On s-g-coc-Proper Functions

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Abstract—*This research introduces a novel classification of s-g-coc-Proper and s-g-coc'-Proper functions within topological spaces. The study focuses on examining the fundamental properties of these functions, such as continuity, closure, and compactness, while analyzing the impact of topological operations like composition, restriction, and product on their behavior. Furthermore, the research investigates the relationships between these functions and other topological concepts, including s-g-coc-Closed and s-g-coc-Compact sets. The research includes rigorous mathematical proofs, establishing new theorems such as the stability of topological properties under composition and the analysis of compact fibers of these functions. Additionally, illustrative examples are provided to highlight specific cases where certain properties fail or results deviate.*

Keywords—s-g-coc-open; s-g-coc-closed; s-g-coc-compact; s-g-coc-continuous; s-g-coc-proper

I. Introduction

Topology, as a foundational branch of mathematics, investigates the properties of spaces that remain invariant under continuous transformations. Supra topological spaces, an extension of classical topological spaces, provide a versatile framework to explore advanced structures and their associated functions. Within this context, the study of s-g-coc -Proper and s-g- coc'-Proper functions introduces a novel perspective that enriches the understanding of continuity, closure, and compactness in supra topological spaces.

This research focuses on the theoretical development of s-g-coc -Proper and s-g- coc'-Proper functions. These functions generalize traditional concepts and offer a rigorous mathematical toolset for analyzing the behavior of mappings under topological operations such as composition, restriction, and product. By establishing a comprehensive theoretical framework, the study aims to clarify the relationship between these functions and other concepts like s-g-coc -Closed and s-g-coc -Compact sets.

The motivation for this research stems from significant prior contributions to topology. For instance, Ref [1] introduced the foundational properties of supra topological spaces, which have since served as a basis for further studies. Similarly, [2], [3] provided critical insights into compactness and continuity, laying the groundwork for advanced topological investigations. Recent studies in [4]. The authors expanded these ideas to include specialized functions, which paved the way for exploring s-g-coc -Proper functions [5].

Moreover, additional theoretical advancements in general topology have been instrumental. For example [6], [7] provided comprehensive treatments of fundamental topological concepts, emphasizing their applicability to complex spaces. In parallel, [8], [9] showcased the growing relevance of topology in modern applications, such as data analysis and network theory. Additionally, algebraic topology has played a crucial role in understanding structural properties of topological spaces, particularly through concepts such as homotopy, covering spaces, and compactness, which are essential in analyzing the behavior of functions and topological mappings [10], [11]. Furthermore, fixed point theory has provided significant insights into the existence and stability of solutions in various mathematical frameworks, highlighting the importance of compact and closed mappings [12]. Although this research primarily focuses on theoretical aspects, the connections to such applications underscore the broader significance of s-g-coc -Proper functions in both pure and applied mathematics.

Through rigorous proofs and illustrative examples, this study contributes to the advancement of topological theory by refining definitions, proving new theorems, and exploring the implications of these functions. By bridging theoretical insights with foundational studies, the research aims to provide a robust basis for further exploration in supra topological spaces, Algebraic topology, and fixed-point theory.

II. RESEARCH OBJECTIVES

1) To introduce precise definitions of s-g-coc -Proper and s-g- coc'-Proper functions within the context of supra topological spaces.
2) To investigate the fundamental properties of these functions, such as continuity, closure, and compactness, and analyze their behavior under various topological operations.
3) To examine the relationships between s-g-coc -Proper and s-g- coc'-Proper functions and other topological concepts, including s-g-coc -Closed and s-g-coc -Compact sets.
4) To establish new theorems that highlight the interaction between these functions and their stability under operations like composition, restriction, and product.
5) To construct illustrative examples, including theoretical and practical cases, that clarify where certain properties hold or fail.
6) To extend the theoretical framework of supra topological spaces, enriching the mathematical foundation and paving the way for future research in this domain.

III. BACKGROUND AND RELATED WORK

The study of s-g-coc-Proper functions is rooted in the broader context of general topology and supra topological spaces. These spaces and functions offer a mathematical framework to analyze advanced topological properties such as closure, continuity, and compactness. This section reviews the foundational work and related studies that have shaped the theoretical foundation for the current research

A. General Topology and Foundational Concepts

General topology provides the primary tools and definitions for studying relationship between topological spaces and their associated functions.

- Ref [2] in Foundation of General Topology introduced fundamental properties of continuity and compactness, emphasizing their role in understanding the behavior of functions between spaces.
- Ref [3] in General Topology expanded on compactness and closure properties, providing the mathematical rigor necessary for studying advanced concepts such as s-g-coc--Proper functions.

These foundational studies laid the groundwork for the analysis of specific classes of functions in topological spaces, including closed, compact, and continuous functions.

B. Supra Topological Spaces

Supra topological spaces generalize classical topological spaces by incorporating additional layers of structure and complexity.

- Ref [1] n their seminal paper On Supra Topological Spaces introduced the key properties and definitions of supra spaces, which serve as the basis for many subsequent studies.
- Ref [4] examined the interaction between supra topology and ordered spaces, highlighting the importance of supra homeomorphisms in understanding complex relationship.

The introduction of supra spaces has broadened the scope of topology, making it possible to explore novel function classes like s-g-coc-Proper in more generalized settings.

C. Development of s-g-coc-proper functions

The concept of s-g-coc-Proper functions emerges from the study of advanced closure and compactness properties:

- Ref [5] explored related classes such as g-closed and g-compact functions, which provided a basis for formalizing s-g-coc-Proper functions.
- Ref [13] in their thesis On Separation Axioms for Supra Topological Spaces, extended these ideas to study axioms and separation properties in supra spaces, laying the groundwork for defining s-g-coc-compactness and related function properties.

These studies demonstrate the interplay between continuity, closure, and compactness in supra spaces, establishing a strong theoretical foundation for the current research.

D. Applications and Relevance

Although primarily theoretical, s-g-coc-Proper functions have potential applications in several fields:

- In dynamic systems, they can model stability by analyzing compact fibers.
- In data analysis, s-g-coc--compactness can be used to study clustering and high-dimensional data structures.
- In network theory, these functions can help optimize communication flows by ensuring stability and balance between connected nodes.

IV. SUMMARY OF RELATED WORK

The existing literature provides a comprehensive theoretical base for understanding s-g-coc-Proper functions and their interaction with supra topological spaces. This research builds upon these studies to propose new properties, refine existing definitions, and explore applications. By bridging the gap between theoretical advancements and practical applications, this work aims to contribute to the ongoing development of advanced topological function analysis.

V. RESEARCH OBJECTIVES

1) Develop precise definitions for s-g-coc -Proper and s-g- coc′ -Proper functions in supra topological spaces.
2) Analyze the core properties of these functions, including continuity, closure, and compactness.
3) Explore relationship with concepts like s-g-coc -Closed and s-g-coc -Compact sets.
4) Prove new theorems that connect these functions to other topological properties.
5) Provide illustrative examples to clarify special cases and failures of certain properties.
6) Expand the theoretical framework of supra topological spaces for future research.

VI. FUNDAMENTAL CONCEPTS

A. Definition

A suppose that the function $f: X \to Y$ maps a space X to space Yif for each s-open set in Y, it is called f as an s-g-coc-continous if f^{-1} (B) is an s-g-coc-open set in X [13].

B. Definition

Let X space and Y topological supra space and f:X→Y, then f is called an s-g-coc-closed function if it was f (B) is an s-g-coc-closed set in Y ∀ s-closed set B in X [13].

C. Definition

If there is a finite sub cover for each s-g-coc-open cover of a topological space X, then X is considered s-g-coc-compact [13].

D. Definition

Suppose that X and Y topological spaces If here $f^{-1}(B)$is s-compact set in X for each s-coc-compact set B in Y,then f:X→Y referred to as an s-g-coc-compact function [13].

E. Definition

i. Topological space X is said to be CC if for each s-compact set in X is s-closed [13].

ii. Topological space X is said to be $CC^{'}$ if for each s-g-coc-compact set in *X* is s-g-coc-closed [13].

F. Remark

For each CC space is CC^′ Space but the converse is not true [13].

G. Proposition

If we assume that X is a supraspace, then the following claims are comparable [13]:

1) X is CC.
2) $\mu=\mu^{gk}$.

H. Definition

A space X is called an s-g-coc-T_0-space if and only if for each $x \neq y$ in X there exist an s-g-coc-open set U such that $x \in U$, $y \notin U$.

I. Theorem

Let $f: X \longrightarrow Y$ be a bijective s-g-coc-continuous function. If Y is an $s-T_0$-space, then X is an s-g-coc-T_0-space.

J. Definition

Suppose that X and Y topological spaces and $f: X \longrightarrow Y$ be a function, then f is said to be an s-g-coc irresolute (s-g–coc′–continuous) function if $f^{-1}(B)$is an s-g-coc-open group in X for every s-g-coc-open set in Y [13].

K. Definition

Suppose that X and Y topological spaces and $f: X \longrightarrow Y$ be a function, then γ is said to be an s-g-coc′-closed function if (B) is an s-g-coc-closed set in Y for all s-g-coc-closed set B in X [13].

L. Definition

Suppose that X and Y topological spaces and $f: X \longrightarrow Y$ be a function, then $f$is called s-g-coc′-compact function if for each s-g-coc-compact set B in Y ,$f^{-1}(B)$ is s-g-coc-compact set in X [13].

M. Theorem

Suppose that $f: X \longrightarrow Y$ be a s-g-coc-closed function then the restriction of f to a closed subset B of X is an s-g-coc-closed of B in to Y [13].

N. Theorem

Suppose that X and Y topological spaces and $f: X \longrightarrow Y$ is an s-g-coc-continuous and B is an s-closed set is not empty in (X, τ) then the restriction $f/_B: (X, \tau/_B) \longrightarrow (Y, \sigma)$ [13].

O. Proposition

If $f: X \longrightarrow Y$ is an s-g-coc-continuous function between topological space X and Y, and $B \subseteq X$ closed subset, then restriction $f/_B$ is also s-g-coc-continuous function.

Proof. To prove that $f/_B$ is s-g-coc-continuous, we must show that the preimage of any s-g-coc-open set in Y under $f/_B$ is s-g-coc-open set in Yunder $f/_B$is s-g-coc-open in B.

Let V be an s-g-coc-open set in Y .By definition of s-g-coc-continuity, the preimage $f^{-1}(v)$ is s-g-coc-open in X.now consider the preimage of V under $f/_B$,which is given by($f/_B)^{-1}(V) = f^{-1}(V) \cap B$, since $f^{-1}(V)$is s-g-coc-open in X and B is a closed subset of X ,the intersection $f^{-1}(V) \cap B$is s-g-coc-open in the subspace B.Therefore $(f/_B)^{-1}(V)$ is s-g-coc-open in B, and so $f/_B$ is s-g-coc-continuous.

P. Theorem

A subset A⊆Xis s-g-coc-compact if and only if it is the intersection of an s-closed subset and an s-g-compact subset of X.

Proof. Let A⊆X be s-g-coc-compact subset of X, and let B⊆X be an s-closed set of X. Thus, A is s-g-compact, meaning that every s-g-coc-open cover of A admits a finite subcover. Furthermore, since A is s-closed, then A∩B is s-closed. As the intersection of two s-closed sets is s-closed since A is s-g-compact, A∩B as a subset of A, is also s-g-compact Thus, A∩B satisfies both conditions.

- It is s-closed.
- It is s-g-compact.

Therefore, $A \cap B$ is s-g-coc-compact.

Q. Proposition

If $f: X \to Y$and $g:Y \longrightarrow Z$ are s-g-coc′-continuous, then(g∘ f) is s-g-coc′-continuous [13].

R. Remark

If $f: X \to Y$ s-g-coc'-continuous and $g: Y \longrightarrow Z$ is s-g-coc-continuous then $(g \circ f)$ is s-g-coc-continuous [13].

S. Proposition

Let $f: X \to Y$ and $g: Y \to Z$ be an s-g -coc'-closed function. Then the composition $(g \circ f)$is s-g- coc'-closed.

Proof. Clear.

T. Proposition

Let $f: X \to Y$ be an s-g -coc'-closed function and $g: Y \to Z$ be an s-g-coc-closed function. Then the composition $(g \circ f)$is s-g- coc'-closed

Proof. Clear.

U. Proposition

Suppose that $f: X \to Y$ be an s-g-coc-continuous, then f^{-1} (β) is an s-g-coc-closed set in X for all β is an s-closed set in Y [13].

We will study the properties of multiplication of groups.

V. Proposition

Let $f_1:(X_i,T_i)\to(Y_i,\sigma_i)$,i=1,2, be s-g-coc-continuous functions. Then their product $f_1\times f_2:(X_1\times X_2,T_{prod})\to(Y_1\times Y_2,\sigma_{prod})$, defined $(X_1\times X_2)(x_1,x_2)=(f_1(x_1),f_2(x_2))$ is also s-g-coc-continuous.

Proof. To prove that $f_1\times f_2$ be s -g-coc -continuous.

We need to verify that the preimage of any s-g-coc-open set in$(Y_1\times Y_2,\sigma_{prod})$ is s-g-coc-open in $(X_1\times X_2,T_{prod})$.

- Let $U\subseteq Y_1\times Y_2$ be an s-g-coc-open set. By definition of product topology σ_{prod}, U can be expressed as a union of basis elements of the from $U_1\times U_2$,where $U_1\subseteq Y_1$and $\subseteq U_2\subseteq Y_2$are s-g-coc-open.

Preimage under:
$(f_1\times f_2)^{-1}(U):(f_1\times f_2)^{-1}(U)(\cup(U_1\times U_2))$

Using properties of the preimage and unions:
$(f_1\times f_2)^{-1}(U):\cup((f_1\times f_2)^{-1}(U)(U_1\times U_2)$

The preimage of Basis Element$(U_1\times U_2)$under $(f_1\times f_2)$is
$(f_1\times f_2)^{-1}(U_1\times U_2):\cup(f_1^{-1}(U_1)\times(f_2^{-1}(U_2)$.

- Since $f_1$and $f_2$are s-g-coc-continuous, $f_1^{-1}(U_1)$ is s-g-coc-open in (X_1,T_1),and $f_2^{-1}(U_2)$is s-g-coc-open in(X_2,T_2).The product of two s-g-coc-open in the product topology T_{prod}.Therefore, $(f_1\times f_2)^{-1}(U_1\times U_2)$is s-g-coc-open in $(X_1\times X_2,T_{prod})$ since$(f_1\times f_2)^{-1}(U)$is a union of s-g-coc-open sets in $(X_1\times X_2,T_{prod})$, it is also s-g-coc-open. Thus $f_1\times f_2$is s-g-coc-continuous.

W. Remark

i.T_{prod}This represents the usual product topology on X×Y, but we are particularly interested in the s-g-coc -open sets under this topology, which form a subfamily of the product topology and are denoted by T_{prod}^{gk}.

ii.$T_{gk-prod}$ Here, T^{gk} and ,$\sigma^{gk}$are specialized topologies on X and Y, respectively, generated by s-g-coc -open sets. The product topology formed from these specialized topologies is denoted by $T_{gk-prod}$

X. Proposition

If that X and Y are topological spaces and $A\times B$ is an s-g-coc-open set in $X\times Y$ then:

i. A is s-g-coc-open sets in X .
ii. B is s-g-coc-open in Y. provided A and B are projection of the product set $A\times B$ onto X and Y, respectively.

Proof. Let $\upsilon\in A$, Then by definition of the product set $A\times B$,there exists $\alpha\in Y$such that $(\upsilon,\alpha)\in A\times B$.

Since $A\times B$ is s-g-coc-open set in $X\times Y$,there exists an s-open set $H\subseteq X\times Y$such that $(\upsilon,\alpha)\in H$ and an s-compact subset $W\in C(X\times Y)$such that $(\upsilon,\alpha)\in H-w\subseteq A\times B$,then $p_{r1}(H)$ is s-open group in X contain υ, $p_{r1}(w)$ s-compact in Y .From the inclusion $H-w\subseteq A\times B$,we have $p_{r1}(H-w)\subseteq A$.Therefore $p_{r1}(H)-p_{r1}(w)\subseteq A$ since$p_{r1}(H)$ is s-open in X, and $p_{r1}(w)$ is s-compact in X, the set A satisfies the definition of s-g-coc-openness in X.

Y. Proposition

If $A\times B\subseteq X\times Y$ is s-g-coc-closed set in $X\times Y$,A and B are projections of $A\times B$ onto X and Y, respectively, then:

i. A is s-g-coc-closed sets in X
ii. B is s-g-coc-open in Y.

Proof. Clear.

Z. Proposition

Let (X,T) and (Y,σ) be two spaces. Then $T_{prod}^{gk}\subseteq T_{gk-prod}$.

Proof. Clear.

AA. Proposition

Let (X,T)& (Y,σ)be cc-spaces then:

i. If A is an s-g-coc-open(s-g-coc-closed) set in (X,T) then $A\times Y$is also s-g-coc-open (s-g-coc-closed) in $(X\times Y,T_{prod})$.
ii. If B is an s-g-coc-open (s-g-coc-closed) set in (Y,σ)then $X\times B$ is ans-g-coc-open (s-g-coc-closed) in $(X\times Y,T_{prod})$.

Proof.

i. Assume $A\subseteq X$ is s-g-coc-open(s-g-coc-closed) in (X, T).

Since (X.T) is a cc-space, we know from (Proposition VI.G), then $T^{gk}=T$ .This means that the s-g-coc-open sets in (X,T) concide with the standard open sets in T .BY the definition of the product topology T_{prod},the set $A\times Y$ is open in $(X\times Y,T_{prod})$ because A is open in (X,T) and Y is the entire space (Y,σ).since A is an open set in (X,T) ,It follows that $A\times Y$inherits the s-g-coc-openness property in $(X\times Y,T_{prod})$, Therefore $A\times Y$is s-g-coc-open in $(X\times Y,T_{prod})$.

ii. Looks like proof in (i).

BB. Proposition

Let X andY be spaces and let $A\subseteq X$ and $B\subseteq Y$ then:

i. If Xand Y be space and it was A,B are s-g-coc-closed subset of X and Y.then $\overline{(A\times B)})^{s-g-coc}=\overline{A}^{s-g-coc}\times\overline{B}^{s-g-coc}$.
ii. if X and Y be space and it was A,αare s-g-coc-open subset of X and Y.Then $(A\times B)^{\circ s-g-coc}=A^{\circ s-g-coc}\times B^{\circ s-g-coc}$.

Proof. Clear.

CC. Theorem

- Let (X,T) and (Y,σ) be two topological CC-space the projection functions p_{r1}:X×Y → X and p_{r2}: $X\times Y\to Y$ are s-g-coc-continuous.

Proof. For p_{r1}:X×Y → X.

- Let $A\subseteq X$be s-g-coc-open in (X,T) .

The preiage of A under p_{r1} is $p_{rx}^{-1}(A)$=A×Y Theorem (VI.Y)

The cartesian product of an s-g-coc-open $A\subseteq X$and the whole space Y is s-g-coc-open in X×Y.Hence $p_{r1}$is s-g-coc-continuous. For $p_{r2}:X\times Y\to Y$.

- Let $B\subseteq Y$,be s-g-coc-open in (Y,σ).

The preimage of B under p_{r2} is $p_{r2}{}^{-1}$(B)= $X \times B$.by theorem VI.X, the cartesian product of the whole space X and an s-g-coc-open in $X \times Y$.Hence p_{r2} is s-g-coc-continuous.

DD. Theorem

Let $pr_i : (X_i, T) \to (Y_i, \sigma_i)$,i=1,2 are function $\ni$ $p_{r1} \times p_{r2} : (X_1 \times X_2, T_{prod}) \to (Y_1 \times Y_2, \sigma_{prod})$is An s-g-coc-compact function then p_{ri} is an s-g-coc-compact function.

Proof. Let $p_{r1} \times p_{r2}$ be s-g-coc-compact.by definition, for every s-g-coc-compact subset $K \subseteq Y_1 \times Y_2$,the preimage $(pr_1 \times pr_2)^{-1}(K)$ is s-g-coc-compact in $X_1 \times X_2$ for proof $Pr_1$is s-g-coc-compact function.

- Suppose that $K \subseteq Y_1$be s-g-coc-compact, and let$\alpha_2 \in Y_2$ The product $K \times \{\alpha_2\}$is s-g-coc-compact$Y_1 \times Y_2$.
- Preimage under $p_{r1} \times p_{r2}$:Since $p_{r1} \times p_{r2}$ is s-g-coc-compact ,the preimage $(p_{r1} \times p_{r2})^{-1}(K \times \{\alpha_2\})$is s-g-coc-compact in $X_1 \times X_2$,but $(p_{r1} \times p_{r2})^{-1}(K \times \{\alpha_2\}) = p_{r_1}^{-1}(k) \times p_{r_2}^{-1}(\{\alpha_2\})$.Therefore $p_{r_1}^{-1}(k) \times p_{r_2}^{-1}(\{\alpha_2\}$ is s-coc-compact in $X_1 \times X_2$ is s-g-coc-compact. Since $p_{r_2}^{-1}(\{\alpha_2\})$ is a singleton in$X_1$the s-g-coc-compactness of the product implies that $p_{r_1}^{-1}(k)$ is s-g-coc-compact inX_1.Hence p_{r1} is s-g-coc-compact. In the same way, we can prove $\gamma_2$is ans-g-coc-compact.

EE. Proposition

If X be s-g-coc-compact then the projection $p_{r2} : X \times Y \to Y$is s-g-coc-closed.

Proof. Clear.

FF. Theorem

Let $p_r : X \to Y$ be a s-g-coc-homeomorphism. If K is a s-g-coc-compact sub set of X then $p_r(K) \subseteq Y$ is also s-g-coc-compact.

Proof. Clear.

GG. Theorem

If every net in a topological space X has a cluster point in X, then X is an s-compact.

Proof. Clear.

HH. Definition

Let (X,τ) be a topological space with s-g -open topology. The space X is said to satisfy the s-g-T_1separation axiom if and only if:

- For any two distinct points x, y $\in$X where x $\neq$y, there exists a set U $\in \tau$such that: Either x $\in$ U and $y \notin U$,or y $\in$U, y $\notin$V and x $\notin$V, where U and V are s-g -open sets.

II. Example

Let X = {a, b, c} with the topology τ = {$\emptyset$, {a}, {b}, {a, b}, X}. Every point can be separated from the others using s-g -open sets.

- For example, the point a belongs to {a}, which does not contain b.

JJ. Definition

The topological space (X,τ) satisfies the s-g-T_2 separation axiom if and only if:

For any two distinct points x, y $\in$X where x $\neq$y, there exist two s-g -open sets U, V such that:x $\in$U, y $\in$ V, and U $\cap$V = $\emptyset$.This means that every pair of distinct points can be separated by two disjoint s-g -open sets.

KK. Example

Let X = {a, b} with the topology $\tau = \{\emptyset$, {a}, {b}, X}

- The points a and b can be separated using the sets {a} and {b}, which are disjoint.

Notes: Every s-g-T_2space satisies s-g-T_1,but the converse is not always true.

LL. Example

Let X= {a, b, c} with the topology: $\tau = \{\emptyset$, {a, b}, {b, c}, X}.

This space is s-g-T_1(Any two point x and y can be separated by an s-g-open set that contains x and does not contain y. However, this space is not s-g-T_2:(it is not possible to find two disjoint s-g-open sets that separate a and c.

- Any s-g-open set containing a (such as {a, b}) will intersect with any s-g-open set containing c (such as {b, c}).

VII. S-G-COC- PROPER FUNCTIONS

This section introduces a new type of functions, namely s-g-coc -proper functions and s-g-coc′ -proper function functions. The study will explore their definitions, properties, and relationships with existing concepts in topology, particularly s-g-coc -closed functions and s-g-coc -compact functions. This investigation aims to establish a deeper understanding of the interplay between these function types and their role in topological structures.

A. Definition

A function $f: X \to Y$ between topological spaces. is called an s-g-coc-proper function. If it satisfies the following conditions:

i. f is an s-g-coc-continuous function.
ii. f is an s-g- coc-closed function.
iii. for every $\alpha \in Y$,the preimage $f^{-1}(\alpha)$ (The fibers of f over α) is an s-g-coc-compact in f.

B. Example

Let X =[0,1] and Y= [0,1], be subset of the real numbers $\mathbb{R}$ with the standard topology. Define the function $f: X \to Y$ by$f(x) = x^2$.hence f is s-g-coc-proper. The following example shows not every function is ans-g-coc-proper function.

C. Example

Let $X = [0, \infty)$and $Y = [0, \infty)$.

- On X: we use the cofinite topology T_{col},where a set is open if its complement is finite or empty.
- On Y: we use the standard topology T_{std},where a set is open if it is a union of open intervals.

Define the function $f: X \to Y \ni f(x) = x$.The function $f(x) = x$ is not s-g-coc-proper because it fails the s-g-coc-continuity conditin and the fibers of some sets are not s-g-coc-compact.

D. Remark

i. Every proper function from $\mathbb{CC}$-space is s-g-coc-proper into any topological space.
ii. Every homeomorphism from $\mathbb{CC}$-space in to any topological space s-g-coc-proper.

From the example below definition (VII.A) we can see that, although every s-g-coc-proper function is s-g-coc-closed, the reverse is not always true.

E. Example

Let $X = \mathbb{Z}$ (the set of integers) equipped with the discrete topology, where every subset is open. let $Y = \mathbb{R}$(the set of real numbers) equipped with the standard topology, where open sets are union of open intervals. The function definition $f: X \to Y$ by $f(n) = n$ for every $n \in \mathbb{Z}$.The function f is s-g-coc-closed because it preserves the closure of sets. However, f is not s-g-coc-proper because it does not satisfy the compactness condition.

F. Theorem

Let $f: X \to Y$ be an s-g-coc-proper function, and let B be a closed subset of the space X then the restriction of f to B, denoted $f|_B: B \to Y$, is also s-g-coc-proper function.

Proof.

- $f|_B$ is s-g-coc-continuous:

By Theorem (VI.O), the restriction of an s-g-coc-continuous function to any subset is also s-g-coc-continuous. Since f is s-g-coc-continuous, $f|_B$ is s-g-coc-continuous.

- $f|_B$ is s-g-coc-closed:

By theorem (VI.M), the restriction of an s-g-coc-closed function to a subset is also s-g-coc-closed. Since f is s-g-coc-closed, $f|_B$ is s-g-coc -closed.

- Compactness condition for $f|_B$, To prove that $f^{-1}|_B\{\alpha\}$ is s-g-coc-compact set in B for every $\alpha \in Y$. Since f be a s-g-coc-proper function, $f^{-1}\{\alpha\}$ is s-g-coc-compact in X. Then by theorem (6.16), the intersection of an s-g-coc-compact set with a closed set is s-g-coc-compact. Therefore, $f|_B{}^{-1}(\{\alpha\}) = f|_B{}^{-1}(\{\alpha\}) \cap B$ is s-g-coc-compact group in B, hence $f|_B$ is s-g-coc-proper.

G. Theorem

Let X and Y be a topological space and $f: X \to Y$ be a s-g-coc-proper function. If B be a closed (open) set of Y, where $f^{-1}(B)$ is closed subset in X , then $f_B: f^{-1}(B) \to B$ is s-g-coc-proper function.

Proof.

Let $C \subset B$ be an arbitrary closed subset of B. Since B is endowed with the subspace topology from Y, there exists a closed set $A \subset Y$ such that $C = A \cap B$.

- By the definition of the restriction f_B, for any $x \in f^{-1}(B)$ we have $f_B(x) = f(x)$. Therefore, the preimage of C under f_B is given by $f_B{}^{-1}(C) = \{x \in f^{-1}(B): f(x) \in C\}$.
- Rewriting C as $A \cap B$, we obtain $f_B{}^{-1}(C) = \{x \in f^{-1}(B): f(x) \in A \cap B\}$.
- But note that for any $x \in f^{-1}(B)$, the condition $f(x) \in A \cap B$ is equivalent f(x) $\in A$($since\ x \in f^{-1}(B)$ already gurantees f(x) $\in B$).
- Hence, $(f_B)^{-1}(C) = \{(x \in f^{-1}(B): f(x) \in A\} = f^{-1}(A) \cap f^{-1}(B)\}$ is closed in X .In addition, because $f^{-1}(A) \cap f^{-1}(B) \subset f^{-1}(B)$ and the topology on $f^{-1}(B)$ is the subspace topology induced from X, the set $f^{-1}(A) \cap f^{-1}(B)$ is closed in $f^{-1}(B)$. Thus, we have shown that for every closed subset $C \subset B$, $(f_B)^{-1}(C) = f^{-1}(A) \cap f^{-1}(B)$ is closed in $f^{-1}(B)$. Therefore, by definition there stricter function $f_B: f^{-1}(B) \to B$ is s-g-coc-proper.

H. Definition

A function f:X→Y is s-g- coc′ - proper if it satisfies the following:

i. f is s-g-coc′-continuous function.
ii. f is s-g-coc′-closed function.
iii. $f^{-1}\{\alpha\}$ is s-g-coc′-compact in ϑ for every $\alpha \in Y$. (The fibers of f are s-g-coc′-compact).

I. Remark

It is not necessary for s-g-coc′-proper function to be composition of two s-g-coc-proper functions. While the composition of s-g-coc′-proper function is guaranteed to be s-g-coc′-proper, the same cannot be said if the functions are only s-g-coc-proper.

Currently, we consider topological spaces and functions under the condition that the composition of two s-g-coc′- proper functions results in an s-g-coc′ -proper function. Additionally, when one function is s-g-coc′-proper and the other is s-g-coc -proper, their composition is s-g-coc -proper.

J. Theorem

Let $f: X \to Y$ and $g: Y \to Z$ be functions, then:

i. If f and g are s-g -coc′-proper functions, then gof is s-g-coc′-proper.
ii. If f is s-g- coc′ -proper and g is s-g-coc-proper functions, then gof is s-g-coc-proper.

Proof.

i. By theorem (VI.Q), (gof) is s-g-coc′-continuous, by theorem (6.20) gof is s-g-coc′-closed. Now, to prove that for any $(gof)^{-1}\{z\} = f^{-1}(g^{-1}\{z\})$. Since g is s-g-coc′ -proper, , $g^{-1}(\{z\})$ is s-g- coc′ -compact set in Y .since f is s-g- coc′ -proper , the preimage $f^{-1}(g^{-1}\{z\})$ is s-g- coc′-compact set in X. Thus, $(gof)^{-1}\{z\}$ is s-g- coc′ –compact. Gof satisfies all three conditions of s-g- coc′ –proper.
ii. By Theorem (VI.R), gof is s-g-coc-continuous and by theorem (VI.T) s-g-coc-closed according gof is s-g-coc-closed .Let z∈Z .the fiber of gof is $(gof)^{-1}\{z\} = f^{-1}(g^{-1}\{z\})$ since g is s-g-coc-proper , $g^{-1}\{z\}$ is s-g-coc-compact in Y .Give that Y is CC –spaces, $g^{-1}\{z\}$ is compact , since f is a s-g- coc′-proper, the preimage $f^{-1}(g^{-1}\{z\})$ is compact set in X. Furthermore, since X is CC – space compactness implies s-g-coc-compactness. Thus, $(gof)^{-1}\{z\}$ is s-g-coc-compact . gof satisfies all three conditions of s-g-coc-proper.

We now analyze the behavior of s-g-coc -proper functions under Cartesian product operations, focusing on whether the resulting function retains s-g-coc –properness.

K. Remark

It's not always the case that the product of two s-g-coc-proper functions is s-g-coc-proper.

L. Example

Let $X_1 = \mathbb{N}$(natural numbers) and $X_2 = \mathbb{N}$, both with the cofinite topology. Let $Y_1 = \Re$ and $Y_2 = \Re$, both with the standard topology.

- Functions: Define $f_1: X_1 \to Y_1$ by $f_1(n) = n$.
  Define $f_2: X_2 \to Y_2$ by $f_2(n) = n^2$

Although f_1 and f_2 are s-g-coc-proper their product $f_1 \times f_2$ is not necessarily s-g-coc-proper due to the lack of s-g-coc-compactness of the fibers in some cases.

M. Proposition

Let $f_i: (X_i, T_i) \to (Y_i, \sigma_i), i = 1,2$,the function such that $f_1 \times f_2: (X_1 \times X_2, T_{prod}) \to (Y_1 \times Y_2, \sigma_{prod})$ is s-g-coc-proper function then each f_i is a s-g-coc-proper function .

Proof. To prove that $f_1: (X_1, T_1) \to (Y_1, \sigma_1)$ is s-g-coc-proper.

a) Since $f_1 \times f_2$ is s-g-coc- continuous then f_1 is s-g-coc-continuous (by proposition VI.V).

b) Since $f_1 \times f_2$ is s-g-coc-closed then γ_1 is s-g-coc-closed) by proposition (VI.Y).

c) Let $\alpha_1 \in Y$then $(\alpha_1, \alpha_2) \in Y_1 \times Y_2$and evaluates the fiber: $(f_1 \times f_2)^{-1}(\alpha_1,\alpha_2) = f_1^{-1}(\alpha_1) \times f_2^{-1}(\alpha_2)$ since $f_1 \times f_2$ is s-g-coc-proper ,the fiber $(f_1 \times f_2)^{-1}(\alpha_1,\alpha_2)$ in $(X_1 \times X_2, T_{prod})$ .By projection , $f_1^{-1}(\alpha_1)$ must also be is s-g-coc-compact in (X_1, T_1).

f_2 is we can prove s-g- coc-proper. From (a), (b) and (c) f_1 s-g-coc-proper and in similar way

N. Theorem

Let $f: X \to Y$be an s-g-coc-continuous function, where X is an s-g-coc-compact cc-space and Y is cc-space. Then, the following statements are equivalent:

i. f is s-g-coc-proper function.

ii. For any net $(\chi_d)_{d\in D}$is a net in X and any $\alpha \in Y$ that is an s-g-coc-cluster point of $\{\gamma(\chi_d)\}_{d\in D}$, there exists an s-g-coc-cluster point $\upsilon \in X$ of $(\chi_d)_{d\in D}$such that $f(\upsilon) = \alpha$

Proof. Clear.

O. Theorem

Let $f: X \to \{w\}$be a function ,where X is a CC −space and $w \notin X$. Then f is s-g-coc-proper if and only if X is s-g-coc-compact space.

Proof. (⇒)if f is s-g-coc-proper, then X is s-g-coc-compact. Let $(\chi_d)_{d\in D}$be a net in X .since $f(\chi_d) = \omega$ for all $d \in D$ it follows that $f(\chi_d) \overset{s.g.coc}{\propto} w$, sinceγis s-g-coc-proper, by theorem(2.3.13ii) , there exists a point $\upsilon \in X$ such that $\chi_d \overset{s.g.coc}{\propto} \iota$ and $f(\upsilon) = \omega$. Since this holds for any net, by theorem (2.14ii), X must be s-g-coc-compact. Conversely:(X is a s-g-coc-compact⇒ f is s-g-coc-proper): The proof assumes X is a cc-space and $f^{-1}\{\omega\} = X$ is s-g-coc-compact, since X is s-g-coc-compact it's also s-g-coc-closed. Hence f is s-g-coc-continuous-g-coc-closed and $f^{-1}\{\omega\}$ s-g-coc-compact. Thus fis s-g-coc-proper.

P. Theorem

Let $f: X \to Y$ be a homeomorphism from s-g-coc-compact CC-space X into spaceY, then $f^{-1}: Y \to X$ is s-g-coc-proper function.

Proof.

To proof that $f^{-1}$is proper, we verify the following conditions:

- s-g-coc-continuous since f is a homeomorphismf^{-1},is continuous. Moreover, $f^{-1}$is s-g-coc-continuous.
- s-g-coc-closed. As f is a homeomorphism, it is closed This implies that f^{-1} is also s-g-coc-closed.

To show that $(f^{-1})^{-1}(\{\upsilon\})$is s-g-coc-compact for any $v \in X$:The preimage $\{\upsilon\} \subseteq X$ is s-g-coc-compact by assumption by corollary (VI. FF) the image $f(\{\upsilon\})$is s-g-coc-compact in Y but $f(\{\upsilon\}) = (f^{-1})^{-1}\{\upsilon\}$. Hence $(f^{-1})^{-1}\{\upsilon\}$is s-g-coc-compact in Y. Hence f^{-1} is s-g-coc-proper.

Q. Corollary

Let f:X →Y be a s-g-coc-proper function, where Xis s-g-coc-compact cc-space. For any $\alpha \in Y$,the restriction of f to$f^{-1}\{\alpha\}$,denoted as $f_{\{\alpha\}}: f^{-1}\{\alpha\} \to \{\alpha\}$is alsi s-g-coc-proper function.

R. Theorem

Let $f: X \to Y$be a s-g-coc-proper function, whereX s-g-coc-compact then f is s-g-coc-compact.

Proof. Let K is s-g-coc- compact set inY and f is s-g-coc-proper. To prove that $f^{-1}(K)$is be s-g-coc-compact in X, let $(\chi_d)_{d\in D}$be a net in $f^{-1}(K)$, thus $f(\chi_d)$has an s-g-coc- cluser point $\alpha \in K$.As f is s-g-coc-proper, there exists $\upsilon \in X$such that $f(\upsilon) = \alpha$. Since $\alpha \in K$,it follows that $\upsilon \in f^{-1}(K)$.Therefore, $f^{-1}(K)$has an s-g-coc-cluster point in itself. By theorem (VI.GG), $f^{-1}(K)$ is s-g-coc-compact in X .since $f^{-1}(K)$ is s-g-coc-compact for any s-g-coc-compact subset $K \subseteq Y$,f is s-g-coco-compact function by definition. The following example demonstrates that the converse of the theorem (VII.Q) is not true in general.

S. Example

Let $X = \Re$ with the standard topology. Let Y= $\Re$ with the discrete topology. The function f:X →Y. As the identity function f(x)=this example demonstrates that while f is s-g-coc-compact, it is not s-g-coc-proper, showing that the converse of Theorem (VII.Q) is not true in general.

T. Theorem

Let X be an s-g-coc-compact cc-space, and Ybe a cc-spaces, then the projection function $P_2: X \times Y \to Y$ is s-g-coc-proper.

Proof. Clear that $P_2: X \times Y \to Y$is s-g-coc-continuous (proposition VI.CC). Since X is s-g-coc-compact, then P_2 is s-g-coc-closed (proposition VI.EE).

Now, to prove that$P_2^{-1}\{\alpha\}$is s-g-coc-compact in $X \times Y$for every $\alpha \in \psi$. Since $P_2^{-1}\{\alpha\} = \vartheta \times \{\alpha\}$, but $X \times \{\alpha\}$is s-g-compact, therefore $P_2^{-1}\{\alpha\}$is s-g-coc-compact. Hence $P_2$is s-g-coc-proper.

In these theorems, we examine the relationship between s-g-coc -Proper functions and s-g -Separation Axioms, such as s-g-T_0, s-g-T_1, and s-g-T_2 , which are fundamental in analyzing the structure of topological spaces.

s-g-coc -Proper functions, defined by their continuity, closure, and compactness properties, significantly influence the topology of spaces. This section highlights how separation axioms affect these functions and their role in shaping the compactness of fibers and the behavior of compositions.

U. Theorem

Let $f: X \to Y$ be an s-g-coc-proper function, and assume X satisfies the s-g-T_1separation axiom. Then:

1. The fibers $f^{-1}(\propto)$ are s-g-coc-compact in X for every$\propto \in Y$.
2. The image Y satisfies the s-g-coc-T_0 separation axiom.

Proof.

1. Since f is s-g-coc-proper, $f^{-1}(\propto)$is s-g-coc-compact in X satisfies s-g-T_1,every point in X can be separated by s-g-open sets. Hence, $f^{-1}(\propto)$ inherits compactness from X.
2. For Y, since f is closed and continuous, separation between points in Y is preserved via the preimage in X, ensuring that Y satisfies s-g-T_0.

V. Theorem

Let $f: X \to Y$ and $g: Y \to Z$ be s-g-coc-proper functions, and assume X satisfies s-g-T_2.Then:

1. The composition $g \circ f: X \to Z$is s-g-coc-proper.
2. If Z satisfies s-g-T_1,then Y also satisfies s-g-T_1.

Proof.

1. The composition $g \circ f$ satisfies: $(g \circ f)^{-1}(z) = f^{-1}(g^{-1}(z))$.

Since f and g are s-g-coc-proper, $f^{-1}(g^{-1}(z))$is s-g-coc-compact. Thus, $g \circ f$ is s-g-coc-proper.

2. If Z satisfies s-g-T_1,sepration in Z is preserved through g, and subsequently through f ensuring Y satisfies s-g-T_1.

W. Theorem

Let $f: X \to Y$ be s-g-coc-proper. if X satisfies the s-g-T_1 separation axiom, then:

1. Y satisfies s-g-T_0.
2. If f is bijective, Y satisfies s-g-T_1.

Proof.

1. Similar to (Theorem VII.T) separation in X is reflected in Y via f, ensuring Y satisfies s-g-T_0.
2. if f is bijective, every point in Y corresponds uniquely to a point in X.

Given X satisfies s-g-T_1,this separation is preserved in Y, ensuring Y satisfies s-g-T_1.

VIII. CONCLUSION

This research introduced the novel concepts of s-g-coc-Proper and s-g- coc´ -Proper functions within the framework of supra topological spaces, providing a comprehensive analysis of their fundamental properties such as continuity, closure, and compactness. By examining the behavior of these functions under various topological operations including composition, restriction, and product we established critical theorems that expand the understanding of these advanced function classes. Additionally, illustrative examples clarified specific cases where these properties hold or fail.

The study emphasized the connection between s-g-coc-Proper functions and other key topological constructs such as s-g-coc-closed and s-g-coc-compact sets, offering new insights into the stability and interaction of these properties. Through rigorous proofs and theoretical developments, the research contributes to the foundation of supra topology, paving the way for further exploration in this domain. While the findings presented here are primarily theoretical, their implications extend to potential applications in areas such as dynamic systems, network theory, and data analysis. The results provide a solid foundation for future studies aiming to generalize these functions, explore their applications in real-world problems, and investigate their interactions with other advanced topological concepts.

In conclusion, the study advances the theoretical understanding of s-g-coc-Proper functions and sets the stage for future research, including the exploration of practical applications, generalizations to more complex spaces, and deeper connections with algebraic and computational topology.

IX. FUTURE RESEARCH DIRECTIONS ON S-G-COC-PROPER FUNCTIONS

Conceptual Expansion: Generalize s-g-coc-Proper functions to non-metric spaces, high-dimensional spaces, or spaces with random structures.

Link with Algebraic Topology: Explore the relationship between s-g-coc-Proper functions and concepts such as homology and fiber bundles.

Analysis under Topological Constraints: Investigate the impact of separation (Separation) and connectedness (Connectedness) on the properties of these functions.

Multi-Valued Functions: Extend these functions to multi-valued mappings and apply them to optimization problems.

Practical Applications: Use these functions in big data analysis, network stability, and dynamic systems.

Composition and Projection: Study the stability of properties under operations like composition, projection, and extension.

Relation to Convergence: Link these functions with convergence theories such as nets and filters to analyze cluster points.

1) Topological Algorithms: Improve algorithms for analyzing high-dimensional data using these functions.
2) Unified Mathematical Framework: Develop a comprehensive theoretical framework that includes special cases and practical applications for these functions.

These points provide broad research directions to deepen the theoretical and practical understanding of s-g-coc-Proper functions.

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