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ON CLOSED BCH-ALGEBRA WITH RESPECT TO AN ELEMENT OF A BCH-ALGEBRA

By

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Abstract

In this paper, we define the concepts of a closed ideal with respect to an element of a BCH-algebra and a closed BCH-algebra with respect to an element of BCH-algebra . We stated and proved some theorems which determine the relationship between these notion and the notions of some ideals of a BCH-algebra.

INTRODUCTION

The notion of BCK- algebras was formulated first in 1966 [11] by (Y.Imai) and (K.Iseki) as a generalization of the concept of set-theoretic difference and propositional calculus. In the same year (K.Iseki) introduced the notion of BCI –algebra [5], which is a generalization of BCK- algebra . Most of the algebras related to the t-norm based logic, such as MTL-algebras, BL-algebras, hoop, MV-algebras ,B-algebra , BQ-algebra ,and Boolean algebras etc .

In 1983, (Q.P.Hu) and (X.Li) introduced the notion of BCH-algebra which are a generalization of BCK/BCI-algebras [10]. After that, many mathematical papers have been published investigating some algebraic properties of BCK\BCI\BCH-algebras and their relationship with other universal structures including lattices and Boolean algebras .

In 1996 , (M. A. Chaudhry) and (H. Fakhar-ud-din) introduced the notion of ideal, closed ideals, filter, closed filter and some type of ideals in BCH-algebra [7] .

In 2010 , (A. B. Saeid) introduced the notions of fantastic ideal in BCI-algebra [1]

In this paper, we introduce the notion of a closed ideal and closed BCH-algebra with respect to an element of a BCH-algebra. We prove some theorems and give some examples to show that the relation of this notion and other types of ideals of BCH-algebra.

1.PRELIMINARIES

In this section we give some basic concept about BCK-algebra , BCI-algebra , BCH-algebra, , p-semi simple BCH-algebra ,medial BCH-algebra , associative BCH-algebra , BCA-part of BCH-algebra , medial part of BCH-algebra, fantastic ideal in BCI-algebra and (subalgebra , ideal , closed ideal , quasi-associative ideal) in BCH-algebra with some theorems, propositions and examples.

Definition (1.1) : [5, 6]

A BCI-algebra is an algebra $(X,*,0)$ of type $(2,0)$, where X is nonempty set, * is a binary operation and o is a constant , satisfying the following axioms: for all x, y, $z\in X$:

1. $((x *y) * (x * z)) * (z * y) = 0$,
2. $(x * (x * y)) * y = 0$,
3. $x * x = 0$,
4. $x * y =0$ and $y * x = 0$ imply $x = y$,

Definition (1.2) : [12]

A BCK-algebra is a BCI-algebra satisfying the axiom:
$0 * x = 0$ for all $x \in X$.

Definition (1.3) : [10]

A BCH-algebra is an algebra $(X,*,0)$ of type $(2,0)$, where X is nonempty set, * is a binary operation and 0 is a constant , satisfying the following axioms: for all x, y, $z\in X$:

1. $x * x = 0$,
2. $x * y =0$ and $y * x = 0$ imply $x = y$,
3. $(x * y) * z = (x * z) * y$.

Definition (1.4) : [3, 12, 13]

In any BCH/BCI/BCK-algebra X, a partial order $\leq$ is defined by putting $x \leq y$ if and only if $x*y = 0$.

Proposition (1.5) : [4, 8, 9]

In a BCH-algebra X, the following holds for all x, y, $z \in X$,

1. $x * 0 = x$,
2. $(x * (x * y)) * y = 0$,
3. $0 * (x * y) = (0 * x) * (0 * y)$,
4. $0 * (0 * (0 * x)) = 0 * x$,
5. $x \leq y$ implies $0 * x = 0 * y$.

Remark(1.6) : [9]

It is known that every BCI-algebra is a BCH-algebra but not conversely, where a BCH-algebra X is called proper if it is not a BCI-algebra.

Definition (1.7) : [2]

A BCH-algebra X that satisfying in condition if $0 * x = 0$ then $x = 0$, for all $x\in X$ is called a ***P-semisimple BCH-algebra.***

Definition (1.8) : [7]

A BCH-algebra X is called ***medial*** if:
$x * (x * y) = y$, for all x , $y \in X$.

Definition (1.9) : [2]

A BCH-algebra X is called an ***associative BCH-algebra*** if:

$(x * y) * z = x * (y * z)$, for all x , y , $z \in X$.

Definition (1.10) : [9]

Let X be a BCH-algebra . Then the set $X_+ = \{ x \in X : 0 * x = 0 \}$ is called ***the BCA-part*** of X .

Remark (1.11) : [9]

The BCA-part X_+ of X is a nonempty Since $0 * 0 = 0$ gives $0 \in X_+$. Further the BCA-part of a BCH-algebra may coincide with the BCH-algebra, but not necessarily with a BCK-algebra.

Definition (1.12) : [9]

Let X be a BCH-algebra . Then the set $med(X) = \{x \in X : 0 * (0 * x) = x\}$ is called ***the medial part of X*** .

Remark (1.13) : [9]

The medial part med(X) of X is nonempty Since $0 * (0 * 0) = 0$ gives $0 \in med(X)$.

Theorem (1.14) : [7]

Let X be a BCH-algebra . Then $x \in med(X)$ if and only if $x*y = 0*(y*x)$ for all $x,y \in X$

Definition (1.15) : [9]

Let X a BCH-algebra and $S \subset X$. Then S is called a ***subalgebra*** of X if $x*y \in S$ for all $x, y \in S$.

Definition (1.16) : [2, 7]

Let X be a BCH-algebra and $\phi \neq I \subset X$. Then I is called an ***ideal*** of X if it satisfies:

i. $0 \in I$.

ii. $x*y \in I$ and $y \in I$ imply $x \in I$.

Definition (1.17) : [7]

Let X be a BCH-algebra and $I \subset X$ be an ideal. Then I is called a ***closed ideal*** of X if $0*x \in I$ for all $x \in I$.

Definition (1.18) : [7]

Let X be a BCH-algebra and $I \subset X$ be an ideal. Then I is called a quasi-associative ideal if $0*(0*x) = 0*x$, for all $x \in I$.

Definition (1.19) : [1]

Let I be an ideal of X. Then I is called a ***fantastic ideal*** of X if $(x*y)*z \in I$ and $z \in I$, then $x*(y*(y*x)) \in I$, for all $x, y, z \in X$.

Proposition (1.20) : [1]

If X is an associative BCI-algebra, then every ideal is a fantastic ideal of X.

Definition (1.21) ; [11]

A mapping $f : (X, *, 0) \rightarrow (Y, *', 0)$ of BCH-algebras is called a homomorphism if: $f(x * y) = f(x) *' f(y)$ for all $x, y \in X$.

Note that

if $f : X \rightarrow Y$ is a homomorphism of BCH-algebras, then $f(0) = 0$.

Definition (1.22) :

A mapping $f : (X, *, 0) \rightarrow (Y, *', 0)$ of BCH-algebras is called an epimorphism if f is a homomorphism and a surjective.

2.THE MAIN RESULTS

In this section we first define the notion of the closed ideal with respect to an element of a BCH-algebra . For our discussion , we shall link this notion with other type of ideals which mentioned in preliminaries.

Definition (2.1):

Let X be a BCH-algebra and I be an ideal of X . Then I is called a *closed ideal with respect to an element* $a \in X$ (denoted *a-closed ideal*) if:
$a * (0 * x) \in I$, for all $x \in I$.

Remark (2.2):

In a BCH-algebra X , the ideal I = {0} is the closed ideal with respect to 0 . Also , the ideal I = X is the closed ideal with respect to all elements of X.

Example (2.3):

Let X = {0 ,a ,b ,c }. The following table shows the BCH-algebra structure on X .

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then I={0,a} is 0, a-closed ideal, Since
1. I is an ideal [Since i. $0 \in I$.
ii. If $x*y \in I$ and $y \in I$ implies $x \in I$.]
2. $0*(0*0)=0 \in I$ and $0*(0*a)=a \in I$

$\Rightarrow$I is 0-closed

$a*(0*0)=a \in I$ and $a*(0*a)=0 \in I$

$\Rightarrow$I is a-closed

Definition (2.4):

Let X be a BCH-algebra and $a \in X$. Then X is called a ***closed BCH-algebra with respect to a***, or ***a-closed BCH-algebra***, if and only if every proper ideal is closed ideal with respect to a.

Example (2.5):

Let X={0, 1, 2, 3, 4, 5}. The following table shows the BCH-algebra structure on X.

*	0	1	2	3	4	5
0	0	0	0	0	4	4
1	1	0	0	1	4	4
2	2	2	0	2	5	4
3	3	3	3	0	4	4
4	4	4	4	4	0	0
5	5	5	4	5	2	0

$I_1=\{0, 1\}$,

$I_2=\{0, 1, 2\}$ and

$I_3=\{0, 1, 2, 3\}$

are all the proper ideals, which are 1-closed ideals of X, since

i. If $x*y \in I_i$ and $y \in I_i \Rightarrow x \in I_i$, $\forall i=1, 2, 3$] .

ii. I_1 is 1-closed ideal [since $1*(0*x) \in I_1, \forall x \in I_1$]

I_2 is 1-closed ideal [since $1*(0*x) \in I_2$,$\forall x \in I_2$]

I_3 is 1-closed ideal [since $1*(0*x) \in I_3$,$\forall x \in I_3$]

Therefore,

X is 1-closed BCH-algebra.

Theorem (2.6) :

Let X is a BCH-algebra . If $X=X_+$, then X is 0-closed BCH-algebra .

Proof

Let I be an ideal of X

To prove that I is 0-closed ideal

Let $x\in I$. Then

$0*(0*x) = 0*0$ [Since$0*x=0$,by definition (1.10)]

$= 0$ [By definition(1.3) of a BCH-algebra]

But $0\in I$ [Since I is an ideal. By definition (1.16)]

$\Rightarrow 0*(0*x) \in I$ [Since $0\in I$, by definition (1.16)]

$\Rightarrow$ I is 0-closed ideal.

Therefore,

X is 0-closed BCH-algebra. ■

Corollary(2.7) :

Every BCK-algebra is 0-closed BCH-algebra

Proof

Let X be a BCK-algebra

$\Rightarrow$ X is BCH-algebra [By remark (1.6)]

But $0*x = 0, \forall x\in X$ [Since X is BCK-algebra.By definition(1.2)]

$\Rightarrow X = X_+$ [By definition(1.10) of X_+]

$\Rightarrow$ By theorem(2.6) we get

X is 0-closed BCH-algebra. ■

Theorem(2.8) :

Let X be a 0-closed BCH-algebra. Then every quasi-associative ideal is closed ideal.

Proof

Let I be a quasi-associative ideal of X

To prove that I is closed ideal

Let $x\in I$

$0*x = 0*(0*x)$ [By definition(1.18)]

But I is 0-closed ideal [Since X is 0-closed BCH-Algebra. By definition(2.4)]

$\Rightarrow 0*(0*x)\in I \Rightarrow 0*x\in I$

Therefore,

I is a closed ideal. ■

Theorem(2.9) :

Let X be a BCH-algebra. Then every quasi-associative ideal is subalgebra.

Proof

Let I be a quasi-associative ideal and $x, y\in I$

To prove that $x*y\in I$

$0*(0*(x*y)) = 0*((0*x)*(0*y))$ [By Proposition (1.5)]

$=(0*(0*x))*(0*(0*y))$[By proposition (1.5)]

$= (0*x)*(0*y)$ [Since $x, y\in I$ and I is a quasi- associative. By definition(1.18)]

$= 0*(x*y)$ [proposition(1.5)]

$\Rightarrow x*y\in I$ [Since I is a quasi-associative ideal]

Therefore,

I is a subalgebra. ■

proposition(2.10) :

Let X be an a-closed BCI-algebra, with $a\in X$. If X is an associative BCI-algebra, then every a-closed ideal is a fantastic ideal of X .

Proof

Let I be a-closed ideal

$\Rightarrow$ I is an ideal [By definition(2.1)]

$\Rightarrow$By proposition(1.20) we get

I is a fantastic ideal of X.

Theorem(2.11) :

Let f: $(X, *, 0) \to (Y, *', 0)$ be a BCH-epimorphism. If I is an ideal of Y, then $f^{-1}(I)$ is an ideal of X.

Proof

Since f is an epimorphism

$\Rightarrow$By definition(1.22) we get

f is a homomorphism and $f(X) = Y$

To prove that $f^{-1}(I)$ is an ideal

i. $f(0) = 0 \in I$ [Since f is a homomorphism and I is an ideal of Y. By definitions (1.21) and (1.16)]

$\Rightarrow 0\in f^{-1}(I)$

ii. let $x, y\in X$ such that $x*y\in f^{-1}(I)$ and $y\in f^{-1}(I)$

$\Rightarrow f(x*y)\in I$ and $f(y)\in I$

But $f(x*y) = f(x) *' f(y)$ [By definition (1.21)]

$\Rightarrow f(x) *' f(y)\in I$ and $f(y)\in I$

$\Rightarrow f(x)\in I$ [Since I is an ideal of Y]

$\Rightarrow x\in f^{-1}(I)$

Therefore,

$f^{-1}(I)$ is an ideal of X. ■

Theorem(2.12) :

Let f: $(X, *, 0) \to (Y, *', 0)$ be a BCH-epimorphism. If K is an ideal of X. Then f (K) is an ideal of Y.

Proof

Since f is an epimorphism

$\Rightarrow$By definition(1.22) we get

f is a homomorphism and $f(X) = Y$

To prove that f(I) is an ideal

i. $0\in K$ [Since K is an ideal of X]

$\Rightarrow f(0) \in f(K)$

But $f(0) = 0$ [By definition(1.21)]

$\Rightarrow 0 = f(K)$

ii. let $x, y\in Y$ such that $x *' y\in f(K)$ and $y\in f(K)$

$\Rightarrow \exists\, x_1, y_1\in X$ such that $f(x_1)=x$ and $f(y_1) = y$

$\Rightarrow f(x_1) *' f(y_1) \in f(K)$ and $f(y_1) \in f(K)$

But $f(x_1*y_1) = f(x_1) *' f(y_1)$ [Since f is a homomorphism.By definition(1.21)]

$\Rightarrow f(x_1*y_1)\in f(K)$ and $f(y_1)\in f(K)$

$\Rightarrow x_1*y_1\in K$ and $y_1\in K$

$\Rightarrow x_1\in K$[Since K is an ideal. By definition(1.16)]

$\Rightarrow f(x_1)\in f(K) \Rightarrow x\in f(K)$

Therefore,

f(K) is an ideal of Y. ■

Theorem(2.13) :

Let f: $(X, *, 0) \to (Y, *', 0)$ be a BCH-epimorphism. if X is an a-closed BCH-algebra, then Y is a f(a)-closed BCH-algebra.

Proof

Since f is an epimorphism

$\Rightarrow$By definition(1.22) we get

f is a homomorphism and a surjective

Let I be an ideal of Y

$\Rightarrow$ By theorem(2.11) we get

$f^{-1}(I)$ is an ideal of X

$\Rightarrow$ $f^{-1}(I)$ is an a-closed ideal of X[Since X is an a-closed BCH-algebra. By definition (2.4)]

To prove that I is a f(a)-closed ideal

Let $y\in I \Rightarrow \exists\, x\in X$ such that $f(x) = y$

$\Rightarrow f(x) \in I \Rightarrow x\in f^{-1}(I)$

$\Rightarrow a*(0*x)\in f^{-1}(I)$ [Since $f^{-1}(I)$ is an a-closed ideal. By definition(2.1)]

$\Rightarrow f(a*(0*x))\in I$

$\Rightarrow f(a)*'f(0*x)\in I$[Since f is a homomorphism. By definition(1.21)]

$\Rightarrow f(a) *' (f(0) *' f(x))\in I$ [Since f is a homomorphism. By definition(1.21)]

$\Rightarrow f(a) *' (0 *' f(x))\in I$ [Since f(0) = 0.By definition(1.21)]

$\Rightarrow f(a) *' (0 *' y)\in I$ [Since f(x) = y]

$\Rightarrow$ I is a f(a)-closed ideal of Y

Therefore,

Y is a f(a)-closed BCH-algebra. ■

Theorem(2.14) :

Let f: $(X, *, 0) \to (Y, *', 0)$ be a BCH-epimorphism. if Y is an a-closed BCH-algebra, then X is a b-closed BCH-algebra, where f(b) = a

Proof

Since f is an epimorphism

$\Rightarrow$By definition(1.22) we get

f is a homomorphism and a surjective

Let K be an ideal of X

$\Rightarrow$ By theorem(2.12) we get

f(K) is an ideal of Y

$\Rightarrow$ f(K) is an a-closed ideal [Since Y is an a-closed BCH-algebra. By definition(2.4)]

To prove that K is a b-closed ideal, where f(b)=a

Let $x \in K$

$\Rightarrow f(x) \in f(K)$

$\Rightarrow a *' (0 *' f(x)) \in f(K)$ [Since f(K) is an a-closed ideal. By definition(2.1)]

$\Rightarrow f(b) *' (0 *' f(x)) \in f(K)$ [Since a = f(b)]

$\Rightarrow f(b) *' (f(0) *' f(x)))\in f(K)$ [Since f (0) = 0. By definition(1.21)]

$\Rightarrow f(b) *' f(0*x)\in K$ [Since f is a homomorphism. By definition(1.21)]

$\Rightarrow f(b*(0*a))\in f(K)$ [Since f is a homomorphism. By definition(1.21)]

$\Rightarrow b*(0*a)\in K$ [Since f is a homomorphism. By definition(1.21)]

$\Rightarrow$ K is a b-closed ideal of X.

Therefore,

Y is a b-closed BCH-algebra. ■

Corollary (2.15) :

Let f: (X, *, 0) → (Y, *', 0) be a BCH-epimorphism. Then X is 0-closed BCH-algebra if and only if Y is 0-closed BCH-algebra.

Proof

Since f is an epimorphism

⇒By definition(1.22) we get

f is a homomorphism and a surjective

Let X be 0-closed BCH-algebra

⇒ By theorem(2.13) we get

⇒ Y is f(0)-closed ideal of X

But f(0) = 0 [Since f is a homomorphism. By definition(1.21)]

Therefore,

Y is a 0-closed BCH-algebra

Conversely

Let Y be 0-closed BCH-algebra

Since f(0) = 0 [Since f is a homomorphism. By definition(1.21)]

⇒ By theorem(2.14) we get

X is 0-closed BCH-algebra,where f(0) = 0. ■

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المستخلص :

في بحثنا هذا عرفنا مفاهيم المثالي المغلق بالنسبة لعنصر في جبر (BCH) وجبر (BCH) المغلق بالنسبة لعنصر في جبر (BCH) حيث ذكرنا وبرهنا عدد من المبرهنات التي حددت العلاقة بين المفاهيم أعلاه وبعض أنواع المثاليات في جبر (BCH).