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Solving Parabolic Partial Delay Differential Equations Using The Explicit Method And Higher Order Differences

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Abstract

In this paper we use the higher order differences for second order (derivative) in solving parabolic partial delay differential equations by using the explicit method and we get results are more closer to the exact values than the results which can be obtained if the familiar second order (derivative) form is used. Finally ,we make a comparison using Matlab between the results through two tables of values form [3] which results from Taylor series expansions of a function centered on the grid point (x_i, t_j) . In this paper, we use the higher order differences for second order instead of the familiar form for solving the parabolic delay differential equations .

1. Introduction

Delay differential equations play an important role in many life applications , say problems of mixing of liquids , population growth, control systems, mechanical and electrical systems. In addition, many physical and engineering problems can be modeled mathematically in the form of partial delay differential equations (PDDE's) . Interest in equations of this type (see for example [1] and [2]) have continued to grow as it has become apparent that they are also of importance in area of biomedical modeling especially physiological and hormonal control systems .

It is well known that PDDE's are differential equations in which the unknown function depends on two or more independent variables and its partial derivatives , with several values of the delay .

Most researches discuss the methods for solving partial delay differential equations of second order by using the familiar second order

2. Classification of Second Order PDDE's

Consider the following partial delay differential equation

$$a\frac{\partial^2}{\partial x^2}u(x,t)+b\frac{\partial^2}{\partial x\partial t}u(x,t)+c\frac{\partial^2}{\partial t^2}u(x,t) + d\frac{\partial}{\partial x}u(x,t)+ e\frac{\partial}{\partial t}u(x,t)+ f\,u(x,t) + hu(x-T_1,t-T_2)=g(x,t) \qquad (2.1)$$

where a,b,c,d,e,f,g,h,T_1 and T_2 are known functions of x and t. The classification of second order partial delay differential equation (2.1) is similar to that in partial differential equations ,

1. If $b^2-4ac>0$, then equation (2.1) is said to be of the hyperbolic type.

2. If $b^2-4ac=0$, then equation (2.1) is said to be of the parabolic type.

3. $b^2-4ac<0$, then equation (2.1) is said to be of the elliptic type.

3. Explicit Method for Solving PDDE's

The explicit method [4],[5] and [6] is one of the numerical methods that use for solving partial differential equations .In this section , we shall use this method in solving the heat equation with constant delay. This equation takes the form

$$u_t(x,t)=\alpha^2 u_{xx}(x,t)+u(x,t-T),$$
$$0\le x\le L, t\ge 0 \tag{3.1}$$

subject to the boundary conditions

$$u(0,t)=0, \quad u(L,t)=0, \qquad t\ge 0 \tag{3.2}$$

and initial condition

$$u(x,0)=f(x), \quad 0\le x\le L. \tag{3.3}$$

Also, initial delay condition is given by

$$u(x,t)=\varphi(x,t), 0\le x\le L, -T\le t\le 0 \tag{3.4}$$

where α is a constant and $T>0$ is the delay term .

To approximate the solution of this problem, we first select two mesh constants h and k, with $m=\frac{L}{h}$ is an integer number .The grid points are (x_i,t_j) where $x_i=ih,\ i=0,1,2,...m,$ and $t_j=jk,\ j=0,1,2,...$

The heat equation (3.1) can be written at the interior grid point (x_i,t_j) , $i=1,2,...m\text{-}1$ and $j=1,2,...$ as

$$u_t(x_i,t_j)=\alpha^2 u_{xx}(x_i,t_j)+u(x_i,t_j-T). \tag{3.5}$$

Derivation of the difference equation in our method is obtained by using the forward difference quotient of $u_t(x_i,t_j)$

$$u_t(x_i,t_j)=\frac{1}{k}[\ u(x_i,t_{j+1})-u(x_i,t_j)\ ]-$$
$$\frac{k}{2}\frac{\partial^2}{\partial t^2}u(x_i,\eta_j) \tag{3.6}$$

such that $\eta_j\in(t_{j-1},t_{j+1})$ and the difference quotient of the fourth order formula of $u_{xx}(x_i,t_j)$

$$u_{xx}(x_i,t_j)=\frac{1}{12h^2}[-u(x_{i-2},t_j)+16u(x_{i-1},t_j)$$
$$-\ 30\,u(x_i,t_j)\ +\ 16\,u(x_{i+1},t_j)-u(x_{i+2},t_j)]-$$
$$\frac{h^2}{12}\frac{\partial^4}{\partial x^4}u(\varepsilon_i,t_j) \tag{3.7}$$

such that $\varepsilon_i\in(x_{i-1},x_{i+1})$. Applying (3.6) and (3.7) in equation (3.5) gives

$$\frac{1}{k}[u_{i,j+1}-u_{i,j}]=\frac{\alpha^2}{12h^2}[-u_{i-2,j}+16u_{i-1,j}-$$
$$30u_{i,j}+16u_{i+1,j}-u_{i+2,j}]+\varphi(x_i,t_j-T)+$$
$$\frac{k}{2}\frac{\partial^2}{\partial t^2}u(x_i,\eta_j)-\frac{\alpha^2h^2}{12}\frac{\partial^4}{\partial x^4}u(\varepsilon_i,t_j). \tag{3.8}$$

Solving equation (3.8) for $u_{i,j+1}$ gives

$$u_{i,j+1}=u_{i,j}+\frac{k\alpha^2}{12h^2}[-u_{i-2,j}+16u_{i-1,j}-30u_{i,j}$$
$$+\ 16\,u_{i+1,j}-\ \ u_{i+2,j}\ ]\ +k\varphi(x_i,t_j-T)\ +\frac{k^2}{2}.$$
$$\frac{\partial^2}{\partial t^2}u(x_i,\eta_j)-\frac{k\alpha^2h^2}{12}\frac{\partial^4}{\partial x^4}u(\varepsilon_i,t_j). \tag{3.9}$$

If $\lambda=\frac{k\alpha^2}{h^2}$ we have

$$u_{i,j+1}=(1-\frac{5\lambda}{2})u_{i,j}-\frac{\lambda}{12}u_{i-2,j}+\frac{4\lambda}{3}u_{i-1,j}+$$
$$\frac{4\lambda}{3}u_{i+1,j}-\frac{\lambda}{12}u_{i+2,j}+k\,\varphi(x_i,t_j-T)\ +\ \frac{k^2}{2}.$$

$$\frac{\partial^2}{\partial t^2}u(x_i,\eta_j)-\frac{k\alpha^2h^2}{12}\frac{\partial^4}{\partial x^4}u(\varepsilon_i,t_j), \quad (3.10)$$

where $i=2,3,\ldots,m-2$, $j=1,2,\ldots$. If we neglect the error term

$$T_{i,j}=\frac{k^2}{2}\frac{\partial^2}{\partial t^2}u\left(x_i,\eta_j\right)-\frac{k\alpha^2h^2}{12}\frac{\partial^4}{\partial x^4}u\left(\varepsilon_i,t_j\right) \quad (3.11)$$

we get

$$w_{i,j+1}=(1-\frac{5\lambda}{2})w_{i,j}-\frac{\lambda}{12}w_{i-2,j}+\frac{4\lambda}{3}w_{i-1,j}+ \frac{4\lambda}{3}w_{i+1,j}-\frac{\lambda}{12}w_{i+2,j}+k\varphi(x_i,t_j-T) \quad (3.12)$$

where $w_{i,j}$ is the approximate solution of (3.1) at the grid point $\left(x_i,t_j\right)$, $i=2,3,\ldots,m-2$, $j=1,2,\ldots$.

Now, the initial condition $u(x,0)=f(x),\ 0\le x\le L,$ implies that $w_{i,0}=f(x_i),\ i=0,1,\ldots,m$ and the boundary conditions $u(0,t)=u(L,t)=0$ imply that $w_{0,j}=w_{m,j}=0,\ \ j=0,1,\ldots$. The explicit nature of the difference method that expressed in equation (3.12) can be written in terms of matrices form as

$$W^{(j+1)}=A\,\mathrm{W}^{(\mathrm{j})}+\mathrm{B},\ \mathrm{j}=0,1,2,\ldots \quad (3.13)$$

such that

$$W^{(j+1)}=\left[w_{2,j+1},w_{3,j+1},\ldots,w_{m-2,j+1}\right]^T \quad (3.14)$$

$$W^{(j)}=\left[w_{2,j},w_{3,j},\ldots,w_{m-2,j}\right]^T \quad (3.15)$$

$$W^{(0)}=\left[f\left(x_2\right),f\left(x_3\right),\ldots,f\left(x_{m-2}\right)\right]^T \quad (3.16)$$

$$A=\begin{bmatrix}
1-\frac{5\lambda}{2} & \frac{4\lambda}{3} & \frac{-\lambda}{12} & 0 & 0 & \cdots & 0 & 0\\
\frac{4\lambda}{3} & 1-\frac{5\lambda}{2} & \frac{4\lambda}{3} & \frac{-\lambda}{12} & 0 & \cdots & 0 & 0\\
\frac{-\lambda}{12} & \frac{4\lambda}{3} & 1-\frac{5\lambda}{2} & \frac{4\lambda}{3} & \frac{-\lambda}{12} & \cdots & 0 & 0\\
0 & \frac{-\lambda}{12} & \frac{4\lambda}{3} & 1-\frac{5\lambda}{2} & \frac{4\lambda}{3} & \cdots & 0 & 0\\
0 & 0 & \frac{-\lambda}{12} & \frac{4\lambda}{3} & 1-\frac{5\lambda}{2} & \ddots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & 1-\frac{5\lambda}{2} & \frac{4\lambda}{3} & \frac{-\lambda}{12}\\
0 & 0 & 0 & 0 & \frac{-\lambda}{12} & \frac{4\lambda}{3} & 1-\frac{5\lambda}{2} & \frac{4\lambda}{3}\\
0 & 0 & 0 & 0 & 0 & \frac{-\lambda}{12} & \frac{4\lambda}{3} & 1-\frac{5\lambda}{2}
\end{bmatrix}$$

$$B=[k\varphi(x_2,t_j-T)+\frac{4\lambda}{3}w_{1,j},k\varphi(x_3,t_j-T) -\frac{\lambda}{12}w_{1,j},k\varphi(x_4,t_j-T),\ldots,k\varphi(x_{m-4},t_j-T), k\varphi(x_{m-3},t_j-T)-\frac{\lambda}{12}w_{m-1,j},k\varphi(x_{m-2},t_j-T) +\frac{4\lambda}{3}w_{m-1,j}]^T, \quad (3.17)$$

It is clear that for finding the solution of (3.1) using equation (3.12), we must have $w_{1,j}$ and $w_{m-1,j}$, $j=0,1,2,\ldots$ . .Therefore we shall use the explicit method [3] without using higher order differences to find these values , as follows

$$W^{(j+1)}=A_1\,W^{(j)}+B_1,\ j=0,1,2,\ldots \quad (3.18)$$

such that

$$A_1=\begin{bmatrix}
1-2\lambda & \lambda & 0 & \cdots & 0 & 0\\
\lambda & 1-2\lambda & \lambda & \cdots & 0 & 0\\
0 & \lambda & 1-2\lambda & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \lambda & 0\\
0 & 0 & 0 & \lambda & 1-2\lambda & \lambda\\
0 & 0 & 0 & 0 & \lambda & 1-2\lambda
\end{bmatrix}$$

$$W^{(j+1)}=\left[w_{1,j+1},w_{2,j+1},\ldots,w_{m-1,j+1}\right]^T \quad (3.19)$$

$$W^{(j)}=\left[w_{1,j},w_{2,j},\ldots,w_{m-1,j}\right]^T \quad (3.20)$$

$$W^{(0)} = \left[f(x_1), f(x_2), \ldots, f(x_{m-1}) \right]^T, \qquad (3.21)$$

$$B_1 = [k\varphi(x_1, t_j - T), k\varphi(x_2, t_j - T), \ldots,$$
$$k\varphi(x_{m-1}, t_j - T)]^T, \qquad (3.22)$$

and $\lambda = \dfrac{k\alpha^2}{h^2}$.

Example 3.1. Consider the parabolic partial delay differential equation

$$\frac{\partial}{\partial t} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) + u(x, t-2),$$
$$0 \le x \le 1, \; t \ge 0 \qquad (3.23)$$

with the boundary and initial conditions

$$u(0,t) = u(1,t) = 0, \; t \ge 0, \qquad (3.24)$$

$$u(x,0) = x - x^2, \quad 0 \le x \le 1, \qquad (3.25)$$

and initial delay condition

$$\varphi(x,t) = (2 + x - x^2)e^{t+2}, \quad 0 \le x \le 1,$$
$$-2 \le t \le 0. \qquad (3.26)$$

The exact solution of this example has the form

$$u(x,t) = (x - x^2)e^t, \quad 0 \le x \le 1, \; t \ge 0. \qquad (3.27)$$

For this PDDE, if we suppose that $h = 0.1$ and $k = 0.001$ then we have

$$A_1 = \begin{bmatrix} \frac{8}{10} & \frac{1}{10} & 0 & \cdots & 0 & 0 \\ \frac{1}{10} & \frac{8}{10} & \frac{1}{10} & \cdots & 0 & 0 \\ 0 & \frac{1}{10} & \frac{8}{10} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \frac{1}{10} & 0 \\ 0 & 0 & 0 & \frac{1}{10} & \frac{8}{10} & \frac{1}{10} \\ 0 & 0 & 0 & 0 & \frac{1}{10} & \frac{8}{10} \end{bmatrix}$$

and

$$A = \begin{bmatrix} \frac{3}{4} & \frac{2}{15} & \frac{-1}{120} & 0 & 0 & \cdots & 0 & 0 \\ \frac{2}{15} & \frac{3}{4} & \frac{2}{15} & \frac{-1}{120} & 0 & \cdots & 0 & 0 \\ \frac{-1}{120} & \frac{2}{15} & \frac{3}{4} & \frac{2}{15} & \frac{-1}{120} & \cdots & 0 & 0 \\ 0 & \frac{-1}{120} & \frac{2}{15} & \frac{3}{4} & \frac{2}{15} & \cdots & 0 & 0 \\ 0 & 0 & \frac{-1}{120} & \frac{2}{15} & \frac{3}{4} & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \frac{3}{4} & \frac{2}{15} & \frac{-1}{120} \\ 0 & 0 & 0 & 0 & \frac{-1}{120} & \frac{2}{15} & \frac{3}{4} & \frac{2}{15} \\ 0 & 0 & 0 & 0 & 0 & \frac{-1}{120} & \frac{2}{15} & \frac{3}{4} \end{bmatrix}$$

First, we must find the values $w_{1,j}$ and $w_{m-1,j}$ when $j = 1, 2, \ldots, 99$ using the above matrix A_1 and the relations (3.18) and (3.20)-(3.22). Then we can use the matrix A and the relations (3.15) -(3.17) to find $w_{i,j}$ where $i = 2, 3, \ldots, m-2$ and $j = 1, 2, \ldots, 99$ given in table 3.1 . The numerical results when $h = 0.1$, $k = 0.001$ and its comparison with the exact solution of the equation (3.23) are given in table 3.1 such that the columns numerical 1[3] and numerical 2 represent the numerical solutions of the equation (3.23) using the explicit method without and with higher order differences respectively .

Similarly , we can find the numerical results given in table 3.2 when $h = 0.1$ and $k = 0.0001$

t	x	Exact	Step length $h = 0.1$, k=0.001 and α=1			
			Numerical 1	Error 1	Numerical 2	Error 2
0.1	0.1	0.0994654	0.09953231	0.00006692	0.09953231	0.00006691
0.1	0.2	0.1768274	0.17694232	0.00011497	0.17682216	0.00000524
0.1	0.3	0.2320859	0.23223291	0.00014702	0.23207882	0.00000708
0.1	0.4	0.265241	0.26540631	0.00016531	0.26523276	0.00000824
0.1	0.5	0.2762927	0.27646394	0.00017124	0.27628406	0.0000086
0.1	0.6	0.265241	0.26540631	0.00016531	0.26523276	0.00000824
0.1	0.7	0.2320859	0.23223291	0.00014702	0.23207882	0.00000708
0.1	0.8	0.1768274	0.17694232	0.00011497	0.17682216	0.00000524
0.1	0.9	0.0994654	0.09953231	0.00006692	0.09953231	0.00006691

Table 3.1

t	x	Exact	Step length $h = 0.1$, k=0.0001 and α=1			
			Numerical 1	Error 1	Numerical 2	Error 2
0.1	0.1	0.0994654	0.09947206	0.00000666	0.09947206	0.00000666
0.1	0.2	0.1768274	0.17683882	0.00001147	0.17682683	0.00000057
0.1	0.3	0.2320859	0.23210056	0.00001466	0.23208519	0.00000071
0.1	0.4	0.265241	0.2652575	0.00001648	0.26524020	0.0000008
0.1	0.5	0.2762927	0.2763098	0.00001707	0.27629186	0.00000084
0.1	0.6	0.265241	0.2652575	0.00001648	0.26524020	0.0000008
0.1	0.7	0.2320859	0. 23210056	0.00001466	0.23208519	0.00000071
0.1	0.8	0.1768274	0. 17683882	0.00001147	0.17682683	0.00000057
0.1	0.9	0.0994654	0.09947206	0.00000666	0.09947206	0.00000666

Table 3.2

It is clear form table 3.1 and table 3.2 that the results in the column numerical 2 is more closer to the exact values than the results in the column numerical 1[3] . In other words, we can get a better results if we use higher order differences for second order instead of the familiar second order differences .

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حل المعادلات التفاضلية التباطؤية الجزئية المكافئة باستخدام الطريقة الصريحة والفروقات من الرتب العليا

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المستخلص

في هذا البحث نستخدم الفروقات من الرتب العليا للرتبة (المشتقة) الثانية في حل المعادلات التفاضلية التباطؤية الجزئية المكافئة باستخدام الطريقة الصريحة, وحصلنا على نتائج تكون اقرب إلى القيم الحقيقية من النتائج التي يمكن الحصول عليها عند استخدام الصيغة المألوفة للرتبة (المشتقة) الثانية . أخيرا عملنا مقارنة بين النتائج من خلال جدولين من القيم باستخدام Matlab .