ON THE GRAPH OF THE MAXIMAL SUBGROUP OF THE CYCLIC GROUP C_n

Ohood A. Hadi
Department of Mathematics
Faculty of Computer Science and Mathematics,
University of Kufa, Najaf, Iraq
ohooda.algburi@uokufa.edu.iq
Orcid.org/0000-0003-2106-978X

Hayder B. Shelash
Department of Mathematics
Faculty of Computer Science and Mathematics,
University of Kufa, Najaf, Iraq
hayder.ameen@uokufa.edu.iq
Orcid.org/0000-0002-0324-740X

DOI: http://dx.doi.org/10.31642/JoKMC/2018/130104
Received Oct. 22, 2025. Accepted for publication Nov. 25, 2025

Abstract— *The maximal graph is a finite graph for any two x, y vertices that are joined if $\langle x, y\rangle$ is a maximal subgroup and is denoted by Γ_{Max}.In this paper, we will present an algorithm for calculating the graph of $\Gamma_{Max} C_n$ .Where we will study several cases of the value of n so there will be an algorithm for each case. To calculate the maximum number of subgroups of the cyclic group C_n . We'll employ the Gap program. and use this to find some of the properties of the maximal graph.*

Keywords— *Maximal subgroup; Max graph; Cyclic group*

I. INTRODUCTION

Let G be a finite group. Computing or studying the probability of satisfying the condition between two elements in a group is an important application of computational group theory. Erfanian In 2013, certain facets of probability within finite groups were presented with others [2], one of which probability to compute the probability that two components from G selected at random, generate a Maximal subgroup, then the Maximal degree is defined by [1]

$$P_{Max}(G) = \frac{\{(x, y) \in G \times G : \langle x, y\rangle \leq_{Max} G, \forall x, y \in G\}}{|G|^2}$$

, where $|G|$ is the order of the group G . We present some results on probability in cyclic groups C_n such that the cyclic group is presented by $C_n = \{e, a, a^2, a^3, \cdots, a^{n-1}\}$, and we examine a straightforward undirected graph devoid of loops and many edges. Suppose that Γ is a graph, the collection of vertices and edges of Γ will be represented by $V(\Gamma)$ and $E(\Gamma)$, correspondingly. $deg(v)$ denotes a vertex's degree,$v \in V(\Gamma)$, and it is commonly known that $deg(v) = |Max(v)|$, where $|Max(v)|$ is denotes the order of the vertex.

Consider a graph with vertices $v_1, \dots, v_n$ and corresponding degree sequence $(\deg(v_1), \dots, \deg(v_n))$. A realisation of $d(\Gamma)$ is any graph with the degree sequence d given by [6, 11]. We can present it by

$$d(\Gamma) = \begin{pmatrix} n_1 & n_2 & \cdots & n_s \\ \mu(n_1) & \mu(n_2) & \cdots & \mu(n_s) \end{pmatrix}$$

, where n_i are degree vertices and $\mu(n_i)$, are multiplicities, and the Max graph $(V, E) = \Gamma_{Max}(G)$ is defined by the set of all vertices $V(\Gamma_{Max}(G)) = \{v : \forall v \in G\}$ and the edge set is defined $E(\Gamma_{Max}(G)) = \{\{x, y\} | \langle x, y\rangle \leq \Gamma_{Max} G\}$, where $\langle x, y\rangle$ is a Maximal subgroup of the group. One of the probabilities to compute the number of maximal degree elements of C_n[10].

II. NUMBER THEORY FUNDAMENTALS

The Euler function, denoted by $\varphi(n)$, counts the number of substantially prime non-negative integers fewer *than* n, or function φ.*[5]* There is a distinct prime factor decomposition for each integer n. $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$ Moreover, this decomposition is referred to as the canonical prime factorization of n, *as* $p_1 < p_2 < \cdots < p_t$.

where $\alpha_i \geq 0 \, \forall i, i = 1, \cdots, t$ *and* p_i is a prime number. Thus, $\varphi(n) = \varphi(p_i^{\alpha_i}) = \Pi(p_i^{\alpha_i} - p_i^{\alpha_{i-1}})$ defines the

Euler function φ , and define the functions $\tau(n)$ and $\sigma(n)$, as follows n be a positive integer :

1- $\tau(n) = a|n, a \in \mathbb{Z}^+$, where a the number of divisors of n;

2- $\sigma(n) = \sum a_i|n$, the sum of divisors of n.

III. REVIEWED LITERATURE

In 1973, Gustafson investigated the question: What proportion of pairs of elements in a group commute? Interest in applying probabilistic methods to finite groups has grown significantly over the past decade compared to the preceding 30 years [1]. In 2013, Erfanian and collaborators introduced certain probabilistic concepts in this context [2]. Shelash et al developed a method to determine the number of subgroups in a finite abelian p-group [8]. In 2019, Lazorec introduced the notion of finite groups, the relative cyclic subgroup commutativity degrees [3]. This article will analyze and compute the number of elements of maximal degree in the groups $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_{p^m q^n}$. Additionally, a formula for calculating the probability of maximal degree, denoted $P_{\max}(G)$, will be derived.

IV. MAXIMALITY GRAPH

In this section, we will present a new graph, which is named the Maximality graph and denoted by $\Gamma_{Max}(G)$. Let $\Gamma(V, E)$ be a finite graph, $V(\Gamma)$ is a set of all vertices and $E(\Gamma)$ is a set of all edges of the graph Γ,

The maximal graph is a finite graph, denoted by $\Gamma_{Max}(V, E)$, where $V(\Gamma_{Max})$ is a set of all vertices that are elementary of the group G and $E(\Gamma_{Max})$ is a set of all edges of the graph Γ_{Max}, for any two elements x and y , they are joined if $\langle x, y\rangle$ is a maximal subgroup of the group G.

We recall that the number of edges of any graph is computed by $|E(\Gamma)| = \frac{\sum_{\forall v} deg_v}{2}$.

Now, we will introduce some of the relations between the number of edges and the probability degree of maximal subgroups.

Definition 4.1 *Let Γ be a finite graph, the degree sequence matrix of the graph Γ is defined by :*

$$\Delta(\Gamma) = \begin{pmatrix} deg_{v_1} & deg_{v_2} & \cdots & deg_{v_k} \\ \lambda_1 & \lambda_2 & \cdots & \lambda_k \end{pmatrix}$$

Where deg_{v_i} is a degree of vertex and λ_i is a multiple of a degree vertex deg_{v_i} , where $i = 1, \cdots, k$

For example $\Delta(K_3) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ Since graph K_3 has three vertices, a degree vertex for any of them $deg_{v_i} = 2$.

Lemma 4.2 *Let G be isomorphic to $C_{p^\alpha}, \alpha \geq 2$, the degree of vertices is given by:*

$$deg(x) = \begin{cases} p^{\alpha-1} - 1 & if \;\; x = a^{tp} \\ \varphi(p^{\alpha-1}) & if \;\; x = a^{tp^r}, r \geq 2 \end{cases}$$

Proof.

The maximal degree elements are defined by $max\, d(x, G) = \{y \in G: \langle x, y\rangle \leq \Gamma_{max} G\}$ .This means, for any elements if p divides the order of it, then it has maximal degree,

$$max\, d(a^{tp}, G) = |\{e, a^p, a^{2p}, \cdots, \}| = p^{\alpha-1}$$

And we must withdraw ι^{tp} to get on the $deg(x) = max\, d(\iota^{tp}, G) - 1 = p^{\alpha-1} - 1$, the count of all such elements equals $\varphi(p^{\alpha-1})$, on the other hand, all elements of type $\iota^{tp^r}, r \geq 2$, have $max\, d(x, G) = \iota^{tp}, \forall t \in \mathbb{Z}^+$.

This mean it has $max\, d(x, G) = \varphi(p^{\alpha-1})$ .There are exactly $p^\alpha - \varphi(p^{\alpha-1})$ elements of this form.

Thus,the $deg(\iota^{tp^r}) = \varphi(p^{\alpha-1})$.

Proposition 4.3 *Let G be isomorphic to $C_{p^\alpha, \alpha\geq 2}$,the following is hold.*

1. $Max\, d(G) = \varphi(p^{\alpha-1})(p^{\alpha-1} + p^{\alpha-2})$;
2. $\sum_{\forall x \in G} \; deg(x) = |E(x)|$;

Proof.

1- $Max\, d(G) = \varphi(p^{\alpha-1}) Max\, d(x, G) \; + \; p^{\alpha-1} - \varphi(p^{\alpha-1} Max\, d(x, G))$

$$= p^{\alpha-1}\varphi(p^{\alpha-1}) + ((p^{\alpha-1} - \varphi(p^{\alpha-1}))\varphi(p^{\alpha-1})$$
$$= \varphi(p^{\alpha-1})(p^{\alpha-1} + p^{\alpha-2})$$

2- see [12]

The table below represents the case $n = p^\alpha = 3^\alpha$

TABLE1 1: MAXIMALITY DEGREE OF $n = 3^\alpha$

α	$n = 3^\alpha$	$E(\Gamma)$
2	9	3
3	27	33
4	81	315
5	3^5	2889
6	3^6	26163
7	3^7	235953
8	3^8	2125035
9	3^9	19129689
10	3^{10}	172180323

Lemma 4.4 *Let Γ be a graph and G a finite group, the $\sum_{\forall x\in G} deg_x = 2Max\ d(x,G)$.*

$$|E(\Gamma)| = \frac{\varphi(p^{\alpha-1})(p^{\alpha-1}+p^{\alpha-2}-1)}{2}$$

Proof.

$$|E(\Gamma)| = \frac{\sum deg(\iota^{tp}) + \sum deg(\iota^{kp^r})}{2}, r \geq 2\ t,k \in \mathbb{Z}^+$$

$$= \frac{\varphi(p^{\alpha-1})(p^{\alpha-1}-1) + \varphi(p^{\alpha-1})(p^{\alpha-1}-\varphi(p^{\alpha-1}))}{2}$$

$$= \frac{\varphi(p^{\alpha-1})p^{\alpha-1} - \varphi(p^{\alpha-1}) + \varphi(p^{\alpha-1})p^{\alpha-1} - \varphi(p^{\alpha-1})\varphi(p^{\alpha-1})}{2}$$

$$= \frac{\varphi(p^{\alpha-1})(2p^{\alpha-1}-p^{\alpha-1}+p^{\alpha-2}-1)}{2}$$

$$= \frac{\varphi(p^{\alpha-1})(p^{\alpha-1}+p^{\alpha-2}-1)}{2}$$

Corollary 4.5 *Let $G \cong C_n$ be a cyclic group, the following three cases are available:*

1- If $n = p^{\alpha}$, then

$$\Delta(\Gamma_{Max})(C_{p^{\alpha}}) = \begin{pmatrix} \varphi(p) & 0 \\ p & \varphi(p^{\alpha}) \end{pmatrix},$$

and the number of edges is equal to :

$$|E(\Gamma_{Max}C_{p^{\alpha}}| = \frac{p\varphi(p)}{2}$$

Thus,

$$P_{Max}(C_{p^{\alpha}}) = \frac{2|E(\Gamma_{Max}C_{p^{\alpha}})| + \varphi(p)}{2}$$

2- If $n = p_1p_2$, then:

$$\Delta(\Gamma_{Max})(C_{p_1p_2}) = \begin{pmatrix} \varphi(p_1)+\varphi(p_2) & \varphi(p_1) & \varphi(p_2) \\ 1 & \varphi(p_1) & \varphi(p_2) \end{pmatrix}$$

and the number of edges is equal to :

$$|E(\Gamma_{Max}C_{p_1p_2})| = \frac{\varphi(p_1)(1+\varphi(p_1)) + \varphi(p_2)(1+\varphi(p_2))}{2}$$

Thus,

$$P_{Max}(C_{p_1p_2}) = \frac{2|E(\Gamma_{Max}(C_{p_1p_2}))| + (\varphi(p_1)+\varphi(p_2))}{|C_{p_1p_2}|^2}$$

3- If $n = p_1p_2p_3$, then explains how to calculate the degree vertex and the simple # is referred to as the product of the degree vertex, as shown in the following table

TABLE 2: MAXIMALITY DEGREE OF $n = p_1p_2p_3$

v_i	$d(v_i)$	#
(e)	$(\tau(n)\Sigma p_i) - 1$	1
(ι^{p_1})	$p_2p_3 - 1$	$\varphi(p_2)\varphi(p_3) - 1$
(ι^{p_2})	$p_1p_3 - 1$	$\varphi(p_1)\varphi(p_3) - 1$
(ι^{p_3})	$p_1p_2 - 1$	$\varphi(p_1)\varphi(p_2) - 1$
$(\iota^{p_1}\iota^{p_2})$	$[\varphi(p_1)+\varphi(p_2)]p_3$	$\varphi(p_3)$
$(\iota^{p_1}\iota^{p_3})$	$[\varphi(p_1)+\varphi(p_3)]p_2$	$\varphi(p_2)$
$(\iota^{p_2}\iota^{p_3})$	$[\varphi(p_2)+\varphi(p_3)]p_1$	$\varphi(p_1)$

and the number edges is equal to :

$$|E(\Gamma_{Max}C_{\Pi p_i})| = \frac{\Sigma deg(v_i)}{2}$$

$$P_{Max}(C_{\Pi p_i}) = \frac{2|E(\Gamma_{Max}(C_{\Pi p_i}))| + \Pi(\varphi(p_i)) - 1}{|C_n|^2}$$

In the following example, we will take three cases,

- If $n = p^{\alpha}$, let $G = C_9$, where

$n = 9 = 3^2 \Rightarrow p = 3, \alpha = 2, \varphi(3) = 2,$

$$C_9 = \{e, \iota, \iota^2, a^3, \cdots, \iota^8\}$$

then the matrix degree of edges is equal to :

$$\Delta(\Gamma_{Max})(C_9) = \begin{pmatrix} 2 & 0 \\ 3 & 6 \end{pmatrix} c$$

TABLE 3: MAXIMALITY DEGREE ELEMENTS OF C_9

	e	ι	ι^2	ι^3	ι^4	ι^5	ι^6	ι^7	ι^8
e	0	0	0	1	0	0	1	0	0
ι	0	0	0	0	0	0	0	0	0
ι^2	0	0	0	0	0	0	0	0	0
ι^3	1	0	0	1	0	0	1	0	0
ι^4	0	0	0	0	0	0	0	0	0
ι^5	0	0	0	0	0	0	0	0	0
ι^6	1	0	0	1	0	0	1	0	0
ι^7	0	0	0	0	0	0	0	0	0
ι^8	0	0	0	0	0	0	0	0	0

Such that,

$$Max_{C_9}(e) = \{\iota^3, \iota^6\} \Rightarrow |MaxC_9(e)| = 2.$$

$$Max_{C_9}(\iota^3) = \{e, \iota^3, \iota^6\} \Rightarrow |Max_{C_9}(\iota^3)| = 2\ ,$$

Without a loop,

$$Max_{C_9}(\iota^6) = \{e, \iota^3, \iota^6\} \Rightarrow \left|Max_{C_9}(\iota^6)\right| = 2$$

Without a loop,

$$\therefore |E(\Gamma_{Max}(C_9))| = 3.$$

Or by using Lemma 4.4,

$$|E(\Gamma)| = \frac{\varphi(p^{\alpha-1})(p^{\alpha-1}+p^{\alpha-2}-1)}{2}$$

$$= \frac{\varphi(3^{2-1})(3^{2-1}+3^{2-2}-1)}{2} = \frac{2(3+0)}{2} = 3$$

Then, by Corollary 4.5(1)

$$P_{Max}(C_{p^{\alpha}}) = \frac{2|E(\Gamma_{Max}C_{p^{\alpha}})| + \varphi(p)}{2}$$

$$P_{Max}(C_{3^2}) = \frac{2*3+2}{9^2} = \frac{8}{81}$$

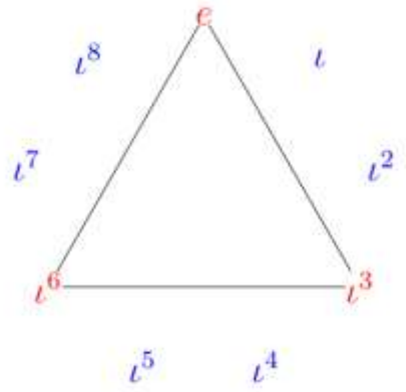

Figure 1: Maximal Graph of C_9

- Let $G = C_{15}$, $p_1 = 5, p_2 = 3$, then the matrix degree of edges is equal to :

$$\Delta(\Gamma_{Max})(C_{15}) = \begin{pmatrix} 6 & 4 & 2 \\ 1 & 4 & 2 \end{pmatrix}$$

by Corollary 4.5(2)

$$P_{Max}(C_{15}) = \frac{\varphi(5)(5+1) + \varphi(3)(3+1)}{(15)^2}$$
$$= \frac{4(6) + 2(4)}{225} = \frac{32}{225}$$

and

$$|E(\Gamma_{Max}(C_{15}))| = \frac{\varphi(5)[1+\varphi(5)] + \varphi(3)[1+\varphi(3)]}{2}$$
$$= \frac{4(1+4) + 2(1+2)}{2} = \frac{26}{2} = 13$$

Thus,

$$P_{Max}(C_{15}) = \frac{2|E(\Gamma_{Max}(C_{15}))| + (\varphi(5) + \varphi(3))}{15^2}$$
$$= \frac{2(13) + 4 + 2}{15^2} = \frac{32}{225}$$

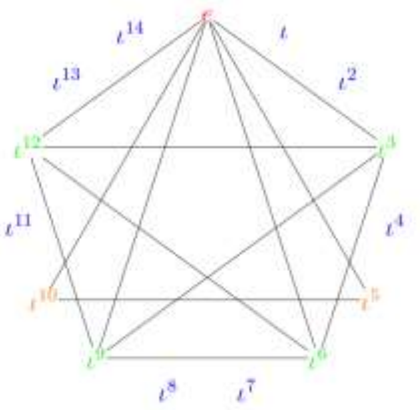

Figure 2: Maximal Graph of C_{15}

Let $G = C_{105}$, where

$$n = 105 = 3 * 5 * 7 \Rightarrow p_1 = 3, p_2 = 5, p_3 = 7,$$
$$C_{105} = \{e, \iota, \iota^2, \iota^3, \cdots, \iota^{104}\}$$

, then the matrix degree of edges is equal to :

$$\Delta(\Gamma_{Max})(C_{105}) = \begin{pmatrix} 44 & 14 & 42 & 30 & 40 & 20 & 34 \\ 1 & 9 & 6 & 2 & 4 & 13 & 25 \end{pmatrix}$$

Such that,

$$|e| = \{\iota^3, \iota^5, \iota^6, \iota^7, \iota^9, \iota^{10}, \iota^{12}, \iota^{14}, \iota^{18}, \iota^{20}, \iota^{24}, \iota^{25}, \iota^{27}, \iota^{28}, \iota^{33}, \iota^{39}, \iota^{40}, \iota^{48}, \iota^{49}, \iota^{50}, \iota^{51}, \iota^{54}, \iota^{55}, \iota^{56}, \iota^{57}, \iota^{65}, \iota^{66}, \iota^{69}, \iota^{72}, \iota^{77}, \iota^{78}, \iota^{80}, \iota^{81}, \iota^{85}, \iota^{87}, \iota^{91}, \iota^{93}, \iota^{95}, \iota^{96}, \iota^{98}, \iota^{99}, \iota^{100}, \iota^{102}, \iota^{104}\} = 44$$

$$|\iota^3| = \{e, \iota^3, \iota^6, \iota^9, \iota^{12}, \iota^{15}, \iota^{18}, \iota^{21}, \iota^{24}, \iota^{27}, \iota^{30}, \iota^{33}, \iota^{36}, \iota^{39}, \iota^{42}, \iota^{45}, \iota^{48}, \iota^{51}, \iota^{54}, \iota^{57}\iota^{60}, \iota^{63}, \iota^{66}, \iota^{69}, \iota^{72}, \iota^{75}, \iota^{78}, \iota^{81}, \iota^{84}, \iota^{87}, \iota^{90}, \iota^{93}, \iota^{96}, \iota^{99}, \iota^{102}\} = 35 \Rightarrow 34 * 25 = 850$$

$$|\iota^5| = \{e, \iota^5, \iota^5, \iota^{10}, \iota^{15}, \iota^{20}, \iota^{25}, \iota^{30}, \iota^{35}, \iota^{40}, \iota^{45}, \iota^{50}, \iota^{55}, \iota^{60}, \iota^{65}, \iota^{70}, \iota^{75}, \iota^{80}, \iota^{85}, \iota^{90}, \iota^{95}, \iota^{100}\}$$
$$= 21 \Rightarrow 20 * 13 = 260$$

$$|\iota^7| = \{e, \iota^7, \iota^{14}, \iota^{21}, \iota^{28}, \iota^{35}, \iota^{42}, \iota^{49}, \iota^{56}, \iota^{63}, \iota^{70}, \iota^{77}, \iota^{84}, \iota^{91}, \iota^{98}\} = 15 \Rightarrow 14 * 9 = 126$$

$$|\iota^{15}| = \{\iota^3, \iota^6, \iota^9, \iota^{12}, \iota^{18}, \iota^{21}, \iota^{24}, \iota^{27}, \iota^{33}, \iota^{36}, \iota^{39}, \iota^{42}, \iota^{48}, \iota^{51}, \iota^{54}, \iota^{57}, \iota^{63}, \iota^{66}, \iota^{69}, \iota^{72}, \iota^{78}, \iota^{81}, \iota^{84}, \iota^{87}, \iota^{93}, \iota^{96}, \iota^{99}, \iota^{102}\iota^5, \iota^{10}, \iota^{20}, \iota^{25}, \iota^{35}, \iota^{40}, \iota^{50}, \iota^{55}, \iota^{65}, \iota^{70}, \iota^{80}, \iota^{85}, \iota^{95}, \iota^{100}\} = 42 \Rightarrow 42 * 6 = 252$$

$$|\iota^{21}| = \{\iota^3, \iota^6, \iota^9, \iota^{12}, \iota^{15}, \iota^{18}, \iota^{24}, \iota^{27}, \iota^{30}, \iota^{33}, \iota^{36}, \iota^{39}, \iota^{45}, \iota^{48}, \iota^{51}, \iota^{54}, \iota^{57}, \iota^{60}, \iota^{66}, \iota^{69}, \iota^{72}, \iota^{75}, \iota^{78}, \iota^{81}, \iota^{87}, \iota^{90}, \iota^{93}, \iota^{96}, \iota^{99}, \iota^{102}, \iota^7, \iota^{14}, \iota^{28}, \iota^{35}, \iota^{49}, \iota^{56}, \iota^{70}, \iota^{77}, \iota^{91}, \iota^{98}\}$$

$$|\iota^{35}| = \iota^5, \iota^{10}, \iota^{15}, \iota^{20}, \iota^{25}, \iota^{30}, \iota^{40}, \iota^{45}, \iota^{50}, \iota^{55}, \iota^{60}, \iota^{65}, \iota^{75}, \iota^{80}, \iota^{85}, \iota^{90}, \iota^{95}, \iota^{100}, \iota^7, \iota^{14}, \iota^{21}, \iota^{28}, \iota^{42}, \iota^{49}, \iota^{56}, \iota^{63}\iota^{77}, \iota^{84}, \iota^{91}, \iota^{98}\}$$
$$= 30 \Rightarrow 30 * 2 = 60$$

by Corollary 4.5(3)

$$|E(\Gamma_{Max}(C_{105}))| = \frac{1752}{2} = 876$$

Thus,

$$P_{Max}(C_{105}) = \frac{2|E(\Gamma_{Max}(C_{105}))| + \Pi(\varphi(p_i)) - 1}{105^2}$$
$$= \frac{2(876) + (\varphi(3)\varphi(5)\varphi(7) - 1)}{105^2}$$
$$= \frac{1752 + 47}{11025} = \frac{1799}{11025}$$

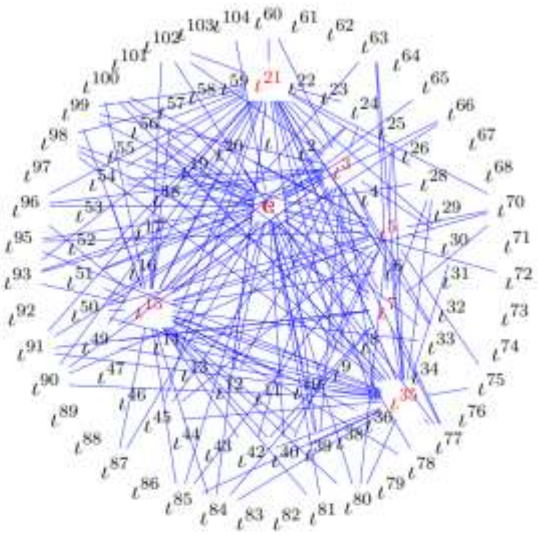

Figure 3: Maximal Graph of C_{105}

V. Conclusions

When n it is large, and computing the Maximal degree directly from its general definition becomes computationally challenging. To address this, we developed three algorithms in this study to compute the Maximal degree probabilities for the cyclic group C_n. Specifically, we consider three cases:

- the first case corresponds to when $n = p$ is prime, yielding the group C_p;
- the second case applies when $n = p_1 p_2$ is a product of two distinct primes, giving the group $C_{p_1 p_2}$;
- the third case handles $n = p_1 p_2 p_3$, the product of three distinct primes, corresponding to $C_{p_1 p_2 p_3}$.

Extending this approach to other finite groups, such as dihedral groups, is expected to further advance the field.

Acknowledgement I extend my sincere thanks to my professor (Dr.Hayder Baqer), who had a great credit in the completion of this research.

References

[1]. W. H. Gustafson, What is the probability that two group elements commute?, *Amer. Math. Monthly* **80** (1973), 1031–1034.

[2]. A. Erfanian, K. Moradipour, and N. H. Sarmin, Finite groups with the same commuting probability, *Res. J. Appl. Sci. Eng. Technol.* **5** (2013), no. 13, 3525–3528.

[3]. M. S. Lazorec, Relative cyclic subgroup commutativity degrees of finite groups, *Filomat* **33** (2019), no. 13, 4021–4032.

[4]. G. Chartrand and P. Zhang, *Introduction to Graph Theory*, Posts and Telecom Press, Beijing, China, 2006.

[5]. D. W. Jensen and M. K. Keane, A number-theoretic approach to subgroups of dihedral groups, *SUT J. Math.* **37** (1990), no. 1, 1–12.

[6]. D. B. West, *Introduction to Graph Theory*, Prentice-Hall of India Pvt. Ltd., New Delhi, 2003.

[7]. G. James and M. Liebeck, *Representations and Characters of Groups*, 2nd ed., Cambridge University Press, New York, 2001.

[8]. H. B. Shelash and A. R. Ashrafi, Wielandt subgroups of certain finite groups, *Math. Notes* **177** (2020), 121–131.

[9]. The GAP Group, *GAP – Groups, Algorithms, and Programming*, Version 4.8.9, 2019. https://www.gap-system.org

[10]. J. M. Oh, Y. Kim, and K. W. Hwang, The number of chains of subgroups in the lattice of subgroups of the dicyclic group, *Discrete Dyn. Nat. Soc.* (2012), Art. ID 760246, 17 pp.

[11]. C. Gary and P. Zhang, Introduction to Graph Theory, Posts and Telecom Press, Beijing, China, 2006.

[12]. S.Naduvath, Lecture notes on Graph Theory, Published by Centre for Studies in Discrete Mathematics, Vidya Academy of Science Technology, Thrissur-680501, Kerala, India.http://www.csdm.org.in November 2017