Approximate solution of integral-algebraic equations of index-2 by numerical multi-step method

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Abstract— This paper presents a numerical approach for solving index-2 integral algebraic equations (IAEs) based on multi-step collocation techniques. The authors first examine the solvability and smoothness of index-2 IAE solutions via the differentiation index framework. The proposed method integrates principles from multi-step methods and collocation, where the approximate solution is determined by satisfying specific matching conditions at designated collocation points. We perform a convergence analysis to prove that the algorithm's approximations converge to the exact solution and to determine their rate of convergence. For comparison, we also solve the equation using a one-step collocation method, thereby highlighting the superior efficiency and precision of our multi-step approach.

Keywords— Integral-algebraic equations, Index, Piecewise polynomial numerical method, Convergence analysis.

I. INTRODUCTION

A system of integral-algebraic equations (IAEs) integrates both first and second-kind integral equations. Its index is defined as the minimum number of times the system must be differentiated to be reformulated as a system of second kind integral equations. In this work, we focus on index-2 IAEs, which are defined by the following system:

$$\begin{cases} z(x) = g_1(x) + \int_0^x Y_{11}(x,y)z(y)dy + \int_0^x Y_{12}(x,y)w(y)dy \ , \\ \\ 0 = g_2(x) + \int_0^x Y_{21}(x,y)z(y)dy \ , \qquad x \in J = [0, X], \end{cases} \tag{1.1}$$

where g_i and $Y_{ij}(i,j = 1,2)$ are the given functions. Similar to differential-algebraic equations (DAEs), the index of an IAE is a measure of its complexity; higher indices correspond to systems that are more challenging to solve numerically. A system of index-2 DAEs can be expressed in the following form:

$$\acute{u} = G(t,w,u), \tag{1.2}$$

$$0 = H(t,u), \tag{1.3}$$

where $H_u\, G_w$ has a bounded inverse. We need this condition for $H_u\, G_w$ because two differentiations of the constraint (1.3) are required to transform it into an ODE system. Differential- and integral-algebraic equations offer a unified framework for modeling applied mathematics problems in fields such as network analysis, rigid-body dynamics, and control systems [2, 9, 10]. In [5], the authors employed a one-step collocation technique to compute approximate solutions for the IAEs (1.1), establishing a global convergence analysis and proving its optimal order of convergence. Also, reference [7] presents an error analysis of a global spectral method for IAEs (1.1),utilizing interpolation approximation within a Sobolev space framework. The authors of [13] established the existence of a unique solution for the IAE system (1.1) under a new index definition and solved it numerically using piecewise continuous collocation methods. The study in [1] introduces a method based on block pulse functions

for numerically solving linear Volterra integral-algebraic equations of index 1. In [4], the authors examined systems of Volterra linear integral equations featuring identically singular matrices in the principal part via numerical methods. In [12], collocation solutions for linear index-1 integral-algebraic equations were analyzed using the tractability index and the ν-smoothing property of a Volterra integral operator. A multi-step collocation method was introduced for solving integral-algebraic equations of index 1 in [22] and the existence and uniqueness of the solution were established. The reader is referred to [3,17,18] for additional research on the study of the IAEs. In [14, 15, 16], the authors studied Haar wavelet numerical method to solve Fredholm integro-differential equation, partial Volterra integro-differential equations and parabolic Volterra integro-differential equations of the second kind. The cubic B-spline method together with convergence analysis was considered to solve Volterra integro-differential equation in [19]. Fourth and sixth-order compact finite difference method (CFDM) based on composite Boole's rule were studied to solve coupled Fredholm integro-differential equations in [20].

By offering an effective balance of high accuracy and computational efficiency, multi-step collocation methods are a powerful class of solvers for ODEs and integral equations. This paper applies multi-step collocation techniques to solve index-2 IAEs. The method achieves higher-order accuracy through optimal collocation points and benefits from favorable stability by combining the strengths of both multi-step and collocation approaches. Here, for characterization of the solution, we consider Theorem 1 which describes solvability and the regularity of the solution. Also, we know that for the multi-step numerical method, the convergence order is high and a high-precision integration method is required. Based on Theorem 1, for the index-2 IAEs (1.1), smooth data led to smooth solution and then we can apply multi-step numerical method to solve it and investigate convergence analysis well. Multi-step methods use information from several previous steps to compute the next value. They are efficient but can be complex to derive for high orders. On the other hand, collocation methods find a polynomial that satisfies the integral equation exactly at a set of collocation points within a single step. A polynomial multi-step collocation method takes a multi-step view by considering a time interval that spans several steps, but then uses the collocation principle to determine the solution across that entire interval. Polynomial multi-step collocation methods are advanced numerical techniques for solving Volterra integral equations (VIEs). These methods combine ideas from collocation and multi-step methods to achieve higher-order accuracy. The paper is organized as follows:

Section 2 examines the existence and uniqueness conditions for the solution of index-2 IAEs (1.1) and introduces a polynomial multi-step collocation method to solve it numerically. In Section 3, we develop and analyze the convergence of numerical solutions. This is succeeded by the examination of two test problems in Section 4 to validate theoretical findings.

II. CHARACTERIZATION OF THE SOLUTION

This section establishes the conditions for the existence and uniqueness of solutions to index-2 IAEs (1.1), as presented in the following theorem.

Theorem 1 Let p ≥ 0 and
1. $Y_{1l} \in C^{p+1}(S) for\ l = 1,2\ and\ S = \{(x,y): 0 \le y \le x \le X\}$,
2. $H_{21} \in C^{p+2}(S)$
3. $|Y_{21}(x,x)Y_{12}(x,x)| \ge y_0 > 0$,
4. $g_1 \in C^{p+1}(J), g_2 \in C^{p+2}(J) and\ g_1(0) = 0$,
then there exists a unique solution $z, w \in C^p(J)$ for the IAE system (1.1).

Proof. A system of index-1 IAEs is derived by differentiating the second equation of (1.1) with respect to x and eliminating z via the first equation. This form aligns with the definition found in [3].

$$\begin{cases} z(x) = g_1(x) + \int_0^x Y_{11}(x,y)z(y)dy + \int_0^x Y_{12}(x,y)w(y)dy , \\ 0 = \bar{g}_2(x) + \int_0^x \bar{Y}_{21}(x,y)z(y)dy + \int_0^x \bar{Y}_{22}(x,y)w(y)dy , \end{cases} \tag{2.1}$$

Where $\bar{g}_2(x) = \acute{g}_2(x) + Y_{21}(x,x)g_1(x)$,

$$\bar{Y}_{21}(x,y) = Y_{21}(x,x)Y_{11}(x,y) + \frac{\partial Y_{21}(x,y)}{\partial x}$$

And $\bar{Y}_{22}(x,y) = Y_{21}(x,x)Y_{12}(x,y)$, $Since\ |\bar{Y}_{22}(x,x)| \ge y_0 > 0$,By re-derivative and replacing z using the first equation, we have

$$\begin{cases} z(x) = g_1(x) + \int_0^x Y_{11}(x,y)z(y)dy + \int_0^x Y_{12}(x,y)w(y)dy , \\ w(x) = \hat{g}_2(x) + \int_0^x \hat{Y}_{21}(x,y)z(y)dy + \int_0^x \hat{Y}_{22}(x,y)w(y)dy, \end{cases} \tag{2.2}$$

$$\hat{g}_2(x) = \frac{1}{\bar{\bar{Y}}_{22}(x,x)} \left(\acute{\bar{g}}_2(x) + Y_{21}(x,x)g_1(x)\right) ,$$

$$\hat{Y}_{21}(x,y) = \frac{1}{\bar{\bar{Y}}_{22}(x,x)} \left(\bar{Y}_{21}(x,x)Y_{11}(x,y) + \frac{\partial \bar{Y}_{21}(x,y)}{\partial x}\right) ,$$

$$\hat{Y}_{22}(x,y) = \frac{1}{\bar{\bar{Y}}_{22}(x,x)} \left(\bar{Y}_{21}(x,x)Y_{12}(x,y) + \frac{\partial \bar{Y}_{22}(x,y)}{\partial x}\right)$$

Since System (2.2) is a regular system of second-kind integral equations, the proof is concluded by applying Theorem 2.1.2 from [3]. ☐

NUMERICAL METHODS

development of a multi-step collocation scheme for equation (1.1) proceeds from the following framework.

$$J_h := \left\{x_n = nh , n = 0,1,\dots,N \left(x_N = X, h = \frac{X}{N}\ N > 0\right)\right\}.$$

Let

$$P_h := \{x_{n,i} = x_n + s_i h : 0 < s_1 < \cdot\ \cdot\ \cdot < s_m \le 1\ (0 \le n \le N-1)\},$$

such that $\{x_{n,i}\} and \{s_i\}$ represent collocation points and parameters, respectively.

The collocation solution is defined on each subinterval $\epsilon_n = [x_n, x_{n+1}] (for\ t \in [0,1] and\ n \geq r-1)$ by the representation:

$$a(x_n + th) = \sum_{i=0}^{r-1} \Gamma_i(t) a_{n-i} + \sum_{k=1}^{m} \bar{\Gamma}_k(t) A_{n,k} \ , \qquad (2.3)$$

$$b(x_n + th) = \sum_{i=0}^{r-1} \Gamma_i(t) b_{n-i} + \sum_{k=1}^{m} \bar{\Gamma}_k(t) B_{n,k} \ , \qquad (2.4)$$

Where

$$A_{n,k} := a(x_{n,k}) \quad , \quad a_{n-i} := a(x_{n-i}) \ ,$$

$$B_{n,k} := b(x_{n,k}) \quad , \quad b_{n-i} := b(x_{n-i}) \ ,$$

And

$$\Gamma_i(t) = \prod_{j=1}^{m} \frac{t - s_j}{-i - s_j} \cdot \prod_{j=0, j\neq i}^{r-1} \frac{t+j}{-i+j} \ , \qquad (2.5)$$

$$\bar{\Gamma}_i(t) = \prod_{i=0}^{r-1} \frac{t+i}{s_k + i} \cdot \prod_{i=1, i\neq k}^{m} \frac{t - s_i}{s_k - s_i} \ , \qquad (2.6)$$

A classical one-step method is employed to compute the starting values $a_1, \ldots, a_{r-1}$ and $b_1, \ldots, b_{r-1}$. Setting $x = 0$ in (2.2) provides the initial conditions:

$$a_0 = z(x_0) = z(0) = g_1(0) \ ,$$

$$b_0 = w(x_0) = w(0) = \hat{g}_2(0).$$

The approximations a and b satisfy the following collocation system:

$$\begin{cases} a(x_{n,i}) = g_1(x_{n,i}) + \int_0^{x_{n,i}} Y_{11}(x_{n,i}, y) a(y) dy \\ \qquad + \int_0^{x_{n,i}} Y_{12}(x_{n,i}, y) b(y) dy \ , \\ 0 = g_2(x_{n,i}) + \int_0^{x_{n,i}} Y_{21}(x_{n,i}, y) a(y) dy \ , \end{cases} \qquad (2.7)$$

Applying the change of variable $y = x_p + ht$ yields:

$$y = x_p \Rightarrow t = 0 \ , y = x_{p+1} \Rightarrow t = 1 \ , dy = hdt,$$

and for $y = x_n + ht$,

$$y = x_n \Rightarrow t = 0, y = x_n + s_i h \Rightarrow t = s_i \ , dy = hdt.$$

The substitution of equations (2.3) and (2.4) into equation (2.7) results in the following linear system for the unknowns $A_{n,i}, B_{n,i}, n = r-1, \cdot \cdot \cdot, N-1, i = 1, \ldots, m$

$$\begin{aligned} A_{n,i} &= g_1(x_{n,i}) + h \sum_{p=0}^{r-2} \int_0^1 \begin{pmatrix} Y_{11}(x_{n,i}, x_p + th) a(x_p + th) \\ + Y_{12}(x_{n,i}, x_p + th) b(x_p + th) \end{pmatrix} dt \\ &+ h \sum_{p=r-1}^{n-1} \int_0^1 Y_{11}(x_{n,i}, x_p + th) \left(\sum_{i=0}^{r-1} \Gamma_l(t) a_{1,p-l} \sum_{k=1}^{m} \bar{\Gamma}_k(t) A_{p,k} \right) dt \\ &+ h \sum_{p=r-1}^{n-1} \int_0^1 Y_{12}(x_{n,i}, x_p + th) \left(\sum_{i=0}^{r-1} \Gamma_l(t) b_{1,p-l} \sum_{k=1}^{m} \bar{\Gamma}_k(t) B_{p,k} \right) dt \\ &+ h \int_0^{s_i} Y_{11}(x_{n,i}, x_n + th) \left(\sum_{l=0}^{r-1} \Gamma_l(t) a_{n-l} \sum_{k=1}^{m} \bar{\Gamma}_k(t) A_{n,k} \right) dt \\ &+ h \int_0^{s_i} Y_{12}(x_{n,i}, x_p + th) \left(\sum_{i=0}^{r-1} \Gamma_l(t) b_{1,p-l} \sum_{k=1}^{m} \bar{\Gamma}_k(t) B_{p,k} \right) dt \end{aligned}$$

$$\begin{aligned} 0 &= g_2(x_{n,i}) \\ &+ h \sum_{p=0}^{r-2} \int_0^1 Y_{21}(x_{n,i}, x_p + th) a(x_p + th) dt \int_0^{x_{n,i}} Y_{21}(x_{n,i}, y) a(y) dy \\ &+ h \sum_{p=r-1}^{n-1} \int_0^1 Y_{21}(x_{n,i}, x_p + th) \left(\sum_{i=0}^{r-1} \Gamma_l(t) a_{p-l} \sum_{k=1}^{m} \bar{\Gamma}_k(t) A_{p,k} \right) dt \\ &+ h \int_0^{s_i} Y_{21}(x_{n,i}, x_n + th) \left(\sum_{l=0}^{r-1} \Gamma_l(t) a_{n-l} \sum_{k=1}^{m} \bar{\Gamma}_k(t) A_{n,k} \right) dt \end{aligned} \qquad (2.8)$$

Theorem 2 Under the assumptions of Theorem 1, there exists a maximal step size $\bar{h} > 0$ such that for any mesh J_h with $h \in (0, \bar{h})$, the linear system (2.8) possesses a unique solution $A_{n,i}, B_{n,i}, n = r-1, \ldots, N-1, i = 1, \ldots, m$. It therefore follows that for every interval $\epsilon_n = [x_n, x_{n+1}] (n \geq r-1)$, the collocation equation (2.7) possesses a unique multi-step collocation solution, given by the representations in (2.3) and (2.4).

Proof. We now express (2.8) in matrix notation as follows:

$$D_1 W_n = G_{n+1} + \sum_{p=r-1}^{n-1} D_2 W_p + \sum_{p=r-1}^{n} D_3 U_p + h \sum_{p=0}^{r-2} V_p, \qquad (2.9)$$

where

$$W_n = \begin{pmatrix} A_n \\ B_n \end{pmatrix}, G_n = \begin{pmatrix} G_{1n} \\ G_{2n} \end{pmatrix}, U_p = \begin{pmatrix} \mathcal{A}_p \\ \mathcal{B}_p \end{pmatrix}$$

And

$$V_P = \begin{pmatrix} V_{11}^{n,p} + V_{12}^{n,p} \\ V_{21}^{n,p} \end{pmatrix}$$

such that $A_n = (A_{n1}, \dots, A_{nm})^T$, $B_n = (B_{n1}, \dots, B_{nm})^T$

and for $l = 1, 2, G_{ln} = \left(g_l(x_{n,1}), \dots, g_l(x_{n,m})\right)^T$

$$A_p = \left(a_{p-r+1}, a_{p-r+2}, \dots, a_r\right)^T,$$

$$B_p = \left(b_{p-r+1}, b_{p-r+2}, \dots, b_r\right)^T,$$

$$D_1 = \begin{pmatrix} I_m - hC_{n,n}^{(1,1)} & -hC_{n,n}^{(1,2)} \\ -hC_{n,n}^{(2,1)} & 0 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} hC_{n,P}^{(1,1)} & hC_{n,P}^{(1,2)} \\ hC_{n,P}^{(2,1)} & 0 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} h\bar{C}_{n,P}^{(1,1)} & h\bar{C}_{n,P}^{(1,2)} \\ h\bar{C}_{n,P}^{(2,1)} & 0 \end{pmatrix}.$$

Also, for (j, q = 1, 2), we have

$$C_{n,P}^{(j,q)} = \begin{cases} \int_0^1 \begin{pmatrix} Y_j q\left(x_{ni}, x_{p+th}\right) \bar{\Gamma}_k(t) dt \\ i, k = 1, \dots, m \end{pmatrix}, p = 0, \dots, n-1 \\ \int_0^{si} \begin{pmatrix} Y_j q\left(x_{ni}, x_{n+th}\right) \bar{\Gamma}_k(t) dt \\ i, k = 1, \dots, m \end{pmatrix}, \quad p = n, \end{cases}$$

$$\bar{C}_{n,P}^{(j,q)} = \begin{cases} \int_0^1 \begin{pmatrix} Y_j q\left(x_{ni}, x_{p+th}\right) \Gamma_k(t) dt \\ i = 1, \dots, m \quad , \quad l = 0, \dots, r-1 \end{pmatrix}, p = 0, \dots, n-1 \\ \int_0^{si} \begin{pmatrix} Y_j q\left(x_{ni}, x_{n+th}\right) \Gamma_k(t) dt \\ i = 1, \dots, m \quad , \quad l = 0, \dots, r-1 \end{pmatrix}, \quad p = n, \end{cases}$$

$$V_{11}^{n,p} = \begin{pmatrix} \int_0^1 Y_{11}\left(x_{n,1}, x_p + th\right) a\left(x_p + th\right) dt, \dots \\ , \int_0^1 Y_{11}\left(x_{n,m}, x_p + th\right) a\left(x_p + th\right) dt \end{pmatrix}^T,$$

$$V_{12}^{n,p} = \begin{pmatrix} \int_0^1 Y_{12}\left(x_{n,1}, x_p + th\right) a\left(x_p + th\right) dt, \dots \\ , \int_0^1 Y_{12}\left(x_{n,m}, x_p + th\right) b\left(x_p + th\right) dt \end{pmatrix}^T,$$

$$V_{21}^{n,p} = \begin{pmatrix} \int_0^1 Y_{21}\left(x_{n,1}, x_p + th\right) a\left(x_p + th\right) dt, \dots \\ , \int_0^1 Y_{21}\left(x_{n,m}, x_p + th\right) a\left(x_p + th\right) dt \end{pmatrix}^T.$$

The invertibility of the matrix D_1 in (2.9) is guaranteed by Lemma 2.1 [21], Theorem 1, under the condition that

$$|Y_{21}(x, x) Y_{12}(x, x)| > 0, \forall x \in J.$$

We have therefore established the existence and uniqueness of the multi-step collocation solution. ☐

III. CONVERGENCE AND ERROR ANALYSIS

According to the framework established in [5] for one-step collocation, the approximations a, b are given by polynomials of degree $m-1$ on every subinterval $\epsilon_n = [x_n, x_{n+1}]$. We represent these approximations by the following expressions:

$$a(x_n + th) = \sum_{k=1}^{m} L_k(t) A_{n,k}, \tag{3.1}$$

$$b(x_n + th) = \sum_{k=1}^{m} L_k(t) B_{n,k}, \tag{3.2}$$

where

$$L_k(t) = \prod_{j \neq k, j=1}^{m} \frac{(t - s_j)}{(s_k - s_j)}.$$

The convergence properties of the method are given by the following theorem.

Theorem 3 [5] Assume that
(1) The hypotheses of Theorem 1 hold.
(2) The functions a and b, defined by equations (3.1) and (3.2) respectively, represent the collocation approximations to the solutions z and w of system (1.1).
(3) For $0 < s1 < s2 < \dots < sm \leq 1$, let

$$\lambda = (-1)^m \prod_{i=1}^{m} \frac{1 - s_i}{s_i}.$$

The stability condition $|\lambda| < 1$ is necessary and sufficient for the convergence of the collocation approximations a, b to the exact solutions z, w of system (1.1) for all $m \geq 2$. Under this condition, the following error bound holds for sufficiently small h :

$$||z - a||_\infty = O(h^m), \quad ||w - b||_\infty = O(h^{m-1}).$$

When $\lambda = -1$, we have

$$||z - a||_\infty = O(h^m), \quad ||w - b||_\infty = O(h^{m-2}),$$

and for $\lambda = 1$,

$$||z - a||_\infty = O(h^m), \quad ||w - b||_\infty = O(h^{m-3}).$$

Using similar strategy to investigate convergence of the multi-step method in [22], if we consider the multi-step method , where the collocation approximations a and b for the exact solutions z and w of (1.1) are defined by (2.3) and (2.4). Furthermore, the primary order of convergence is $m + r$, and the spectral radius of the matrix

$$\tilde{R} = \left[\frac{0_{r-1,1} \qquad\qquad \boldsymbol{I}_{r-1}}{\Gamma_{r-1}(1) \qquad \Gamma_{r-2}(1), \ldots, \Gamma_0(1)\ ,}\right]. \tag{3.3}$$

is less than 1, therefore, the errors obey the following relation:

$$\|w - b\|_\infty = O(h^{m+r}),\ \|w - b\|_\infty = O(h^{m+r-1}). \tag{3.4}$$

IV. NUMERICAL TESTS

To assess the proposed method's performance, we conducted two benchmark tests measuring its accuracy and computational efficiency. The tests were implemented using Wolfram Mathematica. To guarantee consistency with the theoretical framework, the initial conditions for the numerical scheme are derived from the system's exact analytical solutions.

The accuracy of the method is quantitatively demonstrated in Tables 1-4; these Tables record the maximum absolute errors and the corresponding order of convergence at the grid points for the case where $(r, m) = (2, 2)$ and $s_1 = 0.7,\ s_2 = 1$.

The proposed method from the previous section yields significantly more accurate results than the one-step collocation technique in [5], as evidenced by a comparative analysis of Tables 1-4. This advantage is particularly pronounced for smaller step sizes. In one-step collocation methods, from Theorem 3 (with $m = 2$), we know that the convergence orders for z and w are exactly $m = 2$ *and* $m - 1 = 1$, respectively. However for multi-step method from (3.4) $(with\ m = r = 2)$, the convergence orders for z and w are exactly $m + r = 4$ and $m + r - 1 = 3$, respectively. You can see in the Tables 1-4 that the convergence orders tend to these exact values.

Example 1 Consider the index-2 IAEs (1.1) as:

$$\begin{cases} z(x) = g_1(x) + \int_0^x (y^2+1)z(y)dy + \int_0^x e^{x+y}w(y)dy\ , \\ \quad 0 = g_2(x) + \int_0^x e^{1+y}z(y)dy\ , \qquad x \in J = [0,1] \end{cases}$$

where $g_1(t), g_2(t)$ such that the exact solution is:

$$z(x) = \cos x, w(x) = \sin x.$$

Example 2 Consider the index-2 IAEs (1.1) as:

$$\begin{cases} z(x) = g_1(x) + \int_0^x (y^2+x)z(y)dy + \int_0^x (x^2+y+1)w(y)dy\ , \\ \quad 0 = g_2(x) + \int_0^x (x^4 + y^2 + 4)z(y)dy\ , \qquad x \in J = [0,1] \end{cases}$$

where $g_1(t), g_2(t)$ such that the exact solution is:

$$z(x) = \frac{x}{x^4+1}\ , \quad w(x) = arctan(x^2 + 1).$$

TABLE 1: ORDER OF CONVERGENCE and $\| z - a \|_\infty$ in EXAMPLE 1.

N	Numerical method [5]	Order	Multi-step scheme	Order
8	7.67×10^{-4}	–	2.96×10^{-6}	–
16	1.93×10^{-4}	1.97	2.34×10^{-7}	3.65
32	4.86×10^{-5}	1.98	1.63×10^{-8}	3.83
64	1.21×10^{-5}	1.99	1.07×10^{-9}	3.96

TABLE 2: ORDER OF CONVERGENCE and$\| w - b \|_\infty$ in EXAMPLE 1.

N	Numerical method [5]	Order	Multi-step scheme	Order
8	3.83×10^{-2}	–	1.70×10^{-4}	–
16	2.14×10^{-2}	0.83	4.99×10^{-5}	2.24
32	1.13×10^{-2}	0.91	1.05×10^{-5}	2.54
64	5.96×10^{-3}	0.95	1.80×10^{-6}	2.72

TABLE 3: ORDER OF CONVERGENCE and $\| z - a \|_\infty$ in EXAMPLE 2.

N	Numerical method [5]	Order	Multi-step scheme	Order
8	2.37×10^{-3}	–	1.32×10^{-4}	–
16	5.93×10^{-4}	1.95	1.25×10^{-5}	3.39
32	1.50×10^{-4}	1.98	1.02×10^{-6}	3.71
64	3.75×10^{-5}	1.99	7.13×10^{-8}	3.94

TABLE 4: ORDER OF CONVERGENCE and $\| w - b \|_\infty$in EXAMPLE 2.

N	Numerical method [5]	Order	Multi-step scheme	Order
8	6.80×10^{-2}	–	6.93×10^{-3}	–
16	3.57×10^{-2}	0.92	2.10×10^{-3}	2.14
32	1.81×10^{-2}	0.98	4.61×10^{-4}	2.39
64	9.10×10^{-3}	0.99	7.41×10^{-5}	2.73

V. CONCLUSION

This work established the proposed multi-step method as a robust and accurate framework for index-2 IAEs. The close agreement between theoretical predictions and numerical results across various test problems validates its reliability. The proposed scheme effectively managed the convergence analysis. We found that the collocation parameter and the condition $\rho(\tilde{R}) < 1$ significantly influence the convergence rates.

This methodology will now enable the investigation of approximate solutions for other classes of IAEs. In future work, we will develop a multi-step collocation method to solve IAEs with both vanishing and non-vanishing delays of the following form

$$\begin{cases} z(x) = g_1(x) + \int_{x_0}^{x} Y_{11}(x,y)z(y)dy + \int_{x_0}^{x} Y_{12}(x,y)w(y)dy \ , \\ \quad + \int_{x_0}^{\theta(x)} Y_{13}(x,y)z(y)dy + \int_{x_0}^{\theta(x)} Y_{14}(x,y)w(y)dy, \\ 0 = g_2(x) + \int_{x_0}^{x} Y_{21}(x,y)z(y)dy + \int_{x_0}^{\theta(x)} Y_{22}(x,y)z(y)dy, \end{cases}$$

where $x \in J = [x_0, X]$. Unlike the index-2 IAEs in (1.1), where smooth data produce smooth solutions, the introduction of a non-vanishing delay alters this property. Such delays generate primary discontinuity points, leading to solutions that are initially less regular than the given functions.

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