{ for all } a \in R/\{0\}. \quad (2) F(x - y, x - y, t) = F(y - x, y - x, t)$$

Proof: (1) and (2) are directly from axiom (4).

Theorem (2.6): [3] (Schwartz inequality) If $(X, F, *)$ is a fuzzy pre-Hilbert space then:
 $F(x, y, ts) \geq F(x, x, t^2) * F(y, y, s^2).$

3. Main results

Definition (3.1): Let X be a real vector space, $\circ$ be a continuous t-conorm on $I=[0, 1]$. A function $\mathcal{F}: X \times X \times R \rightarrow [0, 1]$ is called a fuzzy co-pre-Hilbert function if satisfies the following axioms for every $x, y, z \in X$ and $s, t, r \in R$:

$$(1) \mathcal{F}(x, x, 0) = 1 \text{ and } \mathcal{F}(x, x, t) < 1 \text{ for all } t > 0;$$

$$(2) \mathcal{F}(x, x, t) \neq \mathcal{H}(t) \text{ for some } t \in R \Leftrightarrow x \neq 0.$$

$$(3) \mathcal{F}(x, y, t) = \mathcal{F}(y, x, t);$$

$$(4) \text{ For any real number } \alpha$$

$$\mathcal{F}(\alpha x, y, t) = \begin{cases} \mathcal{F}(x, y, \frac{t}{\alpha}) & , \alpha > 0 \\ \mathcal{H}(t) & , \alpha = 0 \\ 1 - \mathcal{F}(x, y, \frac{t}{-\alpha}) & , \alpha < 0 \end{cases}$$

$$(5) \mathcal{F}(x, x, t) \circ \mathcal{F}(y, y, s) \geq \mathcal{F}(x + y, x + y, t + s);$$

$$(6) \sup_{s+r=t} (\mathcal{F}(x, z, s) \circ \mathcal{F}(y, z, r)) = \mathcal{F}(x + y, z, t);$$

$$(7) \mathcal{F}(x, y, \cdot): R \rightarrow [0, 1] \text{ is continuous on } R/\{0\};$$

$$(8) \lim_{t \rightarrow \infty} \mathcal{F}(x, y, t) = 0.$$

$(X, \mathcal{F}, \circ)$ is a fuzzy co-pre-Hilbert space.

Note: $\mathcal{H}(t) = \begin{cases} 1 & , t \leq 0 \\ 0 & , t > 0 \end{cases}$
 $= 1 - H(t)$

Lemma (3.2): If $(X, \mathcal{F}, \circ)$ is a fuzzy co-pre-Hilbert space then $\mathcal{F}(x, y, t)$ is non-increasing with respect to t , for each $x, y \in X$.

Proof: Let $t < s$ for all $s, t \in R$, $\mathcal{F}(x, y, t) = \mathcal{F}(x, y, t) \circ 0 = \mathcal{F}(x, y, t) \circ \mathcal{F}(0, 0, s - t) \geq \mathcal{F}(x, y, s).$

Lemma (3.3): (1) $\mathcal{F}(ax, ay, t) = \mathcal{F}(x, y, \frac{t}{a^2})$ for all $a \in R/\{0\}$.

$$(2) \mathcal{F}(x - y, x - y, t) = \mathcal{F}(y - x, y - x, t)$$

Proof: (1) and (2) are directly from axiom (4).

Theorem (3.4): Let $(X, F, *)$ is a fuzzy pre-Hilbert space. Define $\mathcal{F} = 1 - F$ then $(X, \mathcal{F}, \circ)$ is a fuzzy co-pre-Hilbert space.

Proof: For all $x, y, z \in X$ and $t, s, r \in R$.

$$(1) \mathcal{F}(x, x, 0) = 1 - F(x, x, 0) = 1 - 0 = 1 \text{ and } \mathcal{F}(x, x, t) = 1 - F(x, x, t) < 1 - 0 = 1 \text{ for all } t > 0;$$

$$(2) \text{ Suppose } \mathcal{F}(x, x, t) \neq \mathcal{H}(t) \text{ for some } t \in R \Leftrightarrow \mathcal{F}(x, x, t) \neq \mathcal{H}(t) \Leftrightarrow 1 - F(x, x, t) \neq 1 - \mathcal{H}(t) \Leftrightarrow F(x, x, t) \neq \mathcal{H}(t) \Leftrightarrow x \neq 0;$$

$$(3) \mathcal{F}(x, y, t) = 1 - F(x, y, t) = 1 - F(y, x, t) = \mathcal{F}(y, x, t)$$

$$(4) \text{ For any real number } \alpha$$

$$\mathcal{F}(\alpha x, y, t) = 1 - F(\alpha x, y, t)$$

$$= 1 - \begin{cases} F\left(x, y, \frac{t}{\alpha}\right) & , \alpha > 0 \\ \mathcal{H}(t) & , \alpha = 0 \\ 1 - F\left(x, y, \frac{t}{-\alpha}\right) & , \alpha < 0 \end{cases}$$

$$= \begin{cases} \mathcal{F}\left(x, y, \frac{t}{\alpha}\right) & , \alpha > 0 \\ \mathcal{H}(t) & , \alpha = 0 \\ 1 - \mathcal{F}\left(x, y, \frac{t}{-\alpha}\right) & , \alpha < 0 \end{cases}$$

$$(5) \mathcal{F}(x, x, t) \circ \mathcal{F}(y, y, s) = (1 - F(x, x, t)) \circ (1 - F(y, y, s)) = 1 - (F(x, x, t) * F(y, y, s)) \leq 1 - F(x + y, x + y, t + s) = \mathcal{F}(x + y, x + y, t + s);$$

$$(6) \sup_{s+r=t} (\mathcal{F}(x, z, s) \circ \mathcal{F}(y, z, r)) = \sup_{s+r=t} ((1 - F(x, z, s)) \circ (1 - F(y, z, r)))$$

$$= 1 - \sup_{s+r=t} (F(x, z, s) * F(y, z, r)) = 1 - F(x + y, z, s + r) = \mathcal{F}(x + y, z, s + r)$$

$$(7) \mathcal{F}(x, y, \cdot): R \rightarrow [0, 1] \text{ is continuous on } R/\{0\};$$

$$(8) \lim_{t \rightarrow \infty} \mathcal{F}(x, y, t) = \lim_{t \rightarrow \infty} (1 - F(x, y, t)) = 1 - 1 = 0.$$

Therefore $(X, \mathcal{F}, \circ)$ is a fuzzy co-pre-Hilbert space.

Theorem (3.5): (Schwartz inequality) Let $(X, \mathcal{F}, \circ)$ is a fuzzy co-pre-Hilbert space then: $\mathcal{F}(x, y, ts) \leq \mathcal{F}(x, x, t^2) \circ \mathcal{F}(y, y, s^2)$.

$$\text{Proof: } \mathcal{F}(x, y, ts) = 1 - F(x, y, ts) \leq 1 - F(x, x, t^2) * F(y, y, s^2)$$

$$= 1 - (1 - \mathcal{F}(x, x, t^2)) * (1 - \mathcal{F}(y, y, s^2)) = \mathcal{F}(x, x, t^2) \circ \mathcal{F}(y, y, s^2).$$

The orthogonality:

Definition (3.6): [3] Let $(X, F, *)$ be a fuzzy pre-Hilbert space. A subset A of X is called fuzzy orthogonal if $x \perp y$, for each $x, y \in A$ (i.e. $A^\perp = \{x \in X: x \perp y \forall y \in A\}$).

Definition (3.7): [3] Let $(X, F, *)$ be a fuzzy pre-Hilbert space. $x, y \in X$ is said to be fuzzy orthogonal if $\mathcal{F}(x, y, t) = \mathcal{H}(t) \forall t \in R$ and it is denoted by $x \perp y$.

Theorem (3.8): Let $(X, F, *)$ be a fuzzy pre-Hilbert space. And $A \subset X$;

(1) The relation of orthogonality is symmetric (i.e. if $x \perp y$ then $y \perp x$).

(2) If $x \perp y$ then $\alpha x \perp y \forall \alpha \in R$.

(3) Let $A \subset B$ then $B^\perp \subset A^\perp$.

(4) $A \subset A^{\perp\perp}$.

(5) Let $A \subset B^\perp \Leftrightarrow B \subset A^\perp$.

(6) If $x \perp x \Leftrightarrow x = 0 \forall t \in R$.

(7) $X^\perp = \{0\} \forall t \in R$.

(8) $A \cap A^\perp = \{0\} \forall t \in R$.

(9) The zero vector is orthogonal to every vector.

Proof: (1) If $x \perp y$ then by definition (3.7)

$$\begin{aligned} F(x, y, t) &= H(t) \forall t \in R, \text{ but} \\ F(x, y, t) &= F(y, x, t) \text{ then} \\ F(y, x, t) &= H(t) \forall t \in R \Rightarrow \\ &y \perp x. \end{aligned}$$

(2) If $x \perp y$ then $F(x, y, t) = H(t) \forall t \in R, x, y \in X$.

$$\begin{aligned} F(\alpha x, y, t) &= \begin{cases} F\left(x, y, \frac{t}{\alpha}\right) & , \alpha > 0 \\ H(t) & , \alpha = 0 \\ 1 - F\left(x, y, \frac{t}{-\alpha}\right) & , \alpha < 0 \end{cases} \\ &= \begin{cases} H\left(\frac{t}{\alpha}\right) \\ H(t) \\ 1 - H\left(\frac{t}{-\alpha}\right) \end{cases} = H(t) \\ &\Rightarrow \alpha x \perp y \end{aligned}$$

(3) Let $x \in B^\perp \Rightarrow x \perp y \forall y \in B$, since $A \subset B \Rightarrow x \perp y \forall y \in A$

$$x \in A^\perp \Rightarrow B^\perp \subset A^\perp.$$

(4) Let $x \in A$ then $F(x, y, t) = H(t) \forall y \in A^\perp \Rightarrow x \in A^{\perp\perp} \Rightarrow A \subset A^{\perp\perp}$.

(5) Let $A \subset B^\perp \Rightarrow B^{\perp\perp} \subset A^\perp$

$$A^\perp (\text{by 3}) \Rightarrow \text{from (4)} B \subset B^{\perp\perp} \subset A^\perp \Rightarrow B \subset A^\perp.$$

(6) Let $x \in X$ and $x \perp x$ then from definition (3.7) $F(x, x, t) = H(t) \forall t \in R \Rightarrow x = 0 \forall t \in R$ (by axiom2).

Let $x = 0 \Rightarrow F(x, x, t) = F(0, 0, t) = H(t) \forall t \in R$ (by axiom 4) then $x \perp x \forall t \in R$ (by definition 3.7)

(7) Let $x \in X^\perp \Rightarrow F(x, y, t) = H(t) \forall t \in R$ and $\forall y \in X$.

$$\begin{aligned} \text{Since } x \in X \text{ then } F(x, x, t) &= H(t) \\ \Rightarrow x &= 0 \text{ for all } t \in R \Rightarrow X^\perp \subseteq \{0\} \text{ for all } t \in R \end{aligned}$$

$$\text{Since } 0 \in X^\perp \Rightarrow \{0\} \subseteq X^\perp \Rightarrow X^\perp = \{0\} \text{ for all } t \in R.$$

(8) Let $x \in A \cap A^\perp \Rightarrow x \in A$ and $x \in A^\perp \Rightarrow F(x, x, t) = H(t) \forall t \in R$

$$\begin{aligned} \Rightarrow x &\perp x \forall x \in A \Rightarrow x = 0 \Rightarrow x \in \{0\} \\ \Rightarrow A \cap A^\perp &\subseteq \{0\} \text{ for all } t \in R. \end{aligned}$$

$$\begin{aligned} 0 \in A \text{ and } 0 \perp 0 &\Rightarrow 0 \in A^\perp \Rightarrow 0 \in A \cap A^\perp. \text{ Then } \{0\} \subseteq A \cap A^\perp \Rightarrow A \cap A^\perp = \{0\} \text{ for all } t \in R. \end{aligned}$$

(9) For every vector $x \in X$, we have $F(0, x, t) = H(t) \forall t \in R$

$$\text{Therefore } 0 \perp x \forall x \in X.$$

Theorem (3.9): (parallelogram law) Let $(X, F, *)$ is a fuzzy pre-Hilbert space then, $F(x + y, x + y, 4t^2) * F(x - y, x - y, 4s^2) \leq F(x, x, (t + s)^2)$.

Proof: $F(x + y, x + y, 4t^2) * F(x - y, x - y, 4s^2) \leq F(2x, 2x, 4t^2 + 4s^2) = F(2x, 2x, 4t^2 + 4s^2) * 1 = F(x, x, t^2 + s^2) * F(0, 0, 2ts) \leq F(x, x, (t + s)^2)$.

Theorem : (3.10) Let A be a non-empty subset of a fuzzy pre-Hilbert space X . Then $A^\perp$ is closed fuzzy subspace of X .

Proof: Since $F(0, x, t) = H(t) \forall x \in A \Rightarrow 0 \in A^\perp$ then $A^\perp \neq \emptyset$. Let $x, y \in A^\perp$ and $\alpha, \beta, r \in$

$$RF(x, z, r) = H(r) \quad \forall z \in A \text{ and } F(y, z, r) = H(r) \quad \forall z \in A$$

For every $z \in A$ we have: If $\alpha > 0, \beta > 0$

$$F(\alpha x + \beta y, z, r) = \sup_{t+s=r} (F(x, z, \frac{t}{\alpha}) * F(y, z, \frac{s}{\beta})) = H(\frac{t}{\alpha}) * H(\frac{s}{\beta}) = H(t) * H(s) = H(r) \quad \forall r \in R.$$

If $\alpha < 0, \beta < 0$

$$F(\alpha x + \beta y, z, r) = \sup_{t+s=r} \left(\left(1 - F(x, z, \frac{t}{-\alpha}) \right) * \left(1 - F(y, z, \frac{s}{-\beta}) \right) \right) = H(r)$$

If $\alpha = 0, \beta = 0$

$$F(\alpha x + \beta y, z, r) = F(0, z, r) = H(r) \quad \forall r \in R.$$

If $\alpha < 0, \beta = 0$ or $\alpha = 0, \beta < 0$ or $\alpha > 0, \beta = 0$ or $\alpha = 0, \beta > 0$

$$F(\alpha x + \beta y, z, r) = H(r) \Rightarrow \alpha x + \beta y \in A^\perp \Rightarrow A^\perp \text{ is a fuzzy subspace.}$$

Let $x \in \overline{A^\perp} \exists \{x_n\}$ in $A^\perp$ such that $x_n \rightarrow x$

Let $y \in A \Rightarrow F(x_n, y, t) = H(t) \quad \forall n \in \mathbb{Z}^+$ and $t \in R$ ($x_n \in A^\perp \forall n \in \mathbb{Z}^+$).

Since $x_n \rightarrow x \Rightarrow F(x_n, y, t) \rightarrow F(x, y, t) = H(t)$ for all $y \in A$.

$\Rightarrow x \in A^\perp \Rightarrow \overline{A^\perp} = A^\perp \Rightarrow A^\perp$ is closed fuzzy subset of X .

Definition (3.11): Let $(X, \mathcal{F}, \circ)$ be a fuzzy co-pre-Hilbert space. A subset A of X is called fuzzy orthogonal if $x \perp y$, for each $x, y \in A$.

Definition (3.12): Let $(X, \mathcal{F}, \circ)$ be a fuzzy co-pre-Hilbert space. $x, y \in X$ is said to be fuzzy orthogonal if $\mathcal{F}(x, y, t) = \mathcal{H}(t) \quad \forall t \in R$ and it is denoted by $x \perp y$.

Theorem (3.13): Let $(X, \mathcal{F}, \circ)$ be a fuzzy co-pre-Hilbert space. And $A \subset X$;

(1) The relation of orthogonally is symmetric (i.e. if $x \perp y$ then $y \perp x$).

(2) If $x \perp y$ then $\alpha x \perp y \quad \forall \alpha \in R$.

(3) Let $A \subset B$ then $B^\perp \subset A^\perp$.

(4) $A \subset A^{\perp\perp}$.

(5) Let $A \subset B^\perp \Leftrightarrow B \subset A^\perp$.

(6) If $x \perp x \Leftrightarrow x = 0 \quad \forall t \in R$.

(7) $X^\perp = \{0\} \quad \forall t \in R$.

(8) $A \cap A^\perp = \{0\} \quad \forall t \in R$.

(9) The zero vector is orthogonal to every vector.

Proof: (1) If $x \perp y$ then by definition (3.13)

$$\mathcal{F}(x, y, t) = \mathcal{H}(t) \quad \forall t \in R \text{ but}$$

$$\mathcal{F}(x, y, t) = \mathcal{F}(y, x, t) \text{ then } \mathcal{F}(y, x, t) = \mathcal{H}(t) \quad \forall t \in R \Rightarrow y \perp x.$$

(2) If $x \perp y$ then $\mathcal{F}(x, y, t) = \mathcal{H}(t) \quad \forall t \in R, x, y \in X$.

$$\mathcal{F}(\alpha x, y, t) = \begin{cases} \mathcal{F}(x, y, \frac{t}{\alpha}), & \alpha > 0 \\ \mathcal{H}(t), & \alpha = 0 \\ 1 - \mathcal{F}(x, y, \frac{t}{-\alpha}), & \alpha < 0 \end{cases}$$

$$= \begin{cases} \mathcal{H}(t) \\ \mathcal{H}(t) \\ 1 - \mathcal{H}(\frac{t}{-\alpha}) \end{cases} = \mathcal{H}(t)$$

(3) Let $x \in B^\perp \Rightarrow x \perp y \quad \forall y \in B$, since $A \subset B \Rightarrow x \perp y \quad \forall y \in A \Rightarrow x \in A^\perp \Rightarrow B^\perp \subset A^\perp$.

(3) Let $x \in A \Rightarrow \mathcal{F}(x, y, t) = \mathcal{H}(t) \forall y \in A^\perp \Rightarrow x \in A^{\perp\perp} \Rightarrow A \subset A^{\perp\perp}$.

(4) Let $A \subset B^\perp \Rightarrow B^{\perp\perp} \subset A^\perp$
 $A^\perp$ (by 3) $\Rightarrow$ from (4) $B \subset B^{\perp\perp} \subset A^\perp \Rightarrow B \subset A^\perp$.

(5) Let $x \in X$ and $x \perp x$ then from definition (3.13) $\mathcal{F}(x, x, t) = \mathcal{H}(t) \forall t \in R \Rightarrow x = 0 \forall t \in R$ (by axiom2).

Let $x = 0 \Rightarrow \mathcal{F}(x, x, t) = \mathcal{F}(0, 0, t) = \mathcal{H}(t) \forall t \in R$ (by axiom 4) then $x \perp x \forall t \in R$ (by definition 3.13)

(6) Let $x \in X^\perp \Rightarrow \mathcal{F}(x, y, t) = \mathcal{H}(t) \forall t \in R$ and $\forall y \in X$.

Since $x \in X$ then $\mathcal{F}(x, x, t) = \mathcal{H}(t) \Rightarrow x = 0$ for all $t \in R \Rightarrow X^\perp \subseteq \{0\}$ for all $t \in R$. Since $0 \in X^\perp \Rightarrow \{0\} \subseteq X^\perp \Rightarrow X^\perp = \{0\}$ for all $t \in R$.

(7) Let $x \in A \cap A^\perp \Rightarrow x \in A$ and $x \in A^\perp \Rightarrow \mathcal{F}(x, x, t) = \mathcal{H}(t) \forall t \in R$

$\Rightarrow x \perp x \forall x \in A \Rightarrow x = 0 \Rightarrow x \in \{0\} \Rightarrow A \cap A^\perp \subseteq \{0\}$ for all $t \in R$. $0 \in A$ and $0 \perp 0 \Rightarrow 0 \in A^\perp \Rightarrow 0 \in A \cap A^\perp$. Then $\{0\} \subseteq A \cap A^\perp \Rightarrow A \cap A^\perp = \{0\}$ for all $t \in R$.

(8) For every vector $x \in X$, we have $\mathcal{F}(0, x, t) = \mathcal{H}(t) \forall t \in R$. Therefore $0 \perp x \forall x \in X$.

Theorem (3.14): (parallelogram law) Let $(X, \mathcal{F}, \circ)$ be a fuzzy co-pre-Hilbert space then:

$$\mathcal{F}(x + y, x + y, 4t^2) \circ \mathcal{F}(x - y, x - y, 4s^2) \geq \mathcal{F}(x, x, (t + s)^2)$$

Proof: $\mathcal{F}(x + y, x + y, 4t^2) \circ \mathcal{F}(x - y, x - y, 4s^2) \geq \mathcal{F}(2x, 2x, 4t^2 + 4s^2) =$

$$\mathcal{F}(2x, 2x, 4t^2 + 4s^2) \circ 0 = \mathcal{F}(x, x, t^2 + s^2) \circ \mathcal{F}(0, 0, 2ts) \geq \mathcal{F}(x, x, (t + s)^2).$$

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