SOLUTION OF FUZZY INITIAL VALUE PROBLEMS USING LEAST SQUARE METHOD AND ADOMAIN DECOMPOSITION METHOD

Mohammed Ali Ahmed* and Fadhel Subhi Fadhel**

*** Department of Mathematics, College of Education Ibn Al-Haitham, University of Baghdad, Baghdad Iraq.**

**** Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University, Baghdad Iraq.**

Abstract:

In this paper, we will study the numerical solution of fuzzy initial value problems using two methods, namely, the least square method and the Adomain decomposition method. Also, comparison between the obtained results is made, as well as with the crisp solution, when the α-level equals one.

1- Introduction:

Fuzzy set theory had been introduced by Zadeh in 1965, in which, Zadeh's original definition of fuzzy set is as follows:

A fuzzy set is a class of objects with a continuum grades of membership, such a set is characterized by a membership or characteristic function which assigns to each object a grade of membership ranging between zero and one, [Zadeh L. A., 1965].

In 1978 and 1980, Kandel and Byatt applied the concept of fuzzy differential equations to the analysis of fuzzy dynamical problems, but the initial value problem was treated rigorously by Kaleva in 1987 and 1990, [Pearson, D. W., 1997].

Analytical methods for solving fuzzy initial value problems are so difficult in some cases, especially for nonlinear differential equations and therefore numerical and approximate methods seems to be necessary for solving such type of problems, [Al-Ani E. M., 2005], [Wuhaib, S. A., 2005].

2- Preliminaries of Fuzzy Sets:

Let X be a classical set of objects, called the universal set, whose generic elements are denoted by x. The membership in a classical subset A of X is often viewed as a characteristic function χ from X into {0, 1}, such that:

$$\chi(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases}$$

{0, 1} is called a valuation set. If the valuation set is allowed to be the interval [0, 1], then A is called a fuzzy set, which is denoted in this case by $\tilde{A}$ and the characteristic function by $\mu_{\tilde{A}}$, which the grade of membership of x in $\tilde{A}$, [Zadeh L. A., 1965].

Also, it is remarkable that the closer the value of $\mu_{\tilde{A}}$ to 1, the more x belong to $\tilde{A}$. Clearly, $\tilde{A}$ is a subset of X that has no sharp boundary. The fuzzy set $\tilde{A}$ is completely characterized by the set of pairs:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X, 0 \leq \mu_{\tilde{A}}(x) \leq 1\}.$$

In addition, some other basic concepts related to fuzzy sets may be introduced, which may be summarized as follows, [Zimmerman, H. J., 1985], [Kandel A., 1986]:

1. The height of $\tilde{A}$ is given by:

   $$\text{hgt}(\tilde{A}) = \sup_{x \in X} \mu_{\tilde{A}}(x)$$

2. X may be considered as a fuzzy set with membership function $\mu_{\tilde{A}}(x) = 1$, for all $x \in X$, which is denoted by 1_X; while the empty fuzzy set $\tilde{\emptyset}$ is a fuzzy set with membership function $\mu_{\tilde{\emptyset}}(x) = 0$, $\forall\ x \in X$, which is denoted by 0_X.

3. $\tilde{A}$ is said to be normal if there exists $x_0 \in X$, such that $\mu_{\tilde{A}}(x_0) = 1$, otherwise $\tilde{A}$ is subnormal. Also, if a fuzzy set $\tilde{A}$ is subnormal, then it may be normalized by dividing $\mu_{\tilde{A}}$ on $\text{hgt}(\tilde{A}) \neq 0$.

4. $\tilde{A}^c$ is the complement of $\tilde{A}$ which is also a fuzzy set with membership function:

   $$\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x),\ \forall\ x \in X.$$

$\tilde{A} \subseteq \tilde{B}$ if $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$, $\forall\ x \in X$.

5. $\tilde{A} = \tilde{B}$ if $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$, $\forall\ x \in X$.

6. The union of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is also a fuzzy set $\tilde{C}$ and may be associated with the following membership function:

   $$\mu_{\tilde{C}}(x) = \text{Max}\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\},\ \forall\ x \in X$$

7. The intersection of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is also a fuzzy set $\tilde{D}$ and may be defined with the following membership function:

   $$\mu_{\tilde{D}}(x) = \text{Min}\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\},\ \forall\ x \in X$$

8. A fuzzy subset $\tilde{A}$ of ‘ is said to be convex fuzzy set, if:

   $$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \text{Min}\{\mu_{\tilde{A}}(x_1), \mu$$

   for all $x_1, x_2 \in$ ‘ , and all $\lambda \in [0, 1]$, where $\mu_{\tilde{A}}(x)$ is standing for a suitable membership function

In addition, among the basic concepts in fuzzy set theory is the concept of α-level (α-cut) sets of a fuzzy set $\tilde{A}$, which is used as an intermediate set that connects between fuzzy and nonfuzzy sets, [Wuhaib, S. A., 2005].

Given a fuzzy set $\tilde{A}$ defined on a universal X and any number $\alpha \in (0, 1]$ the α-level, A_α is the crisp set that contains all elements of the universal set X, whose membership grades in $\tilde{A}$ are greater than or equal to a pre specified value of α, i.e.,

$$A_\alpha = \{x : \mu_{\tilde{A}}(x) \geq \alpha,\ \forall\ x \in X\}$$

The following properties are satisfied for all $\alpha \in (0, 1]$, which may be proved easily for all $\alpha, \beta \in (0, 1]$:

1. $\tilde{A} = \tilde{B}$ if and only if $A_\alpha = B_\alpha$.
2. If $\tilde{A} \subseteq \tilde{B}$ then $A_\alpha \subseteq B_\alpha$.
3. $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$.
4. $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$.
5. If $\alpha \leq \beta$, then $A_\alpha \supseteq A_\beta$.
6. $A_\alpha \cap A_\beta = A_\beta$ and $A_\alpha \cup A_\beta = A_\alpha$, if $\alpha \leq \beta$.

Another type of fuzzy sets that may be characterized using different notion is the fuzzy subsets of real numbers, which is the so called fuzzy number. Fuzzy numbers are always fuzzy sets while the converse is not true. The definition of a fuzzy number is as follows:

A fuzzy number $\tilde{M}$ is a convex normalized fuzzy set $\tilde{M}$ of the real line ‘ , such that:

1. There exists exactly one $x_0 \in R$, with $\mu_{\tilde{M}}(x_0) = 1$ (x_0 is called the mean value of $\tilde{M}$).
2. $\mu_{\tilde{M}}(x)$ is piecewise continuous.

Now, the following two remarks illustrates the representation of a fuzzy number and fuzzy functions in terms of its α-level sets, because they are more convenient to use in applications.

Remark (1):

A fuzzy number $\tilde{M}$ may be uniquely represented in terms of its α-level sets, as the following closed intervals of the real line:

$$M_\alpha = [m - \sqrt{1-\alpha}\ ,\ m + \sqrt{1-\alpha}\ ]$$

or

$$M_\alpha = [\alpha m,\ \frac{1}{\alpha} m]$$

Where m is the mean value of $\tilde{M}$ and $\alpha \in [0, 1]$. This fuzzy number may be written as $M_\alpha = [\underline{\tilde{M}}, \overline{\tilde{M}}]$, where $\underline{\tilde{M}}$ refers to the greatest lower bound of M_α and $\overline{\tilde{M}}$ to the least upper bound of M_α.

Remark (2):

Similar to the second approach given in remark (1), one can fuzzyfy any crisp or nonfuzzy function f, by letting:

$$\underline{f}(x) = \alpha f(x),\ \overline{f}(x) = \frac{1}{\alpha} f(x),\ x \in X,\ \alpha \in (0, 1]$$

and hence the fuzzy function $\tilde{f}$ in terms of its α-levels is given by $f_\alpha = [\underline{f}, \overline{f}]$.

3- Solution of Fuzzy Ordinary Differential Equations Using the Least Square Method:

Consider the fuzzy differential equation:

$$\tilde{y}'(x) = f(x, \tilde{y}),\ \tilde{y}(x_0)\ ;\ \ \tilde{y}_0,\ x \in [a, b] \ldots\ldots \mathbf{1}$$

where y is a mapping of x, $f(x, \tilde{y})$ is a function of x and y, while $\tilde{y}_0$ is a fuzzy number.

Among the most important methods used to approximate the solution of fuzzy differential equations is that which has the general idea of minimizing the square of the residue error. To illustrate this method, consider the fuzzy initial value problem (1). Hence, in order to solve the fuzzy ordinary differential equation (1) using the least square method, the related α-level crisp differential equations are given by:

$[\tilde{y}'(x)]_\alpha = [f(x, \tilde{y}(x))]_\alpha$, $[\tilde{y}(a)]_\alpha = [\tilde{y}_0]_\alpha$, $x \in [a, b]$……………………….2

Now, the approximated solution will be denoted by:

$[\tilde{y}(x)]_\alpha = [\underline{y}(x, \alpha), \overline{y}(x, \alpha)]$, $\alpha \in (0, 1]$……3

where $\underline{y}$ and $\overline{y}$ refers to the lower and upper nonfuzzy solution related to the fuzzy solution at certain level α. Also, the initial condition may be rewritten as:

$[\tilde{y}(a)]_\alpha = [\tilde{y}_0]_\alpha = [\underline{y}_0(\alpha), \overline{y}_0(\alpha)]$…….4

The general form of the least square method for solving fuzzy ordinary differential equations is given by:

$$[\tilde{y}(x)]_\alpha = [\tilde{\psi}(x)]_\alpha + \sum_{i=0}^{n} a_i [\tilde{\phi}_i(x)]_\alpha \quad \text{…..5}$$

where $\tilde{\psi}$ is a fuzzy function which satisfies the nonhomogeneous conditions and $\tilde{\phi}_i$ is an linearly independent functions satisfying the homogeneous conditions and a_i, are constants to be determined, for all i = 0, 1, …, n. Therefore, substituting eq.(5) back into eq.(1) and minimizing the square of the residual error defined by:

$$\tilde{E}(a_0, a_1, \ldots, a_n) =$$

$$\int_a^b \left[\tilde{\psi}'_\alpha(x) + \left(\sum_{i=0}^{n} a_i \tilde{\phi}_{i_\alpha}(x)\right) - f\left(x, \tilde{\psi}_\alpha(x) + \left(\sum_{i=0}^{n} a_i \tilde{\phi}_{i_\alpha}(x)\right)\right)\right]^2 dx$$

Hence, the problem now is reduced to find the coefficients a_i, i = 0, 1, …, n. A necessary conditions for the coefficients a_i, i = 0, 1, …, n; which minimizes E is that:

$$\frac{\partial \tilde{E}}{\partial a_i} = 0, \text{ for each } i = 0, 1, \ldots, n$$

which will produce a linear system of n + 1 equations, or the residue error given by eq.(6) may be minimized using the direct minimization techniques.

The following examples illustrate the least square method for solving fuzzy differential equations:

Example (1):

Consider the first order linear fuzzy initial value problem:

$$\tilde{y}'(x) = x + \tilde{y}, \quad \tilde{y}(0) = \tilde{1}, \quad x \in [0, 1]$$

Thus, for α ∈ (0, 1], we may write the initial condition which is a fuzzy number in triangular form and α-level sets, as:

$[\tilde{y}_0]_\alpha = [\underline{y}_0(\alpha), \overline{y}_0(\alpha)] = [1 - \sqrt{1-\alpha}, 1 + \sqrt{1-\alpha}]$

One may find that the crisp solution when α = 1, is:

$$y(x) = 2e^x - x - 1$$

Now, using the least square method to find the upper and lower fuzzy solutions, consider the upper case, i.e., consider:

$$\overline{\tilde{y}}'(x) = x + \overline{\tilde{y}}$$

With initial condition:

$$\overline{\tilde{y}}_0 = 1 + \sqrt{1-\alpha}\,,\ \alpha \in (0, 1]$$

Let:

$$\overline{\tilde{y}}_\alpha(x) = \overline{\tilde{\psi}}_\alpha(x) + \sum_{i=0}^{n} a_i \overline{\tilde{\varphi}}_{i\alpha}(x)$$

Where $\overline{\tilde{\psi}}_\alpha(x) = 1 + \sqrt{1-\alpha}$, which satisfies the nonhomogeneous initial condition; while the functions $\overline{\tilde{\varphi}}_{i\alpha}$ which satisfy the homogeneous condition $\underline{\tilde{y}}(0) = 0$, may be chosen as:

$$\overline{\tilde{\varphi}}_{0\alpha}(x) = x,\ \overline{\tilde{\varphi}}_{1\alpha}(x) = x^2,\ \dots,\ \overline{\tilde{\varphi}}_{n\alpha}(x) = x^{n+1}$$

and hence with $n = 5$:

$$\overline{\tilde{y}}_\alpha(x) = 1 + \sqrt{1-\alpha} + a_0x + a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5 + a_5x^6$$

Therefore:

$$\overline{\tilde{E}}(a_0, a_1, \dots, a_5) = \int_0^1 (\overline{\tilde{y}}'_\alpha(x) - \overline{\tilde{y}}_\alpha(x) - x)^2\, dx$$

$$= \int_0^1 [(a_0 + 2a_1x + 3a_2x^2 + 4a_3x^3 + 5a_4x^4 + 6a_5x^5) - (1 + \sqrt{1-\alpha} + a_0x + a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5 + a_5x^6) - x]^2\, dx$$

Therefore, to find $a_0, a_1, \dots, a_5$; either minimize $\overline{\tilde{E}}$ with respect to $a_0, a_1, \dots, a_5$ or evaluate the linear system:

$$\frac{\partial \overline{\tilde{E}}}{\partial a_i} = 0\,,\ \text{for each } i = 0, 1, \dots, 5$$

Similar calculations may be carried for the lower case of solution $\underline{\tilde{y}}_\alpha$. Hence we get the following results presented in Fig.(1) of the fuzzy solution $\tilde{y}$ with different values of $\alpha \in (0, 1]$. Also, the accuracy of the results may be examined with $\alpha = 1$, which are equal and the same results obtained from the crisp solution.

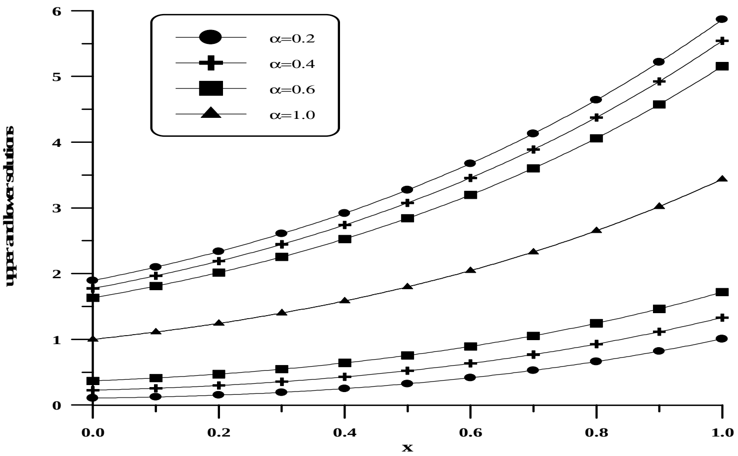

Fig.(1) Upper and lower solution of example (1) with α = 0.2, 0.4, 0.6 and 1.0 using the least square method.

Example (2):

Consider the first order nonlinear fuzzy initial value problem:

$$\tilde{y}'(x) = 1 + \tilde{y}^2,\ \tilde{y}(0) = \tilde{0},\ x \in [0, 1]$$

Thus, for $\alpha \in (0, 1]$, we may represent the initial condition which is a fuzzy number in triangular form and α-level sets, as:

$$[\tilde{y}_0]_\alpha = [\underline{\tilde{y}}_0(\alpha), \overline{\tilde{y}}_0(\alpha)] = [-\sqrt{1-\alpha}\,,\ \sqrt{1-\alpha}\,]$$

One may find that the crisp solution when $\alpha = 1$, is:

$y(x) = \tan(x)$

Now, using the least square method to find the upper and lower fuzzy solutions, consider the lower case, i.e., consider:

$$\underline{\tilde{y}}'(x) = x + \underline{\tilde{y}}$$

With initial condition:

$$\underline{\tilde{y}}_0 = -\sqrt{1-\alpha}\,,\ \alpha \in (0, 1]$$

So:

$$\underline{\tilde{y}}_\alpha(x) = \underline{\tilde{\psi}}_\alpha(x) + \sum_{i=0}^{n} a_i \underline{\tilde{\phi}}_{i\alpha}(x)$$

Where $\underline{\tilde{\psi}}_\alpha(x) = -\sqrt{1-\alpha}$, which satisfies the nonhomogeneous initial condition; while the functions $\underline{\tilde{\phi}}_{i\alpha}$ which satisfy the homogeneous condition $\underline{\tilde{y}}(0) = 0$, may be chosen as:

$$\underline{\tilde{\phi}}_{0\alpha}(x) = x,\ \underline{\tilde{\phi}}_{1\alpha}(x) = x^2,\ \ldots,\ \underline{\tilde{\phi}}_{n\alpha}(x) = x^{n+1}$$

and hence with $n = 5$:

$$\underline{\tilde{y}}_\alpha(x) = -\sqrt{1-\alpha} + a_0x + a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5 + a_5x^6$$

Therefore:

$$\underline{\tilde{E}}(a_0, a_1, \ldots, a_5) = \int_0^1 (\underline{\tilde{y}}_\alpha(x) - \underline{\tilde{y}}_\alpha^2(x) - 1)^2\, dx$$

$$= \int_0^1 [(a_0 + 2a_1x + 3a_2x^2 + 4a_3x^3 + 5a_4x^4 + 6a_5x^5) - (-\sqrt{1-\alpha} + a_0x + a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5 + a_5x^6)^2 - 1]^2\, dx$$

Therefore, to find $a_0, a_1, \ldots, a_5$; either minimize $\underline{\tilde{E}}$ with respect to $a_0, a_1, \ldots, a_5$ or evaluate the linear system:

$$\frac{\partial \underline{\tilde{E}}}{\partial a_i} = 0,\ \text{for each } i = 0, 1, \ldots, 5$$

Similar calculations may be carried for the lower case of solution $\underline{\tilde{y}}_\alpha$. Hence we get the following results presented in Fig.(2) of the fuzzy solution $\tilde{y}$ with different values of $\alpha \in (0, 1]$. Also, the accuracy of the results may be examined with $\alpha = 1$, which are equal the same results obtained from the crisp solution.

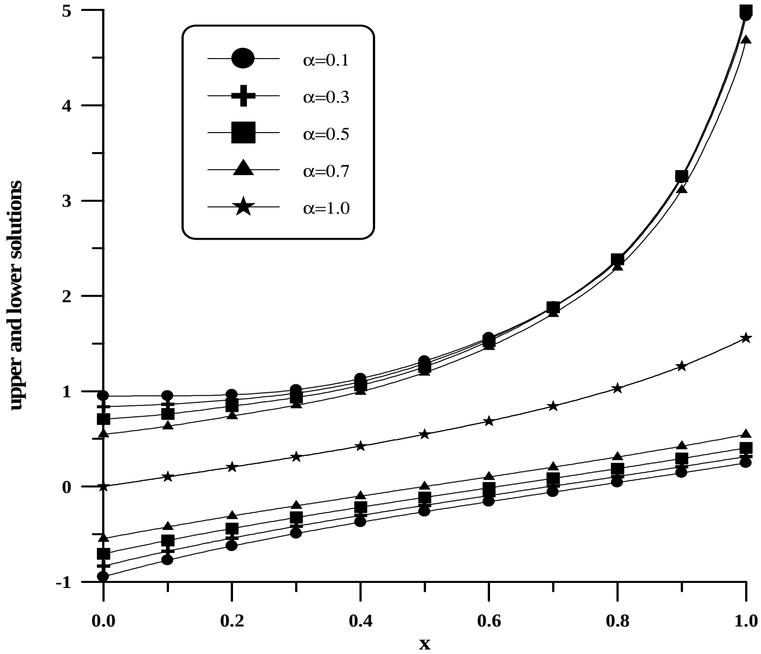

Fig.(2) Upper and lower solution of example (2) with $\alpha = 0.1$, 0.3, 0.5, 0.7 and 1.0 using the least square method.

4- Solution of Fuzzy Ordinary Differential Equations Using Adomian Decomposition Method

Adomian decomposition method (ADM, for short) [Admian G., 1988], [Lesnic D., 2002]; is one of the most modern methods that may be modified and then used for solving fuzzy differential equations of various kinds. To introduce this method, we consider the following fuzzy integral equation related to the fuzzy differential equation (1):

$$\tilde{y}(x) = \tilde{y}_0 + \int_a^x f(s, \tilde{y}(s))\, ds \quad \text{.................}\mathbf{7}$$

Then, extend $\tilde{y}$ using ADM in the series form as:

$$\tilde{y}(x) = \tilde{y}_0 + \sum_{n=0}^{\infty} \tilde{y}_n \quad \text{.......................}\mathbf{8}$$

Where $\tilde{y}_n$, n= 1,2,… are as defined in eq .(11) below and write the fuzzy nonlinear function $f(x, \tilde{y})$ as the series of function:

$$f(x, \tilde{y}) = \sum_{n=0}^{\infty} \tilde{A}_n(x, \tilde{y}_0, \tilde{y}_1, \ldots, \tilde{y}_n) \quad \text{........}\mathbf{9}$$

The dependence of $\tilde{A}_n$ on x and $\tilde{y}_0$ may be non-polynomial. Formally, $\tilde{A}_n$ is obtained by:

$$\tilde{A}_n = \frac{1}{n!}\frac{d^n}{d\varepsilon^n} f\left(x, \sum_{i=0}^{\infty} \varepsilon^i \tilde{y}_i\right)\Bigg|_{\varepsilon=0}, \; n = 0, 1, \quad \text{..}\mathbf{10}$$

where ε is a formal parameter. Functions $\tilde{A}_n$ are polynomials in $\tilde{y}_1$, $\tilde{y}_2$, …, $\tilde{y}_n$, which are referred to as the Adomian polynomials.

$$\tilde{A}_0 = f(x, \tilde{y}_0)$$

$$\tilde{A}_1 = \tilde{y}_1 f'(x, \tilde{y}_0)$$

$$\tilde{A}_2 = \tilde{y}_2 f'(x, \tilde{y}_0) + \frac{1}{2}\tilde{y}_1^2 f''(x, \tilde{y}_0)$$

$$\tilde{A}_3 = \tilde{y}_3 f'(x, \tilde{y}_0) + \tilde{y}_1 \tilde{y}_2 f''(x, \tilde{y}_0) + \frac{1}{6}\tilde{y}_1^3 f'''(x, \tilde{y}_0)$$

$$\tilde{A}_4 = \tilde{y}_4 f'(x, \tilde{y}_0) + \frac{1}{2}\tilde{y}_2^2 f''(x, \tilde{y}_0) + \tilde{y}_1 \tilde{y}_3 f''(x, \tilde{y}_0) + \frac{1}{2}\tilde{y}_1^2 \tilde{y}_2 f'''(x, \tilde{y}_0) + \frac{1}{24}\tilde{y}_1^4 f^{(4)}(x, \tilde{y}_0)$$

where $'$ refers to the partial derivatives of f with respect to y.

Substituting (8) and (9) back into (7) gives a recursive equation for $\tilde{y}_{n+1}$ in terms of $\tilde{y}_1$, $\tilde{y}_2$, …, $\tilde{y}_n$, as:

$$\tilde{y}_{n+1}(x) = \int_a^x \tilde{A}_n(s, \tilde{y}_0(s), \tilde{y}_1(s), \dots, \tilde{y}_n(s))\, ds, \quad n = 0, 1, \dots\dots\dots\dots 11$$

Now, consider the following illustrative examples:

Example (3):

Consider the linear fuzzy ordinary differential equation of the first order:

$$\tilde{y}'(x) = x + \tilde{y}, \quad \tilde{y}(0) = \tilde{1}, \quad x \in [0, 1]$$

Thus, for $\alpha \in (0, 1]$, we may write the initial condition which is a fuzzy number in triangular form and α-level sets, as:

$$[\tilde{y}_0]_\alpha = [\underline{\tilde{y}}_0(\alpha), \overline{\tilde{y}}_0(\alpha)] = [1 - \sqrt{1-\alpha}, 1 + \sqrt{1-\alpha}]$$

Hence, using the ADM, we have to consider first the solution in lower case of the fuzzy solution $\tilde{y}$:

$$(\underline{\tilde{y}}_{n+1}(x))_\alpha = \int_0^x (\underline{\tilde{y}}_n(x))_\alpha\, dx, \quad n \geq 0$$

with initial condition:

$$(\underline{\tilde{y}}_0(x))_\alpha = \frac{x^2}{2} + (1 - \sqrt{1-\alpha})$$

Hence, taking the first ten terms of the solution, we get the following solution in series form:

$$\underline{\tilde{y}}(x, \alpha) = \frac{x^2}{2} + (1 - \sqrt{1-\alpha}) + \frac{x^3 - 6x\sqrt{1-\alpha} + 6x}{6} + \frac{x^2(x^2 - 12\sqrt{1-\alpha} + 12)}{24} + \frac{x^3(x^2 - 20\sqrt{1-\alpha} + 20)}{120} + \frac{x^4(x^2 - 30\sqrt{1-\alpha} + 30)}{720} + \dots$$

and similarly the upper solution may be found to be:

$$\overline{\tilde{y}}(x) = \frac{x^2}{2} + (1 + \sqrt{1-\alpha}) + \frac{x^3 + 6x\sqrt{1-\alpha} + 6x}{6} + \frac{x^2(x^2 + 12\sqrt{1-\alpha} + 12)}{24} +$$

$$\frac{x^3(x^2+20\sqrt{1-\alpha}+20)}{120} + \frac{x^4(x^2+30\sqrt{1-\alpha}+30)}{720} + \ldots$$

The results of the upper and lower solutions for different values of $\alpha \in (0, 1]$ are given in Fig.(3), which agree with the results given in Fig.(1) for the linear case of problems.

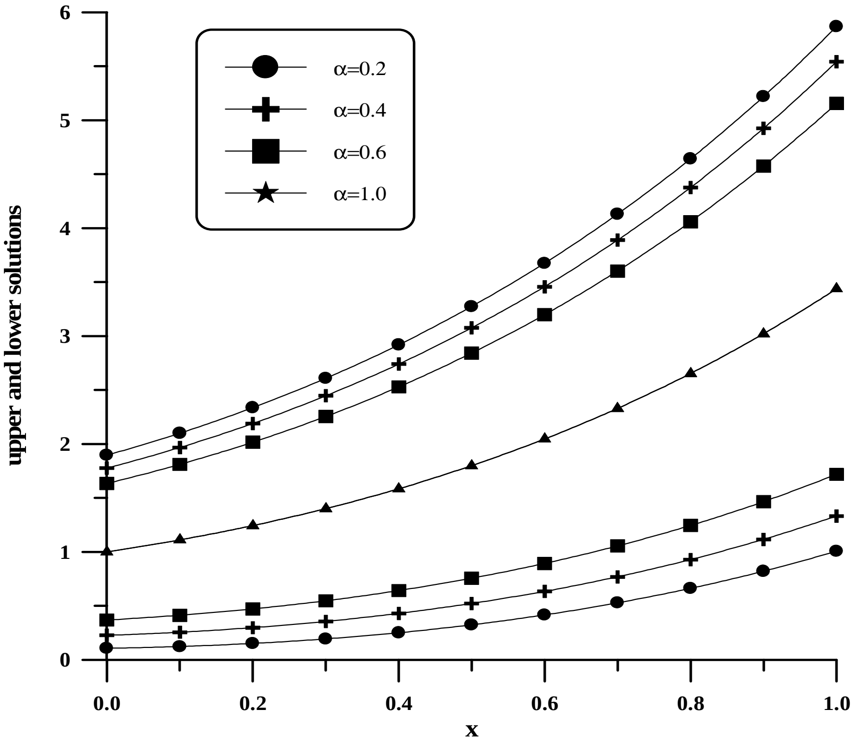

Fig.(3) Upper and lower solution of example (3) with $\alpha = 0.2, 0.4, 0.6$ and 1.0 using ADM.

Example (4):

Consider the first order nonlinear fuzzy initial value problem:

$$\tilde{y}'(x) = 1 + \tilde{y}^2,\ \tilde{y}(0) = \tilde{0},\ x \in [0, 1]$$

Thus, for $\alpha \in (0, 1]$, we may represent the initial condition which is a fuzzy number in triangular form and α-level sets, as:

$$[\tilde{y}_0]_\alpha = [\underline{\tilde{y}}_0(\alpha),\ \overline{\tilde{y}}_0(\alpha)] = [-\sqrt{1-\alpha},\ \sqrt{1-\alpha}\,]$$

Now, using the ADM for solving this nonlinear fuzzy ordinary differential equation, we have for the upper solution:

$$(\overline{\tilde{y}}_{n+1}(x))_\alpha = \int_0^x (\overline{\tilde{A}}_n(x))_\alpha\, dx,\ n \geq 0$$

with the initial condition:

$$(\overline{\tilde{y}}_0(x))_\alpha = x + \sqrt{1-\alpha}$$

where $(\overline{\tilde{A}}_n(x))_\alpha$, $n \geq 0$ are the Adomian fuzzy polynomials that represent the nonlinear terms, in which the first three terms of them are given by:

$$(\overline{\tilde{A}}_0(x))_\alpha = (\overline{\tilde{y}}_0^2(x))_\alpha$$

$$(\overline{\tilde{A}}_1(x))_\alpha = 2(\overline{\tilde{y}}_0(x))_\alpha (\overline{\tilde{y}}_1(x))_\alpha$$

$$(\overline{\tilde{A}}_2(x))_\alpha = 2(\overline{\tilde{y}}_0(x))_\alpha (\overline{\tilde{y}}_2(x))_\alpha + (\overline{\tilde{y}}_1^2(x))_\alpha$$

$$\vdots$$

and hence the upper solution is given by:

$$\overline{\tilde{y}}(x,\alpha) = x + \sqrt{1-\alpha} + \frac{x(3(1-\alpha)+3x\sqrt{1-\alpha}+x^2)}{3} + x^2\left[\left(\sqrt{1-\alpha}\right)^3 + \frac{4(1-\alpha)x}{3} + \frac{2x^2\sqrt{1-\alpha}}{3} + \frac{2x^3}{15}\right] + \ldots$$

Similarly, the lower solution is found to be:

$$\underline{\tilde{y}}(x,\alpha) = x - \sqrt{1-\alpha} + \frac{x(3(1-\alpha)-3x\sqrt{1-\alpha}+x^2)}{3} + x^2\left[-\left(\sqrt{1-\alpha}\right)^3 + \frac{4(1-\alpha)x}{3} - \frac{2x^2\sqrt{1-\alpha}}{3} + \frac{2x^3}{15}\right] + \ldots$$

The results of the upper and lower solutions for different values of $\alpha \in (0, 1]$ are given in Figs.(4) and (5), which shows that the results are accurate in the lower case of solution and inaccurate in the upper case of solution when comparing with those results obtained from the least square method.

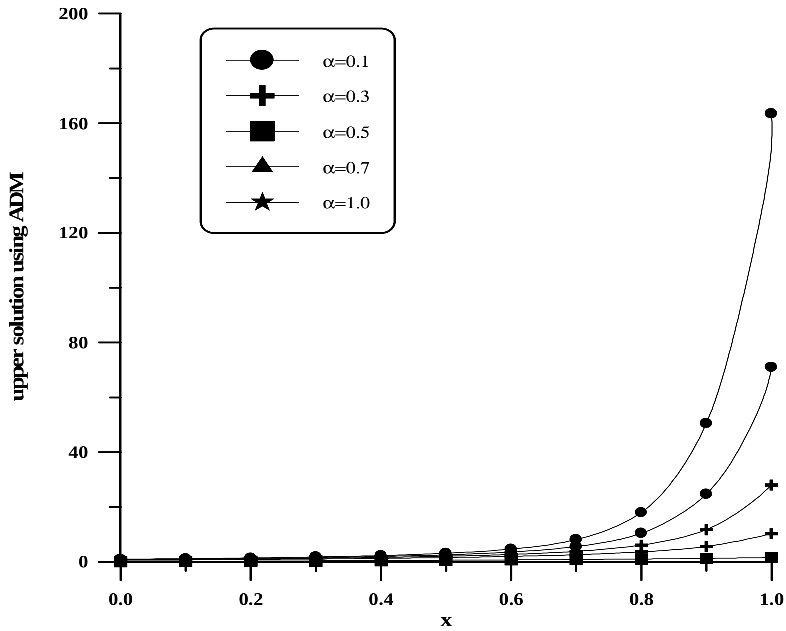

Fig.(4) Upper solution of example (4) with α = 0.1, 0.3, 0.5, 0.7 and 1.0 using ADM.

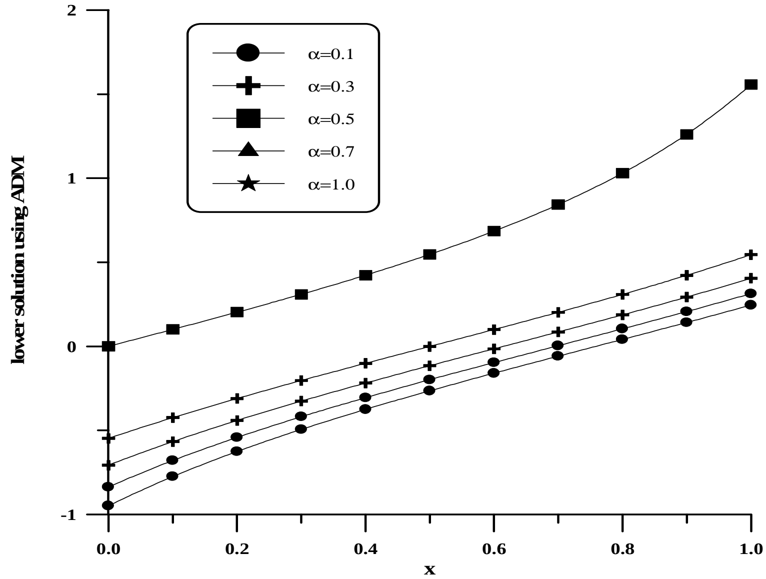

Fig.(5) Lower solution of example (4) with α = 0.1, 0.3, 0.5, 0.7 and 1.0 using ADM.

5- Conclusions:

From the present study of this paper, we may conclude the following:

1. The accuracy of the results may be checked with $\alpha = 1$, in which the upper and lower solutions must be equal.
2. The crisp solution or the solution of the nonfuzzy initial value problem is obtained from the fuzzy solution by setting $\alpha = 1$, and therefore fuzzy initial value problems may be considered as a generalization to the nonfuzzy initial value problems.
3. The ADM given an accurate results when solving linear fuzzy initial value problems, while give inaccurate results when solving nonlinear fuzzy initial value problems. Therefore, the least square method may be considered to be more reliable than the ADM in solving fuzzy initial value problems.

6- References:

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حل مسألة القيمة الإبتدائية الضبابية بإستخدام طريقة المربعات الصغرى وطريقة ادومين للتحليل

* محمد علي احمد ** د. فاضل صبحي فاضل

* جامعة بغداد ، كلية التربية ابن الهيثم ، قسم الرياضيات ،، العراق / بغداد

** جامعة النهرين ، كلية العلوم ، قسم الرياضيات وتطبيقات الحاسوب ،، العراق/ بغداد

المستخلص

في هذا البحث ، سوف نقوم بدراسة الحل العددي لمسألة القيمة الإبتدائية الضبابية بإستخدام طريقتين مختلفتين هما طريقة المربعات الصغرى وطريقة ادومين للتحليل وسوف نقوم أيضاً بمقارنة النتائج التي نحصل عليها مع الحل المضبوط عندما ألفا تساوي واحد فقط .