Some Results on Smarandache Semigroups

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Abstract: We discusse in this paper *a Smarandache semigroups* , *a Smarandache cyclic semigroups* and *a Smarandache lagrange semigroups*.We prove that the Smarandache semigroup Z_{2P} with multiplication modulo 2P where p is an odd prime have two subgroups of order P-1and we prove that Z_{2P} is a S-cyclic semigroup and *Smarandache weakly Lagrange semigruop* and we prove that some results about *a* Smarandache inverse in groups and a Smarandache conjugate in groups .

1.Introduction :

Padilla Raul introduced the notion of *Smarandache semigroups*[1], in the year 1998 in the paper entitled Smarandache Algebraic Structures .Since groups are prefect structures under a single closed associative binary operation ,it have become infeasible to define Smarandache groups . Smarandache semigroups are the analog in the Smarandache ideologies of the groups where Smarandache semigroup is defined to be the semigroup A such that a proper subset of A is a group (with respect to the same binary operation).

In this paper we prove some results in *a Smarandache lagrange semigroups*[1],[2], *a Smarandache cyclic semigroups*,[2] , *a* Smarandache inverse in groups and a Smarandache conjugate in groups,[2], we use Mathlab programming to check our results in this paper .

2.Definitions and Notations:

Definition 2.1:*The Smarandache semigroup* (S-semigroup) is defined to be a semigroup A such that a proper subset of A is a group (with respect to the same binary operation),[2].

Example 2.2: Let $z_{14}=\{0,1, \dots ,13\}$ be the semigroup under multiplication module 14.Clearly the set $A=\{1,13\} \subset z_{14}$ is a group under multiplication modulo 14,so z_{14} is the Smarandache semigroup,[2] .

Definition2.3: Let S be a S-semigroup if every subset A of S which a subgroup is cyclic then we say S is a *Smarandache cyclic* ,[2].

Definition 2.4: Let S be a S-semigroup if there exist at least a proper subset A of S which is a cyclic subgroup under the same operation of S then we say S is a *Smarandache weakly cyclic,* [2] .

Definition 2.5: Let G be a group . An element $x \in G\backslash\{1\}$ is said to have *a Smarandache inverse* y in G if xy=1and for $a,b \in G\backslash\{1,x,y\}$,we have xa=y (or ax=y), yb=x (or by=x) with ab=1 ,[2].

Definition *2.6:* Let G be a group . An element $x \in G$; x is said to have *a Smarandache conjugate* y in G if

1- x is conjugate to y (that is there exist $a \in G$ such that $x=aya^{-1}$).
2- a is conjugate with x and a is conjugate with y ,[2].

Definition 2.7: Let S be a finite S-semigroup .We say S is *a Smarandache non-Lagrange semigroup* if the order of none of the subgroups of S divides the order of the S-semigroup,[2].

Definition 2.8: Let S be a finite S-semigroup.If the order of every subgroups of S divides the order of the S-semigroup S then we say S is *a Smarandache Lagrange semigroup* ,[2].

Definition 2.9: Let S be a finite S-semigroup.If there exist at least one subgroup A that is a proper subset ($A \subset S$) having the same operation of S whose order divides the order of S then we say that S is *a Smarandache weakly Lagrange semigroup* ,[2].

Definition 2.10: Let $(G,*)$ be a finite group and p a prime .A subgroup $(P,*)$ of $(G,*)$ is said to be a Sylow p-subgroup if $(P,*)$ is a p-group and is not properly contained in any other p-subgroup of $(G,*)$ for the same prime number p,[3].

Definition 2.11: Two integers a and b, not both of which are zero, are said to be *relatively prime* whenever g.c.d.(a,b)=1,[4],[5].

Theorem 2.12: The linear congruence $ax \equiv b$ mod n have a solution if and only if d\b ,where d=g.c.d.(a,n). If d\b , then it have d mutually incongruent solutions modulo n, [4].

Theorem 2.13: Let a,b,n $\in Z$,n>0.If g.c.d.(a,n)=1 then the congruence $az \equiv b$ mod n have a unique solution z; moreover , any integer z' is a solution iff $z \equiv z'$ mod n,[6] .
Theorem 2.14:If n is a positive integer then $ax \equiv 1 \bmod n$ have a unique solution iff g.c.d.(a,n)=1,[6],[7].
Theorem (**Sylow*****)2.15:*** Let $(G,*)$ be a finite group of order $p^k q$,where p is a prime not dividing q Then $(G,*)$ have a Sylow p-subgroup of order p^k ,[3],[7],[8].
Theorem 2.16: Let G be a finite group ,H a p-subgroup of G, if H is normal subgroup then H contains in each Sylow p- subgroup in G,[3] .
Theorem2.17:let (G,*) be a group of order pq where p and q prime numbers ,p<q, if $q \not\equiv 1$mod p then (G,*) cyclic,[3],[9] .
Theorem 2.18: If G is an abelian group of order n having at most one cyclic for each prime divisor P of n then G is cyclic ,[10].

3. The Main Results

Proposition 3.1: let Z_{2P} be the semigroup under the multiplication modulo 2P ,P is an odd prime then Z_{2P} have two subsets of order P-1where they are subgroups of it under the same operation .

Proof: Let $(Z_{2P}, \cdot_{2P})$ be the semigroup under multiplication modulo 2P, P is an odd prime ,
the order of Z_{2P} is 2p ,now ,to compuse a subset of Z_{2P} without zero element in it we must cancel zero element and P because that 0**.**x=x**.**0=0 and P**.**x=x**.**P=0 $\forall x \in Z_{2P}$,[4],
so we have all odd and even number in it except 0 and P, now ,we have two subsets ,H and K where H={2n: $n \in Z$ and 0<n$\le$p-1 ,n$\ne$p}
= {2,4, … , 2P-2}
and K={2n+1: $n \in Z$ and $0 \le n \le$P-1 ,2n+1$\ne$P}
= {1,3, ... , 2p-1}.
To prove that $(H, \cdot_{2P})$ is a group ,
If $x,y \in H \subseteq Z_{2P}$ then $xy \in Z_{2P}$,
since x,y even so xy is even and by use the multiplication mod 2P which is even ,since H contains each element of Z_{2P} which is even so $xy \in H$.
It is clear that H is associative under multiplication mod 2P,
now,let $x, 1+P \in H$ where P is an odd prime so x**.**(1+p)=x+xp mod 2p =x , (1+p) .x=x mod 2p since x is even,
so 1+p is the identity element of H.
Now let $x,y \in H$ then $xy \equiv yx \equiv 1+p$ mod 2p have two incongruent solution ,since g.c.d(y,2p)=2,
let the first is x_0 so the other is $x_0 + \frac{2p}{2}$ =x_0+p , ,[4],since p is an odd prime so just one of them must be even ,so just one of them belong to H, therefore H contains one solution of this equation so H have an inverse for each element of it , so $(H, \cdot_{2P})$ is a group.
Now to prove that $(K, \cdot_{2P})$ is a group,
If $x,y \in K \subseteq Z_{2P}$ then $xy \in Z_{2P}$, since x,y odd so xy is odd and by use the multiplication mod 2P xy still odd and since K contains each element of Z_{2P} except p which is odd so $xy \in K$,it is clear that K is associative under multiplication mod 2P and the identity element is 1 ,
now, let x,y $\in K \subseteq Z_{2P}$ then $xy \equiv yx \equiv 1$ mod 2p have a unique solution,since g.c.d.(y,2p)=1,[11], so x have y as inverse so $(K, \cdot_{2P})$ is a group ,it is clear that the order of H =o(Z_{2P})-{odd number}-{0}=2p-$\frac{2p}{2}$-1=p-1,and
the order of K =o(Z_{2P})-{even number}-{p}=2p-$\frac{2p}{2}$-1=p-1.

Remark:
It important to note that any other subgroup of Z_{2P} must be a subset of H or K, since if there is a subset M of Z_{2P} contains element from H and K so there are several cases:
Case 1:If $1 \notin M$ and $x \in K \cap M$ so x have no identity element so M is not a group .
Case 2: if 1 ,$1+p \in M$ and $x \in H \cap M$ so x have two identity element 1 and 1+p so M is not a group .
Case 3: if $1 \in M$ and $1+p \notin M$ and $x,y \in$ M such that $x \in H$ and $y \in$ K then since H is even so xy=z for some $z \in H$,then $y=x^{-1}z$ is even number but $y \in K$ so we have contradiction .

Proposition 3.2: let Z_{2P} be the semigroup under the multiplication modulo 2P ,P is an odd prime then Z_{2P} is a cyclic Smarandache semigroup if p-1$\ne 2^{\alpha}$.

Proof: Let $(Z_{2P}, \cdot_{2P})$ be the semigroup under multiplication modulo 2P, P is an odd prime , and p-1$\ne 2^{\alpha}$.

By remark above there is no subgroup of Z_{2P} not contains in the subgroup of H or K , so we must to prove that H and K cyclic then use the fact that each subgroup of cyclic subgroup is a cyclic subgroup,[3].
First we prove that H is cyclic ,
we know that each $x \in Z$ can be written as a product of prime number , [5],[7],so
$x = p_1^{\alpha_1} p_2^{\alpha_2} ... p_L^{\alpha_L}$ where p_i i=1 ,… ,L , is a prime number , then
O(H)= $2^{\alpha_1} p_2^{\alpha_2} ... p_L^{\alpha_L}$ where o(H) is an even number , then there are two cases:
Case 1: If o(H)= $2^{\alpha_1} p_1^{\alpha_2}$ where $p_1 \geq 3$,
Since $p_1 \geq 3$ and $2 \nmid p_1$ mod 2p ,so H is cyclic,[8],[9].

Case 2: if o(H)= $2^{\alpha_1} p_2^{\alpha_2} ... p_L^{\alpha_L}$ where each $p_i \neq 2$, since H is commutative so each subgroup of it must be normal subgroup ,[8], then by sylow theorem for each p_i , H contains a unique sylow p_i –subgroup (T_i) of order p_i for each i ,
since it is normal and the order of it is prime so it is cyclic.
Now we want to prove that this group is unique for each p_i , suppose that there exist cyclic subgroup of order p_i, say M since it is normal so it is contains in each Sylow p_i -subgroup in Z_{2P},[3], since T_i is Sylow p_i -subgroup so
$M \subseteq T_i$ so $o(m) \leq o(T_i)$
but o(M)= p_i so M= T_i ,therefore this group is unique for each p_i ,so for every p_i divisor of order H there exist a unique cyclic subgroup have order p_i so H is a cyclic subgroup,,[3],[8],
by use the similar way we can prove that K is a cyclic subgroup ,
so Z_{2P} is a Smarandache cyclic semigroup.

Example: let $(z_{14} , \cdot_{14})$ be the semigroup under the multiplication modulo 14,
since 14 =2p where p=7 so by above proposition there are two subsets of it of order p-1=6,
H={2,4,6,8,10,12} and K={1,3,5,9,11,13},the tables of them are given by

$\cdot_{14}$	1	3	5	9	11	13
1	1	3	5	9	11	13
3	3	9	1	13	5	11
5	5	1	11	3	13	9
9	9	13	3	11	1	5
11	11	5	13	1	9	3
13	13	11	9	5	3	1

$\cdot_{14}$	2	4	6	8	10	12
2	4	8	12	2	6	10
4	8	2	10	4	12	6
6	12	10	8	6	4	2
8	2	4	6	8	10	12
10	6	12	4	10	2	8
12	10	6	2	12	8	4

Note that H is a cyclic subgroup generated by 10 and 12 ,i.e. H= <10> = <12>, and 1+p=8 is the identity and K is a cyclic subgroup generated by K= <3> = <5> .

Proposition 3.2: let Z_{2P} be the semigroup under the multiplication modulo 2P ,P is an odd prime then Z_{2P} is a Smarandache weakly lagrange semigroup if $p-1 \neq 2^{\alpha}$.

Proof: $(Z_{2P}, \cdot_{2P})$ be the semigroup under multiplication modulo 2P, P is an odd prime and $p-1 \neq 2^{\alpha}$, by the proposition above there exist two subgroups of Z_{2P} , H and K with order p-1 ,
$o(Z_{2P})=2p$,H and K with order = $p-1 \neq 2^{\alpha}$,
since 2p did not divisible by p-1 so Z_{2P} did not a Smarandache lagrange semigroup but it is weakly Lagrange semigroup since o(H) is even and for each prime q divisor of o(H) there exist a subgroup of order q and since 2 divided o(H) so there exist a subgroup M of H of order 2 ,since $M \subset H \subset Z_{2P}$ so $M \subset Z_{2P}$ and $\frac{o(z_{2p})}{o(M)} = \frac{2p}{2} = p$,so Z_{2P} is Smarandache weakly lagrange semigroup.

Proposition 3.3:Let (G,*) be a group and $x \in G$,if x have Smarandache inverse ,then in general need not have Smarandache conjugate .

Proof: we prove this by an example .Let
G=<a> =<1 2 3 4> be a group with usual composition of mappings , such that (1 2 3 4) $\in S(4)$ which is the semigroup of all mappings from x={1,2, 3,4} to x={1,2,3,4} under the usual composition of mappings .
It is clear that $a^4=I$ where I=(1) and the order of G =4,note that $aa^3=I$ and $a^2 \in G \backslash \{1,a, a^3\}$ such that $a^2a= a^3$ and $a^3 a^2=a$,so a have a Smarandache inverse but there is no $x \in G$ such that x conjugate to a so a have not Smarandache conjugate

Proposition 3.4:Let (G,*) be a group and $x \in G$,if x have Smarandache conjugate ,then in general need not have Smarandache inverse.
Proof: we prove this by an example. Let

$$S_3 = \{I = \begin{pmatrix} 123 \\ 123 \end{pmatrix}, p_1 = \begin{pmatrix} 123 \\ 132 \end{pmatrix}, p_2 = \begin{pmatrix} 123 \\ 321 \end{pmatrix},$$

$$p_3 = \begin{pmatrix} 123 \\ 213 \end{pmatrix}, p_4 = \begin{pmatrix} 123 \\ 231 \end{pmatrix}, p_5 = \begin{pmatrix} 123 \\ 312 \end{pmatrix}\}$$

be the symmetric group of the degree 3 .Now we have p_1 conjugate with p_2 since

$$\begin{pmatrix} 123 \\ 132 \end{pmatrix} = \begin{pmatrix} 123 \\ 213 \end{pmatrix}\begin{pmatrix} 123 \\ 321 \end{pmatrix}\begin{pmatrix} 123 \\ 213 \end{pmatrix},$$

so p_1 conjugate with p_2 by p_3.
And since

$$\begin{pmatrix} 123 \\ 213 \end{pmatrix} = \begin{pmatrix} 123 \\ 312 \end{pmatrix}\begin{pmatrix} 123 \\ 132 \end{pmatrix}\begin{pmatrix} 123 \\ 231 \end{pmatrix},$$ p_3 conjugate with $p_1$by p_5, and

$$\begin{pmatrix} 123 \\ 213 \end{pmatrix} = \begin{pmatrix} 123 \\ 132 \end{pmatrix}\begin{pmatrix} 123 \\ 321 \end{pmatrix}\begin{pmatrix} 123 \\ 132 \end{pmatrix},$$ p_3 conjugate with $p_2$by p_1, so p_1 Smarandache conjugate with p_2,but p_1 did not have Smarandache inverse ,where p_1 p_1=I but there is no $x,y \in S_3 \backslash \{I, p_1\}$ such that $x\,p_1 = p_1$ and $y\,p_1 = p_1$ where xy=I.

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بعض النتائج حول أشباه زمر سماراندش

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تناولنا في هذا البحث أشباه زمر سماركداش , أشباه زمر سمارندانش الدوارة وأشباه زمر سماراندش لاكرانج .أثبتنا أن أشباه زمر سماراندش Z_{2P} معيار 2P حيث P عدد أولي فردي تحوي زمرتين جزئيتين من الرتبة P-1 وبرهنا انها شبه زمرة سماراندش دوارة وكذلك شبه زمرة لاكرانج بضعف كما برهنا بعض النتائج المتعلقة بنظائر سماراندش في الزمر وترافق سماراندش في الزمر.