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Feebly Limit Sets and Cartan G – space

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Abstract

The main goal of this work is to create a general type of G – space , namely, feebly Cartan G – space and a new type of limit sets , namely, feebly limit sets $\Lambda^f(x)$, $J^f(x)$ and, give some properties and some equivalent statement of these concept also we explain the relationship among the definitions feebly Cartan G – space and $\Lambda^f(x)$, $J^f(x)$.

Introduction

One of the very important concepts in topological groups is the concept of group actions and there are several types of these actions. This paper studies an important class of actions namely, feebly Cartan G – space.

Let B be a subset of a topological space (X,T). We denote the closure of B and the interior of B by $\overline{B}$ and B^{o} , respectively. A subset B of (X, T) is said to be semi – open (s.c) if there exists an open subset O of X such that $O \subseteq B \subseteq \overline{O}$ .The complement of a semi–open set is defined to be semi–closed (s.c) and the intersection of all semi – closed subset of X containing B is defined to be semi – closure of B and denote by $\overline{B}^{S}$ .The subset B of (X, T) is called feebly open (f – open) if there is an open set U such that $U \subseteq B \subseteq \overline{U}^{s}$.

The complement of a feebly open set is defined to be a feebly closed (f – closed) [6]. If $B \subseteq \overline{B^{o}}^{o}$ then B is called $\propto$ - open and the family $T^{\propto}$ of all $\propto$ - sets in (X,T) is a topology on X larger than T, [10]. In [5] that the subset B of X is f – open if and only if $B \in T^{\propto}$.In section one, we introduce some definitions, remarks, propositions, theorems which are needed in the next sections. In section two, we define the sets $\Lambda^f(x)$, $J^f(x)$ and prove its properties, also we give some equivalent statement of $\Lambda^f(x)$, $J^f(x)$ .In section three, we defines feebly thin sets and feebly Cartan G – space and give some propositions and theorems which related with this concepts and shown the relationship among the feebly Cartan G – space and the sets $\Lambda^f(x)$ and $J^f(x)$.

1. Preliminaries

__1.1 Definition__ [9]: A subset B of a space X is called feebly open (f – open) set if there exists an open subset U of X such that $U \subseteq B \subseteq \overline{U}^{s}$. The complement of a feebly open set is defined to be a feebly closed (f – closed) set. The collection of all f – open sets in space X is denoted by T^f .

1.2 Definition: A subset B of a space X is called feebly neighborhood (f – neighborhood) of $x\in X$ if there is an open subset O of X such that $x\in O\subseteq B$.

1.3 Definition [8]:A subset A of space X is called f – compact set if every f – open cover of A has a finite sub cover. If $A=X$ then X is called a f – compact space.

1.4 Definition [2]:

(i) A subset A of space X is called f - relative compact if $\overline{A}$ is f – compact.

(ii) A space X is called f – locally f – compact if every point in X has an f – relative compact f – neighborhood.

1.5Definition [7, 10,12]: Let X and Y be spaces and f: $X\to Y$ be a function. Then:

(i) f is called feebly continuous(f – continuous) function if $f^{-1}(A)$ is an f – open set in X for every open set A in Y.

(ii) f is called feebly irresolute (f – irresolute) function if $f^{-1}(A)$ is an f – open set in X for every f- open set A in Y.

1.6 Definition [4]:

Let $(\chi_d)_{d\in D}$ be a net in a space X , $x\in X$. Then :

i) $(\chi_d)_{d\in D}$ f – converges to x (written $\chi_d \xrightarrow{f} x$) if $(\chi_d)_{d\in D}$ is eventually in every f – neighborhood of x . The point x is called an f – limit point of $(\chi_d)_{d\in D}$, and the notation "$\chi_d \xrightarrow{f} \infty$" is mean that $(\chi_d)_{d\in D}$ has no f – convergent subnet.

ii) $(\chi_d)_{d\in D}$ is said to have x as an f – cluster point [written $\chi_d \alpha_f\ x$] if $(\chi_d)_{d\in D}$ is frequently in every f - neighborhood of x .

1.7 Proposition [2]: A space (X, T) is an f – compact space if and only if every net in X has f – cluster point in X.

1.8 Proposition [2]: Let X be a space and $A \subseteq X$, $x\in X$. Then $x\in \overline{A}^{f}$ if and only if there exists a net $(\chi_d)_{d\in D}$ in A and $\chi_d \xrightarrow{f} x$.

1.9 Remark [1]: Let X be a space, then:

(i) If $(\chi_d)_{d\in D}$ is a net in X, $x\in X$ such that $\chi_d \xrightarrow{f} x$ then $\chi_d \to x$.

(ii) If $(\chi_d)_{d\in D}$ is a net in X, $x\in X$ such that $\chi_d\ \alpha\ x$ then $\chi_d\ \alpha_f\ x$.

(iii) If $(\chi_d)_{d\in D}$ is a net in X, $x\in X$. Then $\chi_d \xrightarrow{f} x$ in (X, T) if and only if $\chi_d \to x$ in (X, T^f), and $\chi_d\ \alpha_f\ x$ in (X, T) if and only if $\chi_d\ \alpha\, x$ in (X, T^f).

1.10 Theorem: Let $(\chi_d)_{d\in D}$ be a net in a space (X, T) and x_o in X. Then $\chi_d\ \alpha_f\ x_o$ if and only if there exists a subnet $(\chi_{dm})_{dm\in D}$ of $(\chi_d)_{d\in D}$ such that $\chi_{dm} \xrightarrow{f} x_o$.

Proof: By Remark (1.9,.iii).

1.11 Remark :

(i)A function f:(X, T)$\to$(Y,τ) is f–continuous function if and only if f:(X, T^f)$\to$(Y,τ) is continuous.

(ii)A function f: (X, T) $\to$(Y,τ) is f – irresolute function if and only if f:(X, T^f)$\to$ (Y, τ^f) is continuous.

1.12 Proposition: Let f: $X\to Y$ be a function, $x\in X$. Then:

(i) f is f – continuous at x if and only if whenever a net $(\chi_d)_{d\in D}$ in X and $\chi_d \xrightarrow{f} x$

then $f(\chi_d) \longrightarrow f(x)$.

(ii) f is f – irresolute at x if and only if whenever a net $(\chi_d)_{d\in D}$ in X and $\chi_d \xrightarrow{f} x$

then $f(\chi_d) \xrightarrow{f} f(x)$.

Proof: (i) $\Rightarrow$ Let $x\in X$ and $(\chi_d)_{d\in D}$ be a net in X such that $\chi_d \xrightarrow{f} x$ [To prove that $f(\chi_d) \longrightarrow f(x)$]. Let V be a open neighborhood of $f(x)$. Since f is f – continuous, then $f^{-1}(V)$ is f – neighborhood of x, but $\chi_d \xrightarrow{f} x$, then there is $\beta\in D$ such that $\chi_d \in f^{-1}(V)$, $\forall\ d \geq \beta$. Then $f(\chi_d) \in f(f^{-1}(V)) \subseteq V$. Thus $f(\chi_d)$ is eventually in every open neighborhood of $f(x)$, then $f(\chi_d) \longrightarrow f(x)$.

$\Leftarrow$ Suppose that f is not f – continuous. Then there exists $x\in X$ such that f is not f – continuous at x. Then there exists an open set B in Y such that $f(x) \in B$ and $f(A) \not\subset B$ for each A is an f – open in X such that $x\in A$. Thus there exists $\chi_A\in A$ and $f(\chi_A)\notin B$ for each A is f – open in X . Then $\chi_A \xrightarrow{f} x$. But $f(\chi_A)\notin B$ for each $A\in N_f(x)$, then $f(\chi_A)$ is not convergent to $f(x)$ and this is a contradiction. Then f is f – continuous.

(ii) $\Rightarrow$ Let $x\in X$ and $(\chi_d)_{d\in D}$ be a net in X such that $\chi_d \xrightarrow{f} x$ .Then by Remark(1.9,iii) $\chi_d \longrightarrow x$ in (X, T^f). Since f: $(X,T) \longrightarrow (Y,\tau)$ is f – irresolute (Y,τ), then by Remark(1.11,ii) f: $(X, T^f) \longrightarrow (Y, \tau^f)$ is continuous. Thus $f(\chi_d) \longrightarrow f(x)$ in (Y,τ^f), so by Remark (1.9,iii) $f(\chi_d) \xrightarrow{f} f(x)$.

$\Leftarrow$By Remark (1.9,iii) and Remark (1.11,ii) we have f: $(X, T^f) \longrightarrow (Y, \tau^f)$ is continuous . Then f is f – irresolute.

<u>**1.13 Definition** [3]:</u> A topological transformation group is a triple (G,X,φ) where G is a T_2–topological group, X is a T_2 – topological space and $\varphi : G\times X \rightarrow X$ is a continuous function such that:

(i) $\varphi(g_1,\varphi(g_2, x)) = \varphi(g_1g_2, x)$ for all $g_1,g_2\in G$, $x\in X$.

(ii) $\varphi(e, x) = x$ for all $x\in X$, where e is the identity element of G.

<u>**1.14 Remark** [3]:</u> Let X be a G – space and $x\in X$. Then:

(i) The function φ is called an action of G on X and the space X together with φ is called a G – space (or more precisely left G – space).

(ii) The subspace $\{g.x \ / \ g\in G\}$ is called the orbit (trajectory) of x under G, which denoted by Gx [or $\gamma(x)$], and for every $x\in X$ the stabilizer subgroup G_x of G at x is the set $\{g\in G/\ gx = x\}$.

(iii) $Ag= r_g\ (A) =\{ag{:}a\in A\}$;$Ag$ is called the left translate of A by g.

(ix) $gA= L_g\ (A) =\{ga{:}\ a\in A\}$; gA is called the right translate of A by g.

<u>**1.15 Proposition**:</u>Let G be a topological group and $(g_d)_{d\in D}$ be a net in G. Then:

(i)If $g_d \xrightarrow{f} e$, where e is identity element of G, then $gg_d \xrightarrow{f} g$ (or $g_d\, g \xrightarrow{f} g$) for each $g\in G$.

(ii) If $g_d \xrightarrow{f} \infty$, then $gg_d \xrightarrow{f} \infty$ (or $g_d\, g \xrightarrow{f} \infty$) for each $g\in G$.

(iii) If $g_d \xrightarrow{f} \infty$, then $g_d^{-1} \xrightarrow{f} \infty$.

Proof: i) Since $r_g:G \to G$ is continuous and open , where r_g is right translation by g. then r_g is f – irresolute. Thus by Proposition (1.12,ii) $g_d g \xrightarrow{f} g$ for each $g\in G$.

ii) Let $g_d \xrightarrow{f} \infty$ and $g\in G$. suppose that $g_d g \xrightarrow{f} g_1$, for some $g_1 \in G$. Since r_g is f irresolute, then by Proposition(1.12,ii) $r_g^{-1}(g_d g) \xrightarrow{f} r_g^{-1}(g_1)$ .Then $g_d \xrightarrow{f} g_1 g^{-1}$, a contradiction. Thus $g_d g \xrightarrow{f} \infty$.

iii) Let $g_d^{-1} \xrightarrow{f} g$. Since the inversion map of a topological group G, v: $G \to G$ is f – irresolute, then $g_d \xrightarrow{f} g^{-1}$. Thus if $g_d \xrightarrow{f} \infty$, then $g_d^{-1} \xrightarrow{f} \infty$.

1.16 Proposition: If (G, X, φ) is a topological transformation group, then φ is f – irresolute.

Proof: Let $A\times B$ is an open set in $G\times X$, then $\varphi(A\times B) = AB$. Since $AB = \{ x\in X \,/\, x = ab, a\in A, b\in B\} = \bigcup_{a\in A} aB = \bigcup_{a\in A} \varphi(B)$. Since φ_a: $X \to X$ is homeomorphism from X on itself such that $a\in G$. Then aB is an open set in X, so $\bigcup_{a\in A} aB = AB$ is open. Since φ is continuous and open function, then its clear that the action φ is an f – irresolute function.

2 – Feebly limit sets of a point:

From now on, in this section by G – space is meant a completely regular topological T_2 – space X on which an f – locally f – compact, non – compact, T_2 – topological group G acts continuously on the left.

2.1 Definition: Let X be a G – space and $x\in X$. Then:

(i) $\Lambda^f(x)$ ={$y\in X$: there is a net $(g_d)_{d\in D}$ in G with $g_d \xrightarrow{f} \infty$ such that $g_d x \xrightarrow{f} y$} is called feebly limit set of x.

(ii) $J^f(x)$ ={$y\in X$: there is a net $(g_d)_{d\in D}$ in G and there is a net $(\chi_d)_{d\in D}$ in X with $g_d \xrightarrow{f} \infty$ and $\chi_d \xrightarrow{f} x$ such that $g_d x \xrightarrow{f} y$} is called feebly first prolongation limit set of x.

2.2 Proposition:Let X be a G – space and $x\in X$. Then:

(i) $\Lambda^f(x)$ and $J^f(x)$ are invariant sets under G.

(ii) The orbit $\gamma(x)$ is f – closed if and only if $\Lambda^f(x)$ is a subset of $\gamma(x)$.

(iii) If $x \notin \Lambda^f(x)$, then the stabilizer subgroup G_x of G is f – compact.

(iv) if $\Lambda^f(x) = \phi$, for each $x\in X$. Then the orbit $\gamma(x)$ is not f – compact.

(iv) $\overline{\gamma(x)}^f = \gamma(x) \cup \Lambda^f(x)$

(v) $y\in J^f(x)$ if and only if $x\in J^f(y)$.

(vi) If X is discrete G – space, then $\Lambda^f(x) = J^f(x)$ for each $x\in X$.

(vii) If $x\in J^f(x)$, then for each $y\in\gamma(x)$, $y\in J^f(y)$.

(viii) If $y\in J^f(x)$, then for each $z\in\gamma(x)$, $y\in J^f(z)$.

(x) $g\Lambda^f(x) = \Lambda^f(gx) = \Lambda^f(x)$ and $gJ^f(x) = J^f(gx) = J^f(x)$ for each $g\in G$.

Proof: i) Let $y\in \Lambda^f(x)$ and $g\in G$. Then there is a net $(g_d)_{d\in D}$ in G with $g_d \xrightarrow{f} \infty$ and $g_d x \xrightarrow{f} y$. It is clear that $(gg_d)_{d\in D}$ is a net in G with $gg_d \xrightarrow{f} \infty$.Since the action is f – irresolute, thus

$(gg_d).x \xrightarrow{f} gy$ which implies that $gy \in \Lambda^f(x)$ and hence $\Lambda^f(x)$ is invariant. The proof of $J^f(x)$ is similar.

ii)$\Rightarrow$ Let $y \in \Lambda^f(x)$, then there is a net $(g_d)_{d\in D}$ in G such that $g_d \xrightarrow{f} \infty$ and $g_d x \xrightarrow{f} y$. Since $g_d x \in \gamma(x)$ and $(g_d x)_{d\in D}$ is a net in $\gamma(x)$, then by Proposition (1.8) $y \in \overline{\gamma(x)}^f$. But $\gamma(x)$ is f – closed then $y \in \gamma(x)$, so $\Lambda^f(x) \subseteq \gamma(x)$.

$\Leftarrow$ Let $y \in \overline{\gamma(x)}^f$. Then there exists $(y_d)_{d\in D}$ is a net in $\gamma(x)$ such that $y_d \xrightarrow{f} y$, then $\forall\, d \in D$ there is $g_d \in G$ such that $y_d = g_d x$. Then $(g_d)_{d\in D}$ is a net in G and $g_d x \xrightarrow{f} y$. Now either $g_d \xrightarrow{f} g$ or $g_d \xrightarrow{f} \infty$. If $g_d \xrightarrow{f} g$ then $g_d x \xrightarrow{f} gx = y$, which implies that $y \in \gamma(x)$. If $g_d \xrightarrow{f} \infty$, then $y \in \Lambda^f(x) \subseteq \gamma(x)$, then $\gamma(x)$ is f – closed.

(iii) Let $x \notin \Lambda^f(x)$ and suppose that G_x is not f – compact . Then there is a net $(g_d)_{d\in D}$ in G such that $g_d \xrightarrow{f} \infty$. Since $g_d x = x$, i.e. $g_d x \xrightarrow{f} x$ then $x \in \Lambda^f(x)$ which is a contradiction, thus G_x is f – compact.

(iv) Suppose that $\gamma(x)$ is f – compact. Since $\Lambda^f(x) = \phi$, then there is net $(g_d)_{d\in D}$ in G with $g_d \xrightarrow{f} \infty$, $(g_d x)_{d\in D}$ is a net in $\gamma(x)$. Since $\gamma(x)$ is f – compact, then by Proposition (1.7) $g_d x \xrightarrow{f} y$ for some $y \in X$. Hence $y \in \Lambda^f(x)$, which is a contradiction with $\Lambda^f(x) = \phi$ for each $x \in X$.

(v) The proof of (v) is obvious

(vi) let $y \in J^f(x)$, then there is a net $(g_d)_{d\in D}$ in G with $g_d \xrightarrow{f} \infty$ and there is a net $(\chi_d)_{d\in D}$ in X with $\chi_d \xrightarrow{f} x$ such that $g_d\chi_d \xrightarrow{f} y$. Put $y_d = g_d\chi_d \xrightarrow{f} y$. Then by Proposition (1.15,iii) $g_d^{-1} \xrightarrow{f} \infty$ and $g_d^{-1} y_d = g_d^{-1} g_d \chi_d = \chi_d \xrightarrow{f} x$ thus $x \in J^f(y)$. The converse is similar.

(vii) The proof of (vii) is obvious.

(viii) Let $x \in J^f(x)$ and $y \in \gamma(x)$. Since $J^f(x)$ is invariant, then for each $y \in \gamma(x)$, $y \in J^f(x)$, therefore $x \in J^f(y)$ (by v) and since $J^f(y)$ is invariant, then $y \in J^f(y)$.

(ix) Let $y \in J^f(x)$, from (v), $x \in J^f(y)$. Since $J^f(y)$ is invariant, then for each $z \in \gamma(x)$, $z \in J^f(y)$ and by (v) $y \in J^f(z)$ for each $z \in \gamma(x)$.

(x) The proof of (x) is obvious.

<u>2.3 Proposition</u>: Let X be an f – locally f – compact G – space and $x \in X$, then $x \notin \Lambda^f(x)$ if and only if there is an f – neighborhood U of x and an f – compact f – neighborhood V of e, e is the identity in G, such that $gx \notin U$ for each $g \notin V$.

<u>Proof:</u> Let $x \notin \Lambda^f(x)$ and suppose that the resulting statement is not true. i.e, for each f – neighborhood U of x and for each f – compact f – neighborhood V of e, there is point $g \notin V$ and $gx \in U$. Since X is completely regular, then by [13,Remark(1.1.12),i] there is a sequence $\{U_n\}_{n\in Z^+}$ of an f – open neighborhood of x such that $U_{n+1} \subset U_n \ldots$ and $\bigcap_{n\in Z^+} U_n = \{x\}$. Since $x \notin \Lambda^f(x)$, then G_x is f – compact. Since G is f – locally f – compact, then its clear (G, T^f) is locally compact. So there is V be a compact neighborhood of e such that $G_x \subseteq V$. Thus for each n, there is $g_n \notin V$ and $g_n x \xrightarrow{f} x$. By the hypothesis $x \notin \Lambda^f(x)$, then $(g_n)_{n\in N}$

has an f – convergent a sub net of $(g_n)_{n\in N}$. Say itself. i.e. there is $g\in G$ such that

$g_n \xrightarrow{f} g$, and hence $g_n x \xrightarrow{f} gx = x$ which means that $g\in G_x \subseteq V$ thus for $n_o \in N$, $g_n \in V$ for each $n \geq n_o$, which contradiction that $g_n \notin V$, therefore that the statement is true.

Conversely: Let the statement be true. We suppose that $x\in \Lambda^f(x)$ then there is a net $(g_d)_{d\in D}$ in G such that $g_d \xrightarrow{f} \infty$ and $g_d x \xrightarrow{f} x$, by hypothesis there exists U be an f – neighborhood of x such that $gx\notin U$ for each $g\notin V$. Since $g_d x \xrightarrow{f} x$ then there is $d_o\in D$ such that $g_d x\in U$, for each $d \geq d_o$ therefore that $g_d\in V$, which is an f – compact, thus the net $(g_d)_{d\in D}$ has an f – convergent sub net, say itself, i.e., there is a point $g\in G$ such that $g_d \xrightarrow{f} g$ which is contradiction, since $(g_d)_{d\in D}$ has no f – convergent sub net, thus $x\notin \Lambda^f(x)$.

<u>2.4 Notation</u>: Let X be a G – space and A , B be two subset of X . We mean by $((A, B))$ the set $\{g\in G \,/\, gA\cap B\neq\phi\}$.

<u>2.5 Proposition:</u> Let X be an f – locally f – compact G – space and $x, y\in X$, then $y\notin \Lambda^f(x)$ if and only if there is an f – neighborhood U of y and an f – compact f – neighborhood V of e, such that $gx\notin U$ for each $g\notin V$.

<u>Proof:</u> Let x and y be two points in X such that $y\notin \Lambda^f(x)$. Then either $\gamma(x) = \gamma(y)$ or $\gamma(x)\neq \gamma(y)$. If $\gamma(x) = \gamma(y)$ then $y = gx$, for some $g\in G$, thus we have $x\notin \Lambda^f(x)$ and $y\notin \Lambda^f(y)$ [otherwise $y\in \Lambda^f(x)$].Then by Proposition(2.3) there is an f–neighborhood U_x of x and an f – compact f – neighborhood V_1 of e such that $gx\notin U_x$ for each $g\notin V_1$, Also for y, there is an f – neighborhood U_y of y and an f – compact f – neighborhood W of e, such that $gy \notin U_y$ for each $g\notin W$. Since $((\{x\}, U_x)) \subseteq V_1$ and $((\{y\}, U_y)) \subseteq W$ then we can say that the sets $((\{x\}, U_x))$ and $((\{y\}, U_y))$ have f – compact f – closure.

Let $K_1= U_x\cap\gamma(x)$ and $K_2= U_y\cap\gamma(y)$. Then $((\{x\}, K_1)) = ((\{x\}, U_x))$ and $((\{y\}, K_2)) = ((\{y\}, U_y))$ have f – compact f – closure. Let $V_2 = ((\{x\}, U_y))$ we claim that V_2 has f – compact f – closure. If not there is a net $(g_d)_{d\in D}$ in V_2 such that $g_d \xrightarrow{f} \infty$, i.e. $g_d x\in U_y$, thus $g_d(g^{-1}y) \in ((\{y\},U_y))$. Since $g_d g^{-1} \xrightarrow{f} \infty$ and $g_d(g^{-1}y)\in U_y$, then $g_d g^{-1}\in \overline{((\{y\}, U_y))}^f$ which is f – compact. Thus the net $(g_d)_{d\in D}$ must be f – convergent, which is a contradiction. Then $((\{x\}, U_y))$ has f – compact f – closure. Let $V=V_1\cup V_2$, then V is an f – compact f – neighborhood of e and each $g\notin V$, $gx\notin U_y$ there for the statements is true. If $\gamma(x)\neq \gamma(y)$, then we suppose that the statement is not true. i.e. for each f – neighborhood U of y and for each f – neighborhood V of e there is a point $g\notin V$ and $gx \in U$. Since X is completely regular, then there is $\{U_n\}_{n\in Z^+}$ a sequence of an f – open neighborhood of y such that $U_{n+1}\subset U_n\ldots$ and $\bigcap_{n\in N} U_n = \{y\}$ and V be a f – compact f – neighborhood of e. Then for each n there is $g_n\notin V$ and $g_n x \in U_n$, since $\bigcap_{n\in N}^{\infty} U_n = \{y\}$, then $g_n x \xrightarrow{f} y$. Now, either $g_n \xrightarrow{f} \infty$ or $g_n \xrightarrow{f} g$. If $g_n \xrightarrow{f} \infty$, then $y\in \Lambda^f(x)$, which is contradiction. If $g_n \xrightarrow{f} g$ for some $g\in G$, then $g_n x \xrightarrow{f} gx = y$ which implies $\gamma(x) = \gamma(y)$ which is a contradiction. Therefore the statement is true.

Conversely: Let the statement be true. Suppose that $y\in \Lambda^f(x)$, then there is a net $(g_d)_{d\in D}$ in G with $g_d \xrightarrow{f} \infty$ such that $g_d x \xrightarrow{f} y$. Then by hypothesis there is an f – neighborhood U of y and f – compact f – neighborhood V of e, such that $g_d x \notin U$ for each $g \notin V$. Since $g_d x \xrightarrow{f} y$, then there is $d_o \in D$ such that $g_d x \in U$ for each $d_o \geq d$, therefore $g_d \in V$, which is f – compact, then $(g_d)_{d\in D}$ has an f – convergent subnet, which contradictions that $g_d \xrightarrow{f} \infty$. Hence $y \notin \Lambda^f(x)$.

2.6 Theorem: Let X be f – locally f – compact G – space and $x \in X$. Then $x \notin J^f(x)$ if and only if there is an f – neighborhood U of x and there is an f – neighborhood V of e, where e is the identity element of G, such that $gU \cap U = \phi$ for each $g \notin V$.

Proof: $\Rightarrow$We suppose that the above statement is not true, i.e., for each f – neighborhood U of x and for each f – compact f – neighborhood V of e there is $g \notin V$ such that $gU \cap U \neq \phi$. We can choose $\{U_n\}_{n\in Z^+}$ to be sequence of an f – open neighborhood of x such that $U_{n+1} \subset U_n \ldots$ and $\bigcap_{n\in Z^+} U_n = \{x\}$. Since G is f – locally f – compact , then there is an f – compact f – neighborhood of e , such that $G_x \subset V$. Thus for each n there is $g_n \notin V$ such that $g_n U_n \cap U_n \neq \phi$ i.e., there is $\chi_n \in U_n$ and $g_n \chi_n \in U_n$. Since $\bigcap_{n\in Z^+} U_n = \{x\}$, then we have $\chi_n \xrightarrow{f} x$ and $g_n \chi_n \xrightarrow{f} x$ and by hypothesis the sequence $(g_n)_{n\in N}$ has an f – convergent sub sequence , say itself, thus there is a point $g \in G$ such that $g_n \xrightarrow{f} g$, and by Proposition (1.16) the action is an f – irresolute . Then by Proposition (1.12,ii) $g_n \chi_n \xrightarrow{f} gx = x$ and hence $g \in G_x \subset V$. Therefore $g_n \in V$ for $n \geq n_o$, which is a contradiction. Thus the statement is true.

Conversely: $\Leftarrow$ Let the statement be true, we suppose that $x \in J^f(x)$. Then there is a net $(g_d)_{d\in D}$ in G with $g_d \xrightarrow{f} \infty$ and there is a net $(\chi_d)_{d\in D}$ in X with $\chi_d \xrightarrow{f} x$ such that $g_d \chi_d \xrightarrow{f} x$. Then by hypothesis, there exists U be an f – neighborhood of x and V be an f – compact f – neighborhood such that $gU \cap U = \phi$ for each $g \notin V$. Since$(\chi_d)_{d\in D}$ and $(g_d \chi_d)_{d\in D}$ are f – convergent to x , thus there is $d_o \in D$ such that $\chi_d \in U$ and $g_d \chi_d \in U$ for each $d \geq d_o$ and hence $g_d \in ((U,U))$, therefore $g_d \in V$, which is f – compact , this it must have an f – convergent sub net which is a contradiction $x \notin J^f(x)$.

3 – Feebly Cartan G - space

3.1 Definition: Let X be a G – space .A subset A of X is said to be feebly thin (f – thin) relative to a subset B of X if the set $((A, B)) = \{g \in G \mid gA \cap B \neq \phi\}$ has an f – neighborhood whose closure is f – compact in G. If A is f – thin relative to itself, then it is called f – thin.

3.2 Remark: The f – thin sets have the following properties:

(i) Since $(gA \cap B) = g(A \cap g^{-1}B)$ it follows that if A is f – thin relative to B, then B is f – thin relative to A.

(ii) Since $(gg_1A \cap g_2B) = g_2(g_2^{-1} g\, g_1A \cap B)$ it follows that if A is f – thin relative to B, then so are any translates gA and gB.

(iii) If A and B are f – relative thin and $K_1 \subseteq A$ and $K_2 \subseteq B$, then K_1 and K_2 are f –relatively thin.

(iv) Let X be a G – space and K_1, K_2 be f – compact subset of X, then $((K_1, K_2))$ is f – closed in G.

(v) If K_1 and K_2 are f – compact subset of G– space X such that K_1 and K_2 are f– relatively thin, then $((K_1, K_2))$ is an f – compact subset of G.

Proof: The prove of (i), (ii), (iii) and (v) are obvious.

(iv) Let $g\in \overline{((K_1,K_2))}^f$. Then there is a net $(g_d)_{d\in D}$ in $((K_1, K_2))$ such that $g_d \xrightarrow{f} g$. Then we have net $(k_d^1)_{d\in D}$ in K_1, such that $g_d k_d^1 \in K_2$, since K_2 is f – compact, then by Theorem (1.10) there exists a subnet $(g_{d_m} k_{d_m}^1)$ of $(g_d k_d^1)$ such that $g_{d_m} k_{d_m}^1 \xrightarrow{f} k_o^2$, where $k_o^2 \in K_2$. But $(k_{d_m}^1)$ in K_1 and K_1 is f – compact , thus there is a point $k_o^1 \in K_1$ and a subnet of $k_{d_m}^1$ say itself such that $k_{d_m}^1 \xrightarrow{f} k_o^1$. Then $g_{d_m} k_{d_m}^1 \xrightarrow{f} g k_o^1 = k_o^2$, which mean that $g\in ((K_1, K_2))$, there fore $((K_1, K_2))$ is f – closed in G.

3.3 Theorem: Let X be f – locally f – compact G – space and $x\in X$. Then $x\in J^f(x)$ if and only if x has no f – thin f – neighborhood.

Proof: $\Rightarrow$ Let $x\in J^f(x)$ and suppose that x has f – thin f – neighborhood , there is an f – neighborhood U of x such that the set $((U,U))$ has f – compact closure .By hypothesis $x\in J^f(x)$, then there is a net $(g_d)_{d\in D}$ in G with $g_d \xrightarrow{f} \infty$ and a net $(\chi_d)_{d\in D}$ in X with $\chi_d \xrightarrow{f} x$ such that $g_d\chi_d \xrightarrow{f} x$, since U is a f – neighborhood of x , thus there is $d_o\in D$ such that $\chi_d \in U$ and $g_d\chi_d \in U$ for each $d \geq d_o$. Thus $g_d\in \overline{((U,U))}$,$\forall\ d \geq d_o$, which is f – compact , and hence the net $(g_d)_{d\in D}$ must have an f – convergent subset , which is a contradiction . Therefore x has no f – thin f – neighborhood.

Conversely: $\Leftarrow$ Let x has no f – thin f – neighborhood . We suppose that $x\notin J^f(x)$, then by Theorem (2.6) there is an f – neighborhood U of x and a f – compact f – neighborhood $V$of e such that $gU\cap U = \phi$ for each $g\notin V$. In the other words, if $gU\cap U \neq \phi$, then $g\in V$, thus $((U,U)) \subseteq V$ which is f – compact . Therefore U is an f – thin f – neighborhood of x, which is a contradiction, and hence $x\in J^f(x)$.

3.4 Theorem: Let X be an f – locally f – compact G – space and x, y be two points of X. Then $y\notin J^f(x)$ if and only if there is a f – neighborhood U of x, an f – neighborhood $W$of y and an f – compact f – neighborhood V of e, where e is the identity element of G , such that $gU\cap W = \phi$ for each $g\notin V$.

Proof: $\Rightarrow$ Let $y\notin J^f(x)$:

i) If $\chi(x) = \chi(y)$. Then we have $x\notin J^f(x)$. Then by Theorem(2.6) there is an f – compact f – neighborhood V_1 of e such that $gU\cap U = \phi$ for each $g\notin V_1$. By Theorem(3.3) U is f – thin and if $y = gx$ for some $g\in G$, then gU is an f – neighborhood of y. It is clear that U and gU are f – relative thin ,i.e., $((U,gU))$ has f – compact closure . Put $W=gU$ and $V_2 = \overline{((U,W))}^f$, then $V=V_1\cup V_2$ is an f – compact f – neighborhood of e and for each $g\notin V$ then $gU\cap W = \phi$. Therefore the statement is true.

ii) Let $\chi(x) \neq \chi(y)$. We suppose that the statement is not true, i.e., for each f – neighborhood U of x, for each f – neighborhood W of y and each f – compact f – neighborhood V of e, there is $g\notin V$ and $gU\cap W\neq\phi$. We can choose $\{U_n\}_{n\in N}$ a sequence of an f – open neighborhood of x such that $U_{n+1}\subset U_n\ldots$ and $\bigcap_{n\in N} U_n = \{x\}$ and $\{W_n\}_{n\in Z^+}$ be a sequence of an f – open neighborhood of y such that $W_{n+1}\subset W_n\ldots$

and $\bigcap_{n\in N} W_n = \{y\}$, also we choose V as an f – compact f – neighborhood of e. Thus for each $n\in N$ there is $g_n \notin V$ and $g_n U_n \cap W_n \neq \phi$ i.e., there is $\chi_n \in U_n$ and $g_n\chi_n \in W_n$, since $\bigcap_{n\in N} U_n = \{x\}$, then we have $\chi_n \xrightarrow{f} x$ and $g_n\chi_n \xrightarrow{f} y$, by hypothesis $y\notin J^f(x)$, then the sequence $(g_n)_{n\in N}$ has a f – convergent sub sequence ,(say itself), i.e., there is a point $g\in G$ such that $g_n \xrightarrow{f} g$ thus $g_n\chi_n \xrightarrow{f} gx = y$, which means that $\chi(x) = \chi(y)$, which is a contradiction . Thus the statement is true.

Conversely:$\Leftarrow$ Let the statement be true, suppose that $y\in J^f(x)$. Then there is a net $(g_d)_{d\in D}$ in G with $g_d \xrightarrow{f} \infty$ and a net $(\chi_d)_{d\in D}$ in X with $\chi_d \xrightarrow{f} x$ such that $g_d\chi_d \xrightarrow{f} y$. By hypothesis, there exist U be f – neighborhood of x , W be f – neighborhood of y and V be an f – compact f – neighborhood of e such that $gU\cap W = \phi$ for each $g\notin V$. Thus for $d_o\in D$ we have $\chi_d\in U$ and $g_d\chi_d\in W$ for each $d \geq d_o$, then $g_d\in V$, which is f – compact . Therefore the net $(g_d)_{d\in D}$ has f – convergent subnet , which is contradiction. Thus $y\notin J^f(x)$.

3.5 Proposition: Let X be an f – locally f – compact G – space . Then $J^f(x) = \phi$ for each $x\in X$ if and only if every pair of point of X has f – relatively thin f – neighborhood.

Proof: $\Rightarrow$ Let $J^f(x) = \phi$ for each $x\in X$ and y be any point in X. Thus $(y\notin J^f(x))$. Then by Theorem (3.4) there is an f – neighborhood U of x and an f – neighborhood W of y, and an f – compact f – neighborhood V of e such that $gU\cap W = \phi$ for each $g\notin V$, in the other words, $gU\cap W \neq \phi$ then $g\in V$ i.e., $((U, W))$ has f – compact closure. Therefore U and W are f – relatively thin f – neighborhood. hypothesis, there are f – relative thin f – neighborhood U of x and W of y. Thus $((U, W))$ has f – compact closure. If $V_1 = \overline{((U,W))}^f$ and V_2 be an f – compact f – neighborhood of G_x, then $V = V_1\cup V_2$ is an f – compact f – neighbor hood of e and each $g\in V$, then $gU\cap W\neq\phi$ this means that $y\notin J^f(x)$. But x and y are arbitrary, thus we have $J^f(x)\neq\phi$ for each $x\in X$.

3.6 Definition: A G – space X is said to be an f – Cartan G – space if every point in X has an f – thin f – neighborhood.

3.7 Proposition: If X is f – Cartan G – space, then each orbit of x is f – closed in X and stabilizer group of G is f – compact.

Proof: Let $y\in \overline{\gamma(x)}^f$. Then there is a net $(y_d)_{d\in D}$ in $\gamma(x)$such that $y_d \xrightarrow{f} y$. Since X is an f – Cartan G – space, then y has f – thin f – neighborhood U. Since $y_d\in\gamma(x)$, then there exists a net $(g_d)_{d\in D}$ in G such that $y_d=g_dx$ for each $d\in D$. Fixed d_o and $(g_d g_{d_o}^{-1})(g_{d_o} x) = g_dx$ so $g_d g_{d_o}^{-1}\in((U,U))$, such that $g_d g_{d_o}^{-1} \xrightarrow{f} g$, then $g_dx \xrightarrow{f} g g_{d_o} x$ and $y = g g_{d_o} x$, so $y\in Gx$. Thus $\gamma(x)$ is f – closed in X. Now, let $x\in X$, then there exists an f – thin f – neighborhood V of x. Clearly G_x is f – closed in G and since $G_x \subseteq ((V,V))$. Hence G_x is f – compact.

3.8 Theorem: Let X be a G – space. Then X is f – Cartan G – space if and only if $x \notin J^f(x)$ for each $x\in X$.

Proof: $\Rightarrow$If X is an f – Cartan G – space. Let $x \in J^f(x)$, then there is a net $(g_d)_{d\in D}$ in G with $g_d \xrightarrow{f} \infty$ and there is a net $(\chi_d)_{d\in D}$

Conversely:$\Leftarrow$ Let $x, y\in X$, then by

Since $x\in X$ and X is an f – Cartan G – space, then x has an f – open neighborhood U such that $((U, U))$ is f – relative thin. Then $((U, U))$ is f - relative compact. Thus there is $d\in D$, χ_d and $g_d\chi_d$ are in U. So that g_d is in $((U, U))$.Then $(g_d)_{d\in D}$ contains a convergent subnet, this is contradiction.

$\Leftarrow$ Suppose that X is not an f – Cartan G – space. Then there is a point x in X such that x has no f – neighborhood f – relative thin. Since X is completely regular, then by the point x has a sequence $\{U_n\}_{n\in Z^+}$ of an f – open neighborhood such that $U_{n+1}\subset U_n\ldots$ and $\bigcap_{n\in Z^+} U_n = \{x\}$. Then $((U_n, U_n))$ is not f – relative thin. We can choose an f – open neighborhood U of e in G such that $G_x \subseteq U$ and it is f – relative compact. Then there is a sequence $(g_n)_{n\in N}$ in $((U_n, U_n)) - U$. Since g_n in G then there is a sequence $(\chi_n)_{n\in N}$ in U_n such that $g_n\chi_n$ is in U_n. Since $\bigcap_{n\in Z^+} U_n = \{x\}$, then $\chi_n \xrightarrow{f} x$ and $g_n\chi_n \xrightarrow{f} x$. Since $x\notin J^f(x)$, then $(g_n)_{n\in N}$ has an f – convergent subsequence, say (g_{n_k}) with $g_{n_k} \xrightarrow{f} g$. Thus $g_{n_k}\chi_{n_k} \xrightarrow{f} x$, $\chi_{n_k} \xrightarrow{f} x$ and $g_{n_k} \xrightarrow{f}$ g, imply that $x = xg$. Hence g is in G_x and hence g_{n_k} is in U for large n_k and this is contradiction.

3.9 Proposition: Let X be an f – Cartan$G$ – space, then $\Lambda^f(x) = \phi$ for each $x\in X$.

Proof: Suppose that there is a point $y\in X$ such that $y\in \Lambda^f(x)$. Then there is a net $(g_d)_{d\in D}$ in G with $g_d \xrightarrow{f} \infty$ such that $g_d x \xrightarrow{f} y$. Let U_y be an f – thin f – neighborhood of y. Then there is $d_o\in D$ such that $g_d x\in U_y$ for each $d \geq d_o$, we get $g_d g_{d_1}^{-1} g_{d_1} x = g_d x\in U_y$, thus $g_d g_{d_1}^{-1}\in ((U_y, U_y))$, which has f – compact closure. Hence the net in X with $\chi_d \xrightarrow{f} x$ such that $g_d\chi_d \xrightarrow{f} x$. $g_d g_{d_1}^{-1}$ has f – convergent subnet, say itself, i.e, there is $g_o\in G$ such that $g_d g_{d_1}^{-1} \xrightarrow{f} g_o$, then $g_d \xrightarrow{f} g_o g_{d_1}$, which is a contradiction, therefore $y\notin \Lambda^f(x)$, since y is arbitrary, thus $\Lambda^f(x) = \phi$.

References

[1] AL-Badairy, M. H. ,"on Feebly proper Action" . M.Sc. ,Thesis, University of Al-Mustansiriyah,(2005).

[2] Alyaa , Y.K. , "On Strongly Feebly proper Function", M.Sc., Thesis, University of Kufa,(2008).

[3] Bredon, G.E., "Introduction to compact transformation Groups" Academic press, N.Y., 1972.

[4] Galdas, MGeorgiou,D.N. and Jafari,S. ,"Characterizations of Low separation axiom via α- open sets and α -closer operator" , Bol. Soc. paran. Mat. , SPM,Vol. (21) , (2003) .

[5] Jankovic, D. S. and Reilly, I.L.," On semi – separation properties", Indian J. pur Appl. Math. 16(9), (1985), 957 – 964.

[6] Maheshwari, S. N. and Tapi, U. "On α – irresolute mappings" ,Ann. Univ. Timisoara S. sti.math. 16 (1978),173 – 77.

[7] Maheshwari, S. N. and Thakur, S.S. "On α – irresolute mappings" ,Tamkang math. 11 (1980), 209 – 214.

[8] Maheshwari, S. N. and Thakur, S.S. "On α – compact spaces" , Bulletin of the Institute of Mathematics, Academia Sinica,Vol.13,No.4,Dec. (1985),341– 347.

[9] Navalagi, G.B., "Definition Bank in General Topology", (54) G (1991).

[10] Navalagi, G.B., "Quasi α – closed , strongly α – closed and weakly α – irresolute mapping" , National symposium Analysis and its Applications, Jun. (8 – 10), (1998).

[11] Njastad , O.,"On some classes of nearly open sets",pacific J. Math.,15 (1965),961 – 970.

[12] Reilly, I.L. and Vammanamurthy, M.K.," On α – continuity in Topological spaces", Acta mathematics Hungarica, 45 , (1985), 27 – 32.

[13] Saddam, J.S.," On Strongly proper Actions", M.Sc., Thesis, University of Al-Mustansiriyah,(2000).

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المستخلص

أن الهدف الرئيسي من هذا البحث هو تقديم نوع جديد من فضاءات – G سمي فضاء –G كارتان الضئيل وكذلك نوع جديد من مجموعات الغاية أسميناه مجموعات الغاية الضئيلة $J^f(x)$ و $\Lambda^f(x)$ وأعطينا خصائص وبعض مكافئات تلك المفاهيم ثم بينا العلاقة بين فضاء –G كارتان الضئيل و بين المجموعتين $J^f(x)$ و $\Lambda^f(x)$.