Bifurcation Analysis of the Generalized Computer Virus Propagation Model

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Abstract— *This research examines the generalized Computer Virus Propagation Model by analyzing its local bifurcation behavior, with particular attention to Hopf and transcritical bifurcations. The stability of equilibrium points and the emergence of periodic solutions are investigated through the computation of the first Lyapunov coefficient. A detailed exploration of the system's dynamics reveals the presence of both stable and unstable periodic solutions, depending on specific parameter choices. To support the theoretical results, numerical simulations are provided, highlighting how parameter variations significantly influence the nature of the bifurcating periodic solution.*

Keywords— Computer Virus Propagation, Stability Theory, Hopf bifurcation, Transcritical bifurcation, First Lyapunov Coefficient.

I. INTRODUCTION

In the theory of dynamical systems, bifurcation refers to a qualitative change in the structure of solution trajectories that occurs as certain system parameters are varied. This concept is frequently employed in the study of nonlinear dynamical systems to characterize alterations in the stability of their solutions, as discussed in [1]. In recent years, the phenomenon of Hopf bifurcation in well-known chaotic systems has attracted considerable research interest due to its critical role in explaining stability transitions and the emergence of complex dynamical behaviors. Consequently, it has emerged as one of the most thoroughly studied topics in the field [4-6]. A bifurcation arises when an arbitrarily small variation in a parameter produces a sudden and fundamental modification in the system's behavior or topology. Such phenomena typically involve shifts in the local stability of equilibrium states, periodic orbits, or other invariant sets. In particular, when this transition results in the emergence or vanishing of periodic solutions, it is identified as a Hopf bifurcation [7]. The analysis of Hopf bifurcation generally focuses on examining the existence, non-existence, direction and stability of the periodic solutions that emerge from it. A Hopf bifurcation occurs when a periodic solution either arises from or vanishes at an equilibrium point as a system parameter passes through a bifurcation value. This type of bifurcation has been instrumental in advancing the theory of dynamical systems across multiple dimensions. Its distinctiveness lies in two key characteristics that set it apart from other common bifurcations, such as saddle-node, transcritical, or pitchfork bifurcations. First, Hopf bifurcation cannot manifest in one-dimensional systems, as it requires a minimum of two dimensions. Second, it is specifically associated with the occurrence or disappearance of periodic solutions [8]. Hopf bifurcations are classified into two categories: supercritical, in which stable periodic solutions form around an unstable equilibrium and subcritical, where unstable periodic solutions arise around a stable equilibrium [9]. The stability of these cycles is determined by the first Lyapunov coefficient. Several approaches are available for the analysis of Hopf bifurcation, including the use of bifurcation formulas [4, 10-12], Lyapunov quantities [13, 14] and focus quantities [15, 16].

The rapid growth of computer technology, online applications and the Internet of Things has significantly increased the risk of computer viruses spreading through digital networks. These harmful programs, including both viruses and network worms, have the ability to replicate themselves and move quickly across wired or wireless systems, creating serious security concerns for individuals and organizations alike [17-19]. The large number of such threats and their destructive capacity make it necessary to study how they spread and to design effective strategies for controlling them. Since the transmission of computer viruses shares many features with the spread of biological diseases, mathematical epidemic models have often been adapted to describe their behavior [20]. Well-known approaches include the SIRS model [21-23], the SEIR model [24], the SEIRS model [25] and the SEIQV model [26]. These models usually assume that an infected computer passes through a latent stage before it can infect others [27, 28]. By using such models, researchers can better explain the mechanisms of computer virus transmission and propose strategies to slow down or prevent their spread.

A computer can be classified as either internal or external depending on its current internet connection status. In this study, only internal computers are considered. These internal machines are grouped into three distinct classes. The first class consists of uninfected or virus-free computers, denoted as susceptible. The second category refers to infected computers that are not yet spreading the virus, commonly called latent computers. The third class represents seizing computers, which are actively distributing the virus across the network. As the number of computers worldwide continues to grow and gradually approaches saturation, it is reasonable to assume that the total population of computers remains constant over time. Let $x(t)$, $y(t)$ and $z(t)$ represent, at time t, the numbers of susceptible, latent and seizing internal computers, respectively. Consequently, the total number of internal computers is expressed as $x(t) + y(t) + z(t) \equiv 1$. Unless otherwise stated, the notations x, y and z will be used in place of their time-dependent forms. Based on the above explanation, in 2013, Yang and colleagues [27] introduced the following computer virus propagation model:

$$\begin{cases} \dot{x} = \delta - \beta x(y+z) + \theta y + \mu z - \delta x, \\ \dot{y} = \beta x(y+z) - (\theta + \alpha + \delta)y, \\ \dot{z} = \alpha y - (\delta + \mu)z, \end{cases} \tag{1}$$

where x, y and z are the states described above and δ, β, θ, μ and α are constant parameters. For a derivation of the model, see [27, 29]. A key challenge in analyzing the qualitative behavior of this model is finding appropriate Lyapunov functions. To address this, they employed linear combinations of quadratic functions of the independent variables as potential Lyapunov functions. Yang [18, 30] studied nonlinear infection and network topology. More recently, Parsaei [31] analyzed global stability using Lyapunov and Volterra–Lyapunov methods. In this paper, we are interested in studying local bifurcations, such as Hopf bifurcation and transcritical bifurcation, where the parameters are real.

The structure of this article is organized as follows. Section 2 focuses on determining the equilibrium points of system (1) and analyzing their stability. Section 3 addresses the Hopf bifurcation, discussing the conditions under which it arises, as well as the stability and direction of the resulting periodic solution, which are examined through the first Lyapunov coefficient. This section also explores the transcritical bifurcation. Finally, the concluding remarks are presented in the last section.

II. Equilibrium Points and Their Stability

Equilibrium points serve as the foundational elements in the analysis of any dynamical system. To find the equilibrium points of system (1), we set the right-hand sides of its equations equal to zero:

$$\delta - \beta x(y+z) + \theta y + \mu z - \delta x = 0,\ \beta x(y+z) - \theta y - \alpha y - \delta y = 0,\ \alpha y - \delta z - \mu z = 0.$$

By solving above equations, the following two equilibrium points of system (1) are obtained: $E_1 = (1,0,0)$ and $E_2 = (x_0, y_0, z_0)$, where $x_0 = \frac{1}{r_0} = \frac{(\mu+\delta)(\theta+\alpha+\delta)}{\beta(\mu+\alpha+\delta)}$, $y_0 = \frac{(r_0-1)(\mu+\delta)}{r_0(\mu+\alpha+\delta)}$ and $z_0 = \frac{\alpha(r_0-1)}{r_0(\mu+\alpha+\delta)}$ provided that $r_0 = \frac{\beta(\mu+\alpha+\delta)}{(\mu+\delta)(\theta+\alpha+\delta)}$ and $\beta(\mu+\alpha+\delta) \neq 0$.

The following formula defines the Jacobian matrix of system (1) at (x^*, y^*, z^*)

$$J(x^*, y^*, z^*) = \begin{pmatrix} -\beta(y^*+z^*) - \delta & -\beta x^* + \theta & -\beta x^* + \mu \\ \beta(y^*+z^*) & \beta x^* - (\theta+\alpha+\delta) & \beta x^* \\ 0 & \alpha & -(\delta+\mu) \end{pmatrix}$$

and the corresponding characteristic equation is

$$\lambda^3 - \Omega_1\lambda^2 - \Omega_2\lambda - \Omega_3 = 0 \tag{2}$$

where

$$\Omega_1 = -\left(\beta\left(-x^* + y^* + z^*\right) + \alpha + 3\delta + \mu + \theta\right),$$
$$\Omega_2 = -(-\alpha\beta x^* + \alpha\beta y^* + \alpha\beta z^* - 2\beta\delta x^* + 2\beta\delta y^* + 2\beta\delta z^* - \beta\mu x^* + \beta\mu y^* + \beta\mu z^* + 2\alpha\delta + \alpha\mu + 3\delta^2 + 2\delta\mu + 2\theta\delta + \theta\mu),$$
and
$$\Omega_3 = -(-\alpha\beta\delta x^* + \alpha\beta\delta y^* + \alpha\beta\delta z^* + 2\alpha\beta\mu y^* + 2\alpha\beta\mu z^* - \beta\delta^2 x^* + \beta\delta^2 y^* + \beta\delta^2 z^* - \beta\delta\mu x^* + \beta\delta\mu y^* + \beta\delta\mu z^* + \alpha\delta^2 + \alpha\delta\mu + \delta^3 + \delta^2\mu + \delta^2\theta + \delta\mu\theta).$$

The following propositions outlines the stability conditions for the equilibria E_1 and E_2. For convenience, we define the following set of parameters: $\Delta = \beta - (\alpha + 2\delta + \theta + \mu)$ and $\psi = \alpha^2 + 2(\beta - \mu + \theta)\alpha + (\mu + \beta - \theta)^2$.

Proposition 1. The following statements describe the stability of the equilibrium point E_1 of system (1):

1. When $\delta < 0$, then E_1 is unstable.
2. When $\psi < 0;\ \delta > 0$,
   i. If $\beta > (\alpha + 2\delta + \theta + \mu)$, then E_1 is unstable.
   ii. If $\beta < (\alpha + 2\delta + \theta + \mu)$, then E_1 is asymptotically stable.
3. When $\psi > 0;\ \delta > 0$,
   i. If $\beta > (\alpha + 2\delta + \theta + \mu)$, then E_1 is unstable.
   ii. If $\beta < (\alpha + 2\delta + \theta + \mu)$ and $|\Delta| > \sqrt{\psi}$, then E_1 is asymptotically stable.
   iii. If $\beta < (\alpha + 2\delta + \theta + \mu)$ and $|\Delta| < \sqrt{\psi}$, then E_1 is unstable.
4. When $\psi = 0; \delta > 0$,
   i. If $\beta > (\alpha + 2\delta + \theta + \mu)$, then E_1 is unstable.
   ii. If $\beta < (\alpha + 2\delta + \theta + \mu)$, then E_1 is asymptotically stable.

Proof.

At the equilibrium point E_1, equation (2) takes the following form:

$$(\lambda^2 - \Delta\lambda + \tfrac{1}{4}(\Delta^2 - \psi))(\lambda + \delta) = 0, \quad (3)$$

and its roots are:

$$\lambda_{1,2} = \frac{\Delta \pm \sqrt{\psi}}{2} \text{ and } \lambda_3 = -\delta.$$

1. When $\delta < 0$, λ_3 be a positive eigenvalue. Thus, the equilibrium point E_1 is unstable.

   However, when $\delta > 0$, we note that λ_3 be always negative; therefore, in this case, the other two roots determine the stability of the equilibrium point.

2. When $\psi < 0$, the values of $\lambda_{1,2}$ are complex.
   i. If $\beta > (\alpha + 2\delta + \theta + \mu)$, then $\lambda_{1,2}$ have positive real parts. Therefore, E_1 is an unstable equilibrium point (see Figure 1-(a)).
   ii. If $\beta < (\alpha + 2\delta + \theta + \mu)$, then $\lambda_{1,2}$ have negative real parts. Consequently, all eigenvalues are negative. Therefore, E_1 is asymptotically stable (see Figure 1-(b)).

3. When $\psi > 0$, the values of $\lambda_{1,2}$ are real.
   i. If $\beta > (\alpha + 2\delta + \theta + \mu)$, then at least one of the roots is positive. Therefore, E_1 is unstable (see Figure 2-(a)).
   ii. If $\beta < (\alpha + 2\delta + \theta + \mu)$ and $|\Delta| > \sqrt{\psi}$, then all eigenvalues are negative. Therefore, E_1 is asymptotically stable (see Figure 2-(b)).
   iii. If $\beta < (\alpha + 2\delta + \theta + \mu)$ provided that $|\Delta| < \sqrt{\psi}$, then one of the roots is positive. Therefore, E_1 is unstable (see Figure 2-(c)).

4. When $\psi = 0$, the values of $\lambda_{1,2}$ are repeated real numbers.
   i. If $\beta > (\alpha + 2\delta + \theta + \mu)$, then $\lambda_{1,2}$ are positive. Therefore, E_1 is unstable (see Figure 3-(a)).
   ii. If $\beta < (\alpha + 2\delta + \theta + \mu)$, then $\lambda_{1,2}$ are negative. Consequently, all the roots are negative and E_1 is asymptotically stable (see Figure 3-(b)).

At the second equilibrium point, E_2, equation (2) becomes

$$\lambda^3 - T\lambda^2 - K\lambda - D = 0, \quad (4)$$

where

$T = \frac{-1}{(\alpha+\delta+\mu)(\delta\alpha+2\mu\alpha+\delta^2+\mu\delta)}(\delta\,\alpha^3 + 2\mu\,\alpha^3 + \alpha^2\beta\delta + 3\delta^2\alpha^2 + 7\mu\delta\,\alpha^2 + \alpha^2\delta\theta + 2\mu^2\alpha^2 + 2\alpha^2\mu\theta + 2\alpha\beta\,\delta^2 + 2\alpha\beta\delta\mu + 3\delta^3\alpha + 8\mu\,\delta^2\alpha + 7\mu^2\delta\alpha + 2\mu^3\alpha + \beta\,\delta^3 + 2\beta\,\delta^2\mu + \beta\delta\,\mu^2 + \delta^4 + 3\mu\,\delta^3 + 3\mu^2\delta^2 + \mu^3\delta - \delta\theta(\delta+\mu)^2)$,

$K = \frac{-\delta}{(\alpha+\delta+\mu)(\delta\alpha+2\mu\alpha+\delta^2+\mu\delta)}(\alpha^3\beta + \mu\,\alpha^3 + 4\alpha^2\beta\delta + 3\alpha^2\beta\mu - \delta^2\alpha^2 + \alpha^2\mu\theta + 5\alpha\beta\,\delta^2 + 8\alpha\beta\delta\mu + 3\alpha\beta\,\mu^2 - 2\delta^3\alpha - 3\mu\,\delta^2\alpha - 2\alpha\,\delta^2\theta - 4\alpha\delta\mu\theta + \mu^3\alpha - 2\alpha\,\mu^2\theta + 2\beta\,\delta^3 + 5\beta\,\delta^2\mu + 4\beta\delta\,\mu^2 + \beta\,\mu^3 - \delta^4 - 2\mu\,\delta^3 - 2\delta^3\theta - \mu^2\delta^2 - 5\delta^2\mu\theta - 4\delta\,\mu^2\theta - \mu^3\theta)$,

and $D = -\delta\big(\beta(\alpha+\delta+\mu) - (\delta+\mu)(\delta+\alpha+\theta)\big)$.

Proposition 2. The following statements describe the stability of the equilibrium point E_2 of system (1):

1. The equilibrium point E_2 is unstable when $\delta\,\beta(\alpha + \delta + \mu) < \delta(\mu + \delta)(\theta + \alpha + \delta)$.
2. The equilibrium point E_2 is asymptotically stable if and only if $\delta\beta(\alpha + \delta + \mu) > \delta(\mu + \delta)(\theta + \alpha + \delta)$, $T < 0$ and $TK + D > 0$.

Proof.

1. When $\beta(\alpha + \delta + \mu) < (\mu + \delta)(\theta + \alpha + \delta)$, it follows that the determinant D is positive and at least one root of equation (4) is positive, indicating that the equilibrium point E_2 is unstable (see Figure 1-(a)).
2. According to Hurwitz's theorem, the equilibrium point E_2 is asymptotically stable if and only if $T < 0$, $D < 0$ and $TK + D > 0$. (see Figure 2-(c)).

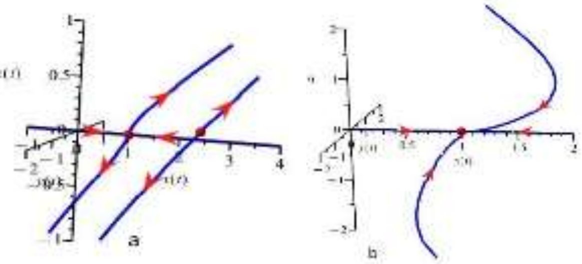

Fig. 1 The phase portrait of system (1) when $\alpha = 1$, $\delta = 1$ and $\beta = -2$. (a) shows that both equilibrium points, $E_1$and E_2, are unstable when $\mu = -5$ and $\theta = -6$. (b) shows that the axial equilibrium point, E_1, is asymptotically stable when $\mu = 2$ and $\theta = 1$. The red balls indicate the equilibrium points.

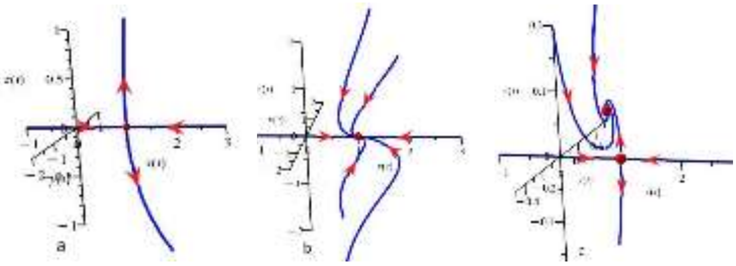

Fig. 2 The phase portrait of system (1) when $\alpha = 1$and $\delta = 1$. (a) shows that the axial equilibrium point, E_1, is unstable when $\beta = 3$, $\mu = -3$ and $\theta = 1$. (b) shows that the axial equilibrium point, E_1, is asymptotically stable when $\beta = 1$, $\mu = 1$ and

$\theta = 3$. (c) shows that the axial equilibrium point, E_1, is unstable while the equilibrium point, E_2, is asymptotically stable when $\beta = 4$, $\mu = 2$ and $\theta = 1$. The red balls indicate the equilibrium points.

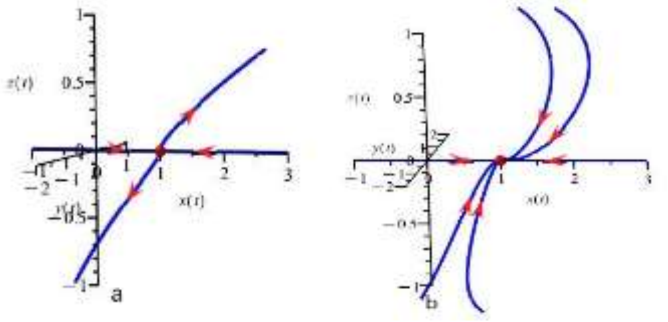

Fig. 3 The phase portrait of system (1) when $\alpha = 1$, $\delta = 1$, and $\beta = -1$. (a) shows that the axial equilibrium point, E_1, is unstable when $\mu = \theta = -3$. (b) shows that the axial equilibrium point, E_1, is asymptotically stable when $\mu = \theta = 1$. The red balls indicate the equilibrium points.

III. BIFURCATION ANALYSIS

In this section, we present a detailed investigation of the Hopf and transcritical bifurcations associated with system (1), focusing particularly on the specific conditions required for their emergence.

III.1 Hopf Bifurcation

The focus of this section is on exploring the Hopf bifurcation and assessing the stability of the periodic solutions that arise in system (1) using the first Lyapunov coefficient. For more detailed information on this topic, see [1, 8, 12, 32, 33].

A. Hopf Bifurcation at E_1

This subsection is dedicated to analyzing the Hopf bifurcation in system (1) at the equilibrium point E_1. The proposition below outlines the primary result concerning the emergence of the Hopf bifurcation.

Proposition 3. At $\alpha = \beta - (\mu + 2\delta + \theta)$, a Hopf bifurcation occurs in system (1) at the equilibrium point E_1 when the parameters satisfy the following conditions: $\delta \neq 0$ and $\theta\beta + \mu(\beta - 2\delta - \mu) - (\beta - \delta)^2 > 0$. (5)

Proof.

By letting $\alpha = \beta - (\mu + 2\delta + \theta)$, equation (3) can be reformulated as follows:

$$(\delta + \lambda)(\beta^2 - 2\beta\delta - \beta\mu - \beta\theta + \delta^2 + 2\delta\mu - \lambda^2 + \mu^2) = 0$$

It is evident that equation (3) admits a pair of complex conjugate roots $\lambda_{1,2} = \pm\omega i$ where $\omega = \sqrt{\theta\beta + \mu(\beta - 2\delta - \mu) - (\beta - \delta)^2}$; $\theta\beta + \mu(\beta - 2\delta - \mu) - (\beta - \delta)^2 > 0$ along with a real root $\lambda_3 = -\delta$. This confirms that the first condition of the Hopf bifurcation theorem is fulfilled. Noting that, in general, $\lambda = \lambda(\alpha)$ depends on the parameter α, we proceed to establish the corresponding relation from equation (3).

$$f(\lambda(\alpha), \alpha) = \lambda^3(\alpha) - (-3\delta - \mu + \beta - \theta - \alpha)\lambda^2(\alpha) - (\alpha\beta - 2\alpha\delta - \alpha\mu + 2\beta\delta + \beta\mu - 3\delta^2 - 2\theta\delta - \theta\mu)\lambda(\alpha) - \alpha\beta\delta + \alpha\delta^2 + \alpha\delta\mu - \beta\delta^2 - \beta\delta\mu + \delta^3 + \delta^2\mu + \delta^2\theta + \delta\mu\theta$$

Hence, the root $\lambda(\alpha)$ derived from equation (3) satisfies the relation below:

$$f(\lambda(\alpha), \alpha) = 0 \quad (6)$$

The differentiation of equation (6) with respect to the parameter α leads to:

$$\frac{\partial f}{\partial \lambda} \cdot \frac{d\lambda(\alpha)}{d\alpha} + \frac{\partial f}{\partial \alpha} = 0,$$

which in turn implies that

$$\frac{d\lambda(\alpha)}{d\alpha} = -\frac{\frac{\partial f}{\partial \alpha}}{\frac{\partial f}{\partial \lambda}} = -\frac{A^*}{B^*} \quad (7)$$

where $A^* = \lambda^2(\alpha) - (\beta - 2\delta - \mu)\lambda(\alpha) - \beta\delta + \delta^2 + \delta\mu$

and $B^* = 3\lambda^2(\alpha) - 2(-3\delta - \mu + \beta - \theta - \alpha)\lambda(\alpha) - \alpha\beta + 2\alpha\delta + \alpha\mu - 2\beta\delta - \beta\mu + 3\delta^2 + 2\delta\mu + 2\theta\delta + \theta\mu$.

Taking the root $\lambda(\alpha) = \lambda_{1,2}(\alpha)$, evaluating at $\alpha = \beta - (\mu + 2\delta + \theta)$ so that $\lambda_{1,2}(\alpha) = \pm\omega i$, in this case $\theta = \frac{\beta^2 - 2\beta\delta - \beta\mu + \delta^2 + 2\delta\mu + \mu^2 + \omega^2}{\beta}$; $\beta \neq 0$, and substituting the value of λ_1 for equation (7), we find that

$$\frac{d}{d\alpha} Re(\lambda_{1,2}(\alpha))\Big|_{\alpha = \beta - (\mu + 2\delta + \theta)} = d = -\frac{1}{2} \neq 0,$$

Thus, the second condition of the Hopf bifurcation theorem is fulfilled. Consequently, system (1) undergoes a Hopf bifurcation at the equilibrium point E_1 when the parameters satisfy conditions (5), giving rise to periodic solution in the neighborhood of this equilibrium.

We now proceed with a detailed analysis of the Hopf bifurcation in system (1), focusing on determining the stability of the resulting a periodic solution through the computation of the first Lyapunov coefficient. To initiate this analysis, the equilibrium point $E_1 = (1,0,0)$ of system (1) is shifted to the origin, denoted as $E_{10} = (0,0,0)$, by applying the linear transformation $x = X_1 + 1$, $y = X_2$ and $z = X_3$. As a result, system (1) is transformed into the following form:

$$\begin{cases} \dot{X}_1 = \delta - \beta(X_1 + 1)(X_2 + X_3) + \theta X_2 + \mu X_3 - \delta(X_1 + 1), \\ \dot{X}_2 = \beta(X_1 + 1)(X_2 + X_3) - (\theta + \alpha + \delta)X_2, \\ \dot{X}_3 = \alpha X_2 - (\delta + \mu X_3). \end{cases} \quad (8)$$

At this stage, it is necessary to derive the equation governing the two-dimensional dynamics on the center manifold at the bifurcation point. System (8) has been brought into its standard (canonical) form through the application of the following linear transformation.

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = T \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix},$$

where $T = \begin{pmatrix} \frac{-\delta^2+(\beta-2\mu)\delta+\beta\mu-\mu^2-\omega^2}{\delta^2+2\delta\mu+\mu^2+\omega^2} & -\frac{\beta\omega}{\delta^2+2\delta\mu+\mu^2+\omega^2} & 1 \\ -\frac{\beta(\delta+\mu)}{\delta^2+2\delta\mu+\mu^2+\omega^2} & \frac{\beta\omega}{\delta^2+2\delta\mu+\mu^2+\omega^2} & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

The form of system (8) can be expressed as:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & -\delta \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{(\eta_2 z_1 - \eta_1 z_3 + \beta\omega z_2)(\eta_2 z_1 + \beta\omega z_2)}{\eta_1\omega} \\ 0 \end{pmatrix}, \quad (9)$$

where $\eta_1 = (\delta+\mu)^2+\omega^2$ and $\eta_2 = \eta_1 - \beta(\delta+\mu)$. According to the center manifold theorem, there exists a positive constant $\epsilon > 0$ and a function H_1 defined in a neighborhood of the origin, which characterizes the local center manifold

$W_1^c(0) = \{(z_1, z_2, z_3) : z_3 = H_1(z_1, z_2) \text{ for } \| (z_1, z_2) \| < \epsilon_1\}$,

The following expression is proposed for determining the center manifold of system (9):

$$H_1(z_1, z_2) = \Phi_1 z_2^2 + \Phi_2 z_1 z_2 + \Phi_3 z_1^2 + O(3), \quad (10)$$

where $O(3)$ means terms of orders z_1^3, $z_1^2 z_2$, $z_1 z_2^2$ and z_2^3. The computation of the coefficients Φ_1, Φ_2 and Φ_3 proceeds as follows:

$$\dot{z}_3 = \frac{\partial H_1}{\partial z_1}\dot{z}_1 + \frac{\partial H_1}{\partial z_2}\dot{z}_2. \quad (11)$$

After calculating the partial derivatives of H_1, they are substituted together with $\dot{z}_1$, $\dot{z}_2$ and $\dot{z}_3$ from system (9) into equation (11). Solving the resulting system provides:

$$\Phi_1 = 0, \Phi_2 = 0 \quad \text{and} \quad \Phi_3 = 0$$

An additional implication of the center manifold theorem is that the dynamics on the center manifold are governed by the following system of differential equations:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} F^1(z_1, z_2) \\ F^2(z_1, z_2) \end{pmatrix}, \quad (12)$$

with

$\| (z_1, z_2) \| < \epsilon$, $F^1(0,0) = F^2(0,0) = 0$, $DF^1(0,0) = DF^2(0,0) = 0$ and by system (9)

$$F^1(z_1, z_2) = 0$$

$$F^2(z_1, z_2) = -\frac{\eta_2^2 z_1^2 + 2\beta\omega\eta_2 z_1 z_2 + \beta^2\omega^2 z_2^2}{\eta_1\omega} + O(3)$$

In the work of [7], a nonlinear coordinate transformation is introduced that allows systems with the structure of system (12) to be rewritten in the form

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} (l_1 u - bv)(u^2+v^2) + O(4) \\ (l_1 u + bv)(u^2+v^2) + O(4) \end{pmatrix}, \quad (13)$$

This system can be expressed in polar coordinates as

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} l_1 r^3 + \cdots \\ \omega + br^2 + \cdots \end{pmatrix}. \quad (14)$$

The approach detailed in [7] also includes the computation of the first Lyapunov coefficient, denoted by l_1, which is given by the following formula

$$l_1 = \frac{1}{16}\left(R_1 + \frac{R_2}{\omega}\right) \quad (15)$$

with the components and defined as

$$R_1 = F^1_{z_1 z_1 z_1} + F^1_{z_1 z_2 z_2} + F^2_{z_1 z_1 z_2} + F^2_{z_2 z_2 z_2},$$
$$R_2 = F^1_{z_1 z_2}\left(F^1_{z_1 z_1} + F^1_{z_2 z_2}\right) - F^2_{z_1 z_2}\left(F^2_{z_1 z_1} + F^2_{z_2 z_2}\right) - F^1_{z_1 z_1} F^2_{z_1 z_1} + F^1_{z_2 z_2} F^2_{z_2 z_2}.$$

Therefore,

$$l_1 = -\frac{\beta\eta_2(\beta^2\omega^2 + \eta_2^2)}{4\eta_1^2\omega^2}.$$

Since $\omega > 0$, $\eta_1 > 0$ and $(\beta^2\omega^2 + \eta_2^2) > 0$ then the sign of l_1 is determined by the sign of $\beta\eta_2$. Thus, the entire preceding analysis of the Hopf bifurcation can be encapsulated in the following theorem.

Theorem 1. System (8) exhibits a Hopf bifurcation at the equilibrium E_1, The bifurcation behavior is characterized by the following conditions

i. If $\beta\eta_2 > 0$, then $l_1 < 0$. This implies that the bifurcation is of supercritical type and the resulting periodic orbit is stable. These solutions exist when $\alpha < \beta - (\mu + 2\delta + \theta)$.

ii. If $\beta\eta_2 < 0$, then $l_1 > 0$. This implies that the bifurcation is of subcritical type and the resulting periodic orbit is unstable. These solutions exist when $\alpha > \beta - (\mu + 2\delta + \theta)$.

Numerical Example 1. If we fix $\beta = 2$, $\delta = 1$, $\mu = 2$, $\theta = \frac{3}{2}$ and $\alpha = -\frac{5}{2}$, then $\eta_1 = 5$, $\eta_2 = 1$, $\omega = 1$ and $l_1 = -\frac{1}{10}$. Since $l_1 < 0$, the equilibrium point E_1 is stable and system (8) has a stable periodic solution with a period of 2π. The Hopf bifurcation is of the supercritical type. Due to $l_1 < 0$ and $d < 0$, this implies that $\frac{d}{l_1} > 0$. Therefore, this periodic solution occurs for $\alpha < -\frac{5}{2}$ and in this interval, both equilibrium points E_1 and E_2 are unstable.

Numerical Example 2. If we set $\beta = 4$, $\delta = 1$, $\mu = 2$, $\theta = \frac{5}{2}$ and $\alpha = -\frac{5}{2}$, then we have $\eta_1 = 10$, $\eta_2 = -2$, $\omega = 1$, and $l_1 = \frac{2}{5}$. Given that $l_1 > 0$, the equilibrium point E_1 is unstable and system (8) exhibits an unstable periodic solution with a period of 2π. The Hopf bifurcation is of the subcritical type.

Because $l_1 > 0$ and $d < 0$, it follows that $\frac{d}{l_1} < 0$. Consequently, this periodic solution exists for $\alpha > -\frac{5}{2}$.

B. Hopf Bifurcation at E_2

This subsection examines the Hopf bifurcation behavior of system (1) at the equilibrium point E_2. The proposition below outlines the key finding related to when the Hopf bifurcation occurs.

Proposition 4. When $\theta = \theta_0 = -\frac{\alpha^2+\alpha\beta+\alpha\delta+\beta\delta+\beta\mu+\delta\mu+\mu^2}{\alpha-\delta-\mu}$, system (1) experiences a Hopf bifurcation at the equilibrium point E_2 when the parameters satisfy the following conditions: $\delta \neq 0$ and $\frac{(\alpha\beta+\delta^2+2\delta\mu+\mu^2)(\alpha+\delta+\mu)}{\alpha-\delta-\mu} > 0$. (16)

Proof.

Setting $\theta = \theta_0$, equation (4) can be expressed in the following form:

$$(\delta+\lambda)((\alpha-\delta-\mu)\lambda^2 + (\alpha+\delta+\mu)(\alpha\beta+\delta^2+2\delta\mu+\mu^2)) = 0. \quad (17)$$

It is evident that equation (17) possesses a pair of purely imaginary conjugate roots, $\lambda_{1,2} = \pm\omega i$ with $\omega = \sqrt{\frac{(\alpha\beta+\delta^2+2\delta\mu+\mu^2)(\alpha+\delta+\mu)}{\alpha-\delta-\mu}}$, along with a real root $\lambda_3 = -\delta$. Therefore, the first requirement of the Hopf bifurcation theorem is satisfied. It is important to note that, in general, λ is a function of θ, $\lambda = \lambda(\theta)$. From equation (4), we can establish the following relation:

$$f(\lambda(\theta),\theta)\& = (\alpha+\delta+\mu)\lambda^3 + (\alpha^2+\alpha\beta+2\alpha\delta+\theta\alpha+\beta\delta+\beta\mu+\delta^2+2\delta\mu-\delta\theta+\mu^2-\mu\theta)\lambda^2 + (\alpha^2\beta-\alpha^2\mu+3\alpha\beta\delta+2\alpha\beta\mu-\alpha\,\delta^2-3\alpha\delta\mu-\alpha\,\mu^2-\alpha\mu\theta+2\beta\,\delta^2+3\beta\delta\mu+\beta\,\mu^2-\delta^3-\delta^2\mu-2\delta^2\theta-3\delta\mu\theta-\mu^2\theta)\lambda + (\alpha+\delta+\mu)\delta(\alpha\beta-\alpha\delta-\alpha\mu+\beta\delta+\beta\mu-\delta^2-\delta\mu-\delta\theta-\mu\theta).$$

Therefore, any root $\lambda(\theta)$ of equation (4) must fulfill the following relation:

$$f(\lambda(\theta),\theta) = 0 \quad (18)$$

Differentiating equation (18) with respect to θ gives:

$$\frac{\partial f}{\partial\lambda}\cdot\frac{d\lambda(\theta)}{d\theta} + \frac{\partial f}{\partial\theta} = 0,$$

which consequently indicates that

$$\frac{d\lambda(\theta)}{d\theta} = -\frac{\partial f}{\partial\theta}\left(\frac{\partial f}{\partial\lambda}\right)^{-1}. \quad (19)$$

At $\theta = \theta_0$,

$$\beta = -\frac{\alpha\,\delta^2+2\alpha\delta\mu+\alpha\,\mu^2-\alpha\,\omega^2+\delta^3+3\delta^2\mu+3\delta\,\mu^2+\delta\,\omega^2+\mu^3+\mu\,\omega^2}{(\alpha+\delta+\mu)\alpha}; \alpha \neq 0$$

and $\lambda = i\omega$, the following is obtained:

$$\frac{d}{d\alpha}Re(\lambda_{1,2}(\theta))\Big|_{\theta=\theta_0} = -\frac{\alpha-\delta-\mu}{2(\alpha+\delta+\mu)} = d \neq 0.$$

Thus, the second condition for the Hopf bifurcation is met. As a result, at E_2, system (1) undergoes a Hopf bifurcation when the parameters satisfy conditions (16), giving rise to a periodic solution in the neighborhood of the equilibrium point.

A detailed examination of the Hopf bifurcation in system (1) is now carried out, focusing on determining the stability of the resulting periodic solution through the first Lyapunov coefficient. To start the analysis, the equilibrium point E_1 of system (1) is shifted to the origin $E_{20} = (0,0,0)$ using the linear transformation $x = X_1 + x_0$, $y = Y_1 + y_0$ and $z = Z_1 + z_0$. Under this transformation, system (1) takes the following form:

$$\dot{X}_1 = \frac{1}{\alpha(\alpha+\delta+\mu)}(-(\delta^2+(\mu+\alpha)\delta+\omega^2)\alpha X_1 - \alpha(\alpha^2+(2\mu+3\delta)\alpha+\mu^2+3\delta\mu+2\delta^2+\omega^2)Y_1 + (\mu^3+(2\alpha+3\delta)\mu^2+(\alpha^2+3\alpha\delta+3\delta^2+\omega^2)\mu+\delta(\alpha\delta+\delta^2+\omega^2))Z_1 + (\delta^3+(3\mu+\alpha)\delta^2+(2\alpha\mu+3\mu^2+\omega^2)\delta+\mu^3+\alpha\,\mu^2+\mu\,\omega^2-\alpha\,\omega^2)X_1Y_1 + (\delta^3+(3\mu+\alpha)\delta^2+(2\alpha\mu+3\mu^2+\omega^2)\delta+\mu^3+\alpha\,\mu^2+\mu\,\omega^2-\alpha\,\omega^2)X_1Z_1),$$

$$\dot{Y}_1 = \frac{1}{\alpha(\alpha+\delta+\mu)}(\alpha\,\omega^2X_1 + \alpha\big((\mu+\delta)\alpha+\mu^2+2\delta\mu+\delta^2+\omega^2\big)Y_1 - (\delta^2+(2\mu+\alpha)\delta+\mu^2+\alpha\mu+\omega^2)(\mu+\delta)Z_1 + (-\delta^3+(-3\mu-\alpha)\delta^2+(-2\alpha\mu-3\mu^2-\omega^2)\delta-\mu^3-\alpha\,\mu^2-\mu\,\omega^2+\alpha\,\omega^2)X_1Y_1(-\delta^3+(-3\mu-\alpha)\delta^2+(-2\alpha\mu-3\mu^2-\omega^2)\delta-\mu^3-\alpha\,\mu^2-\mu\,\omega^2+\alpha\,\omega^2))X_1Z_1,$$

$$\dot{Z}_1 = \alpha Y_1 - (\mu+\delta)Z_1. \quad (20)$$

Now, it is essential to obtain the equation that describes the two-dimensional behavior on the center manifold at the bifurcation point. System (20) has been transformed into its canonical form by applying the following linear transformation.

$$\begin{pmatrix}X_1\\Y_2\\Z_3\end{pmatrix} = M\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix},$$

where $M = \begin{pmatrix}\frac{-\delta-\mu-\alpha}{\alpha} & \frac{\omega}{\alpha} & \frac{(\delta^2+(\mu+\alpha)\delta+\omega^2)\delta}{\alpha\,\omega^2}\\ \frac{\delta+\mu}{\alpha} & -\frac{\omega}{\alpha} & \frac{\mu}{\alpha}\\ 1 & 0 & 1\end{pmatrix}$.

System (20) can be represented in the following form:

$$\begin{pmatrix}\dot{y}_1\\\dot{y}_2\\\dot{y}_3\end{pmatrix} = \begin{pmatrix}0 & -\omega & 0\\ \omega & 0 & 0\\ 0 & 0 & -\delta\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix} + \begin{pmatrix}0\\ \eta3\big((\alpha+\delta+\mu)y_1-\omega y_2+(\mu+\alpha)y_3\big)((-\alpha\,\omega^2-\delta\,\omega^2-\mu\,\omega^2)y_1 + \omega^3y_2+(\alpha\,\delta^2+\delta^3+\delta^2\mu+\delta\,\omega^2)y_3)\\ 0\end{pmatrix} \quad (21)$$

where $\eta3 = \frac{\alpha\,\delta^2+2\alpha\delta\mu+\alpha\,\mu^2-\alpha\,\omega^2+\delta^3+3\delta^2\mu+3\delta\,\mu^2+\delta\,\omega^2+\mu^3+\mu\,\omega^2}{(\alpha+\delta+\mu)\alpha^2\omega^3}$.

By virtue of the center manifold theorem, there exists a $\epsilon > 0$ and a function H_2 defined within a neighborhood around the origin, which characterizes the local center manifold.

$W_2^c(0) = \{(y_1, y_2, y_3): y_3 = H_2(y_1, y_2)$ for $\| (y_1, y_2) \| < \epsilon_2\}$.

To compute the center manifold of system (21), the following expression is proposed

$$H_2(y_1, y_2) = \nu_1 y_2^2 + \nu_2 y_1 y_2 + \nu_3 y_1^2 + \mathcal{O}(3), \tag{22}$$

The coefficients ν_1, ν_2 and ν_3 are determined through the following derivation

$$\dot{y}_3 = \frac{\partial H_2}{\partial y_1}\dot{y}_1 + \frac{\partial H_2}{\partial y_2}\dot{y}_2. \tag{23}$$

The partial derivatives of H_2 are computed and together with $\dot{y}_1$, $\dot{y}_2$ and $\dot{y}_3$ from system (21), are substituted into equation (23). Solving the resulting system yields

$$\nu_1 = \nu_2 = \nu_3 = 0.$$

Another implication of the center manifold theorem is that the dynamics on the center manifold are governed by the following system of differential equations:

$$\begin{pmatrix}\dot{y}_1\\ \dot{y}_2\end{pmatrix} = \begin{pmatrix}0 & -\omega\\ \omega & 0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix} + \begin{pmatrix}G^1(y_1, y_2)\\ G^2(y_1, y_2)\end{pmatrix}, \tag{24}$$

with

$\| (y_1, y_2) \| < \epsilon$, $\quad G^1(0,0) = G^2(0,0) = 0$, $\quad DG^1(0,0) = DG^2(0,0) = 0$ and by system (21)

$$G^1(y_1, y_2) = 0,$$
$$G^2(y_1, y_2) = -\eta 3\, \omega^2(\alpha y_1 + \delta y_1 + \mu y_1 - \omega y_2)^2.$$

Using the transformation and formula described in equations (13)–(15), where

$$R_1 = G^1_{y_1y_1y_1} + G^1_{y_1y_2y_2} + G^2_{y_1y_1y_2} + G^2_{y_2y_2y_2},$$

$$R_2 = G^1_{y_1y_2}\left(G^1_{y_1y_1} + G^1_{y_2y_2}\right) - G^2_{y_1y_2}\left(G^2_{y_1y_1} + G^2_{y_2y_2}\right) - G^1_{y_1y_1}G^2_{y_1y_1} + G^1_{y_2y_2}G^2_{y_2y_2}$$

the following first Lyapunov coefficient is obtained:

$$l_1 = \frac{\omega^4\eta_3^2(\alpha + \delta + \mu)((\alpha + \delta + \mu)^2 + \omega^2)}{4}.$$

When $\eta_3 \neq 0$, the sign of l_1 is determined by the sign of the term $(\alpha + \delta + \mu)$. Thus, the entire discussion on the Hopf bifurcation can be concisely summarized in the following theorem.

Theorem 2. For system (20) satisfying conditions (16) with $\alpha \neq 0$, a Hopf bifurcation occurs at the equilibrium point E_2 , with the following characteristics:

i. If $\alpha + \delta + \mu < 0$, then $l_1 < 0$. This implies that the bifurcation is of supercritical type and the resulting periodic orbit is stable.

ii. If $\alpha + \delta + \mu > 0$, then $l_1 > 0$. This implies that the bifurcation is of subcritical type and the resulting periodic orbit is unstable.

- When $\frac{d}{l_1} > 0$, a periodic solution occurs in the region $\theta < \theta_0$. Conversely, when $\frac{d}{l_1} < 0$, a periodic solution occurs in the region $\theta < \theta_0$ [9].

Numerical Example 3. If we take $\beta = 0.1$, $\delta = -2$, $\mu = 1$, $\alpha = -4$ and $\theta = \frac{15}{2}$, we obtain $\eta_3 = \frac{1}{40}$, $\omega = 1$ and $l_1 = -\frac{213}{640}$. Since $l_1 < 0$, the equilibrium point E_1 is stable and system (8) exhibits a stable periodic solution with a period of 2π. The Hopf bifurcation is of the supercritical type. Because $l_1 < 0$ and $d < 0$, it follows that $\frac{d}{l_1} > 0$. Consequently, this periodic solution exists for $\theta < \frac{15}{2}$ and in this range, both equilibrium points E_1 and E_2 are unstable.

Numerical Example 4. If we set $\beta = -4.6$, $\delta = 1$, $\mu = 2$, $\alpha = 2$ and $\theta = -11$, we find that $\eta_3 = 2.3$, $\omega = 1$, $d = \frac{1}{10}$ and $l_1 = \frac{6877}{40}$. Since $l_1 > 0$, the equilibrium point E_2 is unstable and system (8) has an unstable periodic solution with a period of 2π. The Hopf bifurcation is of the subcritical type. Given that $l_1 > 0$ and $d > 0$, it follows that $\frac{d}{l_1} > 0$. Therefore, this periodic solution occurs for $\theta < -11$ and within this range, the equilibrium point E_2 is stable, while the equilibrium point E_1 is unstable.

a. Transcritical Bifurcation

Consider the following differential equation defined in $\mathbb{R}^n$:

$$\dot{x} = f(x, a), \tag{25}$$

where $a \in \mathbb{R}$ represents a parameter and f is assumed to be differentiable. The system in (25) is said to undergo a transcritical bifurcation when the requirements of Sotomayor's theorem are fulfilled. This result provides sufficient criteria to confirm the presence of a transcritical bifurcation in a dynamical system [7, 34].

Theorem 3. (Sotomayor Theorem). Consider system (25) and let there be a point $p_0 \in \mathbb{R}^n$ such that $f(p_0, a) = 0$ for all a, i.e., p_0 is an equilibrium point of the system. Moreover, if $a = a_0$, assume the following condition satisfies:

i. The Jacobian matrix $J = Df(p_0, a_0)$ has a zero eigenvalue with an eigenvector v, and J^T has an eigenvector w corresponding to zero eigenvalue. Furthermore, J has k eigenvalues with negative real parts and has $n - k - 1$ eigenvalues with positive real parts, where $0 \leq k \leq n - 1$.

ii. $w^T f_a(p_0, a_0) \neq 0$.

iii. $w^T[D^2 f(p_0, a_0)(v, v)] \neq 0$.

Then system (25) exhibits a saddle-node bifurcation at the equilibrium p_0 as μ passes through $a = a_0$.

Remark 1. In Theorem 3, if the conditions (ii) and (iii) are changed to:

$$w^T f_a(p_0, a_0) = 0, \quad w^T[Df_a(p_0, a_0)v] \neq 0, \quad \text{and} \quad w^T[D^2 f(p_0, a_0)(v, v)] \neq 0, \tag{26}$$

then system undergoes a transcritical bifurcation at the equilibrium point p_0 when the parameter α passes through the bifurcation value $\alpha = \alpha_0$ [36].

Proposition 5. When $\theta_0 = \frac{(\alpha\beta - \alpha\delta - \alpha\mu + \beta\delta + \beta\mu - \delta^2 - \delta\mu)}{\mu+\delta}$, system (1) exhibits a transcritical bifurcation at the equilibrium point E_1 as the parameter θ passes through θ_0, provided that $\alpha\beta\delta(\mu+\delta)(\alpha\beta + (\mu+\delta)^2)(\alpha+\delta+\mu) \neq 0$.

Proof.

We use the notation from Remark 1, setting $\alpha = \theta$ and $x = (x, y, z) \in \mathbb{R}^3$. System (1) has two equilibrium points: $E_1 = (1,0,0)$ and $E_2 = (x_0, y_0, z_0)$. These equilibrium points collide at E_1 when $\theta = \theta_0$.

By linearizing system (1) at the equilibrium point E_1 when $\theta = \theta_0$, the corresponding Jacobian matrix is derived as follows:

$$J = \begin{bmatrix} -\delta & \dfrac{-\delta^2 + (-\alpha-\mu)\delta - \alpha(-\beta+\mu)}{\mu+\delta} & -\beta+\mu \\ 0 & -\dfrac{\alpha\beta}{\mu+\delta} & \beta \\ 0 & \alpha & -\delta-\mu \end{bmatrix}.$$

The corresponding characteristic equation can then be written in the form:

$$\lambda(\lambda+\delta)\big((\mu+\delta)\lambda + (\mu+\delta)^2 + \alpha\beta\big) = 0.$$

It has a simple zero with two non-zero solutions: $-\delta$ and $-\frac{(\alpha\beta + (\mu+\delta)^2)}{\mu+\delta}$. The following vectors

$$v = \begin{pmatrix} -\dfrac{\alpha+\delta+\mu}{\alpha} \\ \dfrac{\mu+\delta}{\alpha} \\ 1 \end{pmatrix} \quad \text{and} \quad \omega = \begin{pmatrix} 0 \\ \dfrac{\mu+\delta}{\beta} \\ 1 \end{pmatrix},$$

serve as eigenvectors of the Jacobian J as well as its transpose J^T, both corresponding to the eigenvalue $\lambda_1 = 0$. Based on Theorem 3, the following relations can be expressed:

$$\omega^T f_\theta(E_1, \theta_0) = 0, \quad \omega^T Df_\theta(E_1, \theta_0) = -\frac{(\mu+\delta)^2}{\alpha\beta} \neq 0, \quad \text{and}$$

$$\omega^T(D^2 f(E_1, \theta_0)) = -\frac{2(\mu+\delta)(\alpha+\delta+\mu)^2}{\alpha^2} \neq 0.$$

Consequently, all the hypotheses stated in Remark 1 are fulfilled. As a result, the system exhibits a transcritical bifurcation at E_1 when the bifurcation value is $\theta = \theta_0$.

Numerical Example 5. If we set the following parameters: $\alpha = 2$, $\beta = 3$, $\mu = 1$ and $\delta = 1$, the bifurcation parameter θ_0 becomes 3, resulting in only one equilibrium point at $E_1 = (1,0,0)$. This indicates that the two equilibrium points are colliding. At this equilibrium point, all the conditions of the Sotomayor Theorem are satisfied:

$$\omega^T f_\theta(E_1, \theta_0) = 0, \qquad \omega^T Df_\theta(E_1, \theta_0) = -1 \neq 0$$

and $\omega^T(D^2 f(E_1, \theta_0)) = -24 \neq 0$.

The stability of the equilibrium points is as follows:

- When $\theta < 3$, the equilibrium point E_1 is unstable and E_2 is stable.
- When $\theta = 3$, the dynamics on the center manifold reveal that the single equilibrium point E_1 is unstable.
- When $\theta > 3$, the equilibrium point E_1 is stable, while E_2 is unstable.

Thus, we observe a change in the stability of the equilibrium points at $\theta = 3$.

IV. CONCLUSION

An analysis of the dynamical behavior of the generalized Computer Virus Propagation Model under variations in key parameters is provided in this study. The stability of the equilibrium points is systematically investigated using the corresponding characteristic equations, with a specific focus on the emergence of Hopf bifurcations under certain parameter regimes. It is identified that a Hopf bifurcation occurs at equilibrium E_1 when $\alpha = \beta - (\mu + 2\delta + \theta)$ and at equilibrium E_2 when $\theta = \theta_0$. To characterize the oscillatory dynamics near the Hopf bifurcation, the first Lyapunov coefficient is computed, confirming the existence of periodic solutions originating from these equilibria. Additionally, numerical simulations are conducted to illustrate the coexistence of stable and unstable periodic solutions, reinforcing the theoretical findings. The occurrence of transcritical bifurcations is also verified under suitable parameter conditions, consistent with Sotomayor's theorem. A numerical example is also presented to support its theoretical result.

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