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Uniformly Metrizable Bornologies: A Unified Characterization and Applications to Fundamental Bornologies in Metric Spaces

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Abstract**—***This paper presents a study of bornological structures within metric spaces, with a central focus on the property of being a "uniform metric" bornology. The primary objectives are to characterize such bornologies and to establish necessary and sufficient conditions for this property to hold for several fundamental types. We provide a characterization theorem for uniform metric bornologies and apply this framework to analyze specific cases, including the compact, totally bounded, and Bourbaki-bornological structures. Furthermore, we demonstrate that several previously known results emerge as direct corollaries of our general theorems. The analysis employs tools from mathematical analysis and topology, utilizing the construction of characteristic functions alongside concepts of metric extensions and uniform restrictions. This work yields a deeper theoretical understanding of uniform metric bornologies and provides applicable tools for their investigation in varied contexts. The results are subsequently applied to the study of metric spaces possessing special properties, such as locally bounded and complete spaces.*

Keywords—Bornology; Uniform Metrizability; Characteristic Function; Compact Bornology; Totally Bounded Bornology; Bourbaki-Bounded Set; Heine-Borel Property.

I. INTRODUCTION AND PRELIMINARIES

The concept of boundedness is fundamental in analysis and topology. A bornology in provides an abstract formulation of this concept. Now mentioned some previously studied related points such as: Atsuji [1] established the properties of Bourbaki-bounded subsets within metric spaces: a subset B is Bourbaki-bounded iff any real-valued uniformly continuous mappings remains bounded when restricted to B, which holds precisely when X is small-determined [7]. This applies to normed spaces, length spaces, quasi-convex spaces, and all small-determined spaces.

Small-determined spaces (Garrido & Jaramillo [7]) allow uniform approximated of uniformly continuous mappings by Lip.functions, ensuring such functions remain bounded on bounded sets. The converse fails (e.g., ℕ with standard metric). Hejcman [11] studied when Bourbaki-bounded sets coincide with bounded sets for some uniform pseudometric, defining a uniform space as B-simple. This occurs iff X is σ-Bourbaki-bounded and B-conservative (i.e., ∃ entourage U s.t$U[B]$ is Bourbaki-bounded of any Bourbaki-bounded$B$). For metric spaces, this means ∃δ>0 s.t B^δ is Bourbaki-bounded.
By specifying the precise conditions that allow the existence of equivalent uniform metrics ensuring ideal properties such as finite bounded structure or the Heine–Borel property, its significance lies in linking the abstract concepts of bornology with concrete geometric properties. This provides a theoretical framework that simplifies the analysis of complex spaces and opens new horizons for research.

Definition 1.1[6]: A bornology ß on X is a collection for subsets of a nonempty set that satisfy axioms:
(i) ß covers X,
(ii) If $\beta 1, \beta 2, \dots .., \beta n$ ∈ ß, then $\bigcup_{k=1}^{n} \beta_k$ ∈ ß, $n \in \mathbb{N}$
(iii) If β ∈ß and $\beta' \subseteq$ β, then β' ∈ ß
A pair (X, ß) we say that bornological space, if, in addition, the follow axioms hold, then (X, ß) is called a bornological vector space such that is a vector space over K and satisfying the conditions

(iv) If $N_1, N_2 \in B, then\ N_1 + N_2$ ∈ß

(v) If $\gamma \in K, N$ ∈ß $then\ \gamma N$ ∈ß

(vi) If N ∈ß, $then\ \bigcup_{|\gamma| \leq 1} N$ ∈ß.

Definition 1.2[6]: bornological universe (, ß) , s.t X is a non-emptyset & ß is a collection for subsets of X satisfying:

1. $\forall x \in X$,.⇒ $\{x\}$ ∈ ß.
2. If was B ∈ ß & $A \subset B$, then be A ∈ ß.

3. If was , $B \in ß$, then be $A \cup B \in ß$.

Theelements of B are called bounded sets.

The canonical example arises from metric spaces. the metric space(X , ρ), the collection $ß_\rho$ (X) of every ρ-bounded subsets for a bornology.

This work focuses in metrizable bornological algebras (mba), denoted by the triple (, ß ,d), where d d is a metrik inducing the topology & ß is thebornology.

Definition1.3[6]: Let (X, d) is a metric space and let B sub set of X such that $B\ X$ then B called metric bornonlogy if : $sup_{b,t\in B}\ d(b,t) < \infty$ is finite.

Definition1.4[6]: Let (X, d) is a metric space and let B family of subsets of X such that $ß\ is\ bornology\ on\ X$ then B called metric bornonlogy if : $sup_{b,t\in ß}\ d(b,t) < \infty$ is finite.

Definition 1.5[5]. The bornology B in mba (, ß,d) to be uniformly metrizable if has metric ρ in X, uniformly equivalentto d, s.t ß = $ß_\rho$ (X).

The base C for the bornology B to be subcollection $C \subset ß$ s.t every $B \in ß$ to be contained at some $C \in C$.The subset $A \subset X$ and ε>0, ε-enlargement is we define by $\varepsilon = \{x \in X{:}d(x,A) < \varepsilon\}$.

II. A CHARACTERIZATION OF UNIFORMLY METRIZABLE BORNOLOGIES

A key tool A key tool for characterizing uniform metrizability is the characteristic function.

Definition 2.1. A characteristic mapping for the bornological universe $(X$, ß) to be a continuous mapping $\chi : X \to [\,0 , \infty\,)$ s.t a $E \subset X$ is bounded iff χ (E) to be bounded on [0 , ∞) .

The following proposition links the existence of a uniformly continuous characteristic function to a specific property of a bornological base.

Proposition 2.2. If was (, ß, d) is mba. Then is (, ß) admits a uniformly continuous characteristic mapping iff ß be have countable base { $B\ n$}; $n \in N$ s.t to any δ>0, the following condition holds:

$B^{\delta}_{\mathrm{n}} \subset B_{\mathrm{n}+1}$ for every $n \in N$.

Proof. First, assume that ß has a countable base satisfying the above hypotheses. Then, for any $n \in N$, we define the mapping $\emptyset_n{:}X \to [0,1]$ by

$$\emptyset_n(x) = min\left\{1, \frac{1}{\delta} d(x, B_n)\right\}.$$

Clearly, $\emptyset_n(B_n) = 0, \emptyset_n(X - B_{n+1}) = 1 and \emptyset_n$is Uc, in fact it is a $1/\delta - Lip$ function. Now if taking $B_0 = \emptyset$ and $\emptyset_0(x) = 1$, for any $x \in X$, We study $\chi{:}X \to [0, \infty)$ defined by,

$$\chi(x) = n - 2 + \emptyset_{n-1}(x)$$

when $x \in B_n - B_{n-1}, n \in N$. In this way, χ is a Uc characteristic mapping of $(X, ß)$. Indeed, if $d(x,y) < \delta$, there is $n \in N$ s.t both x, y are in $B_n - B_{n-1}$, $or\ x \in B_n - B_{n-1}$and $y \in B_{n+1} - B_n$. In every case, it is easy to check that

$$|\chi(x) - \chi(y)| \le |\phi_{n-1}(x) - \phi_{n-1}(y) + \phi_n(x) - \phi_n(y)| \le \frac{2}{\delta} d(x,y).$$

And so, χ is not only uniformly continuous but it is in fact $\frac{1}{\delta} - Lipschitz$ on each $\delta - ball$. In particular, χ is what is called a Lipschitz in the small function.

Finally, if $E \in ß$, there exists $n \in N$ s.t $E \subset B_n$, which implies $\chi(E) \subset [0, n-1]$.

Conversely, if $\chi(E)$ is bounded, then $E \subset B_n$for some $n \in N$. Otherwise, for every $n \in N$, there would exist $x \in E$ with $\chi(x) \ge n$.

Now, suppose $(X, ß)$ admits a uniformly continuous characteristic function χ. By uniform continuity, there exists $\delta > 0$ s.t $d(x,y) < \delta$ implies $|\ \chi(x) - \chi(y)\ | \le 1$.

Defining

$B_n = \chi^{-1}([0, n])$ for $n \in N$, the family $\{B_n\}_n$, $n \in N$ forms a countable base for ß that satisfies the required property for this δ . □

This leads to the central characterization theorem.

Theorem 2.3. Let the (, ß,d) if is a mba. Then the following holds be equivalent:

1. ß to be uniformly metrizable.
2. ß be have countable base{ B_{n} : $n \in N$ } s.t to any δ>0, $B^{\delta}_{\mathrm{n}} \subset B_{\mathrm{n}+1}$ to any $n \in N$.
3. Ther exist a uniformly continuous characteristic mapping to $(X$, ß).

Proof. The equivalence of (2) and (3) is established in Proposition 2.2.

(1) ⇒ (2): If ß = $ß_\rho$ (X) to themetric ρ uniformly equivalent for d , the collection ρ-balls $B_{\mathrm{n}} = \{x{:}\ \rho(x,x_0) < n\}$ be a base satisfying the condition.

(2) ⇒ (1): Using the characteristic function χ nfrom Proposition 2.2, and the metric$\rho(\ x,y) = \min\{1, d(x,y)\ \} \vee |\ \chi\ (\ x\ ) - \chi\ (\ y\ )\ |$ to be uniformlyequivalent for d d and satisfies ß = $ß_\rho$ (X). □

Remark 2.4. The construction can be refined so that the new metric ρ is uniformly locally identical to , meaning they coincide on a neighborhood of the diagonal.

this framework establishes the criteria and the constructive methodology for governing a space's boundedness structure by an appropriate metric, thereby facilitating a more tractable analysis of its topological and algebraic characteristics in advanced mathematics.

III. APPLICATIONS TO SPECIFIC BORNOLOGIES

The Finite Bornology: Let $F\text{ß}(X)$ denote abornology for all finite subsets for X . the metricspace (X, d) is considered uniformly locally finite provided that there is a positive real element δ>0 for which any openball for radiu δ has a finite number of points.

Theorem 3.1. If was (X, ß,) is a mba. The following holds are equivalently:

1. F ß ( ) is uniformly metrizable.
2. There exists a metric , uniformly equivalentto d , s.t (X , ρ) is locally finite.
3. (, d) to be countable & uniformlylocally finite.

Proof.(1) ⇔ (2): By definition.
(2) ⇒ (3): A locally finite space is countable. Uniform equivalence implies uniform local finiteness of (X,).
(3) ⇒ (1): Using the countable base B_n ={ x_n } ∪ (B_{n-1}) δ , one can apply Theorem 2.3. □

IV. THE COMPACT BORNOLOGY AND THE HEINE-BOREL PROPERTY

Let $C\text{ß}_d$ ( ) denote the bornology generated of compact subsets for (X,d). A metric space (, ρ) possesses Heine–Borel characteristic precisely when each of its closed & bounded subsets to be compact.i.e., $C\text{ß}\rho$ (X) = $\text{ß}\rho(X)$.
A space qualifies as hemicompact when one can find a sequence of its compact subsets s.t any compact subset of space X will be contained within some member of this sequence.
Themetric space is termed uniformly locally compact if one can find a number δ > 0 s.t every closed ball with a radius of δ is compact.

Theorem 4 .1. If we have (, ß, d) is mba. Then thefollowing holds are equivalently:

1. C ß_d ( ) will be uniformly metrizable.
2. There exists the metric ρ, uniformly equivalent to , with the Heine-Borel property.
3. (,d) is Lindelöf & uniformlylocally compact.

Proof.
(1) ⇒ (2): If $C\text{ß}\rho$ ( ) = $\text{ß}\rho(X)$ for a uniformly equivalent metric ρ , then ρ be have Heine-Borel property.
(2) ⇒ (3): A space with the Heine-Borel property is uniformly locally compact and σ-compact (hence Lindelöf).
(3) ⇒ (1): A Lindelöf, locally compact space is hemicompact. One can construct a countable base { B_n } for $C\text{ß}_d$ ( ) satisfying $B_n^{\frac{\delta}{2}} \subset B_{n+1}$, allowing an application of Theorem 2.3. □

Corollary 4.2. the metric space(X,d) admits a metric ρ , uniformly locally identical to d , with the Heine-Borel property iff (X , d) to be uniformlylocally compact and σ-compact.

V. THE TOTALLY BOUNDED BORNOLOGY

Let $T\text{ß}_d$ ( ) denote the bornology generated by the totally bounded (also known as precompact) subsets of a metric space (X, d). A metric space is classified as uniformly locally totally bounded if one can find a positive real number δ such that every open ball of radius δ is totally bounded.

Theorem 5.1. Let (, ß , d) be a mba and let ($\tilde{X}$, $\tilde{d}$) be its completion. The following holds are equivalent:

1. $T\text{ß}_d$ ( ) is uniformlymetrizable.
2. $T\text{ß}_{\tilde{d}}$ ($\tilde{X}$) is uniformly metrizable.
3. ($\tilde{X}$, $\tilde{d}$) is Lindelöf and uniformly locally compact.
4. (, d) is Lindelöf and uniformly locally totally bounded.

Proof. (1) ⇒ (2). From Theorem 2.4, condition (1) implies that $T\text{ß}_d(X)$admits a countable base $\{B_n\}_n$ s.t , for $\delta > 0$, ${B_n}^{\delta} \subset B_{n+1}$, for all n. Now, according to Lemma 4.1, we have that $\{cl_{\tilde{X}} B_n\}_n$is a countable base for $T\text{ß}_{\tilde{d}}(\tilde{X})$.

Moreover, this base

$$(cl_X B_n)^{\delta/2} \subset cl_X B_{n+1}$$

for every n. Indeed, it is not difficult to prove that in fact when $A, B \subset X$ with
$A^{\delta} \subset B$, then $(cl_{\tilde{X}} A)^{\delta/2} \subset cl_{\tilde{X}} B$. And finally, again from Theorem 2.4, we deduce e e

that $T\text{ß}_{\tilde{d}}(\tilde{X})$ is uniformly metrizable.

(3)⇒ (1). If $\hat{d}$is a metric on $\tilde{X}$ uniformly equivalent to $d˜$ such that $T\text{ß}_{\tilde{d}}(\tilde{X}) = T\text{ß}_{\hat{d}}(\tilde{X})$, then it is clear that the restriction to$'X, d' = \hat{d}|X \times X$is d' uniformly equivalentto d, s.t $T\text{ß}_d(X)$ = $\text{ß}_d(X)$. Hence (1) follows.

(2) ⇔ (3). This equivalence follows directly from Theorem

3.1, sincefor($\tilde{X}, d˜$)we have that$T\text{ß}_{\tilde{d}}(\tilde{X}) = C\text{ß}_{\tilde{d}}(\tilde{X})$.

(3) ⇒ (4). It is clear.

(4) ⇒ (3). Firstly, note that X is Lindel¨of if, and only if, $\tilde{X}$ is Lindel¨of. On the other hand, let $\delta > 0$ such that every $\delta - ball\ in\ X$ is totally bounded. Then every closed $\delta/2 - ball\ in\ X$ is compact. Indeed, let $y \in X$, then ther is $x \in X$ with $d\tilde{}(x,y) < \delta/2$. Then we have,

$B_{\tilde{d}}[y,\delta/2] \subset B_{\tilde{d}}(x,\delta) \subset cl_{\tilde{X}}[B_{\tilde{d}}(x,\delta) \cap X] = cl_{\tilde{X}}B_d(x,\delta)$.

And we finish since $cl_{\tilde{X}}B_d(x,\delta)$ is totally bounded in $\tilde{X}$ and then it is compact. □

This theory presents the equivalence of four topological properties, providing a significant contribution to the understanding of the structures of metric spaces and their completions through the concepts of regular measurability, the Lindelöf property, and local compactness.

VI. THE BOURBAKI-BOUNDED BORNOLOGY

A sub set B for a uniform space $(X, \mathcal{U})$ to be Bourbaki-bounded if any entourage $U \in \mathcal{U}$, ther exict the finite set $F \subset X$ and $m \in N$ s.t $B \subset U^m[F]$,. where$U^1 = U, U^2 = U \circ U$, and in general $U^m = U \circ (U^{m-1})$. In a metric space (X,d), B is Bourbaki-bounded if every $\varepsilon > 0$ ther are a set $F = \{x_1, \dots, x_n\} \subset X$ and some $m \in N$ such that

$$B \subset \bigcup_{i=1}^{n} B(x_i,\varepsilon)^m.$$

Here a point y A sub set B for $(X, \mathcal{U})$ the uniform space to be Bourbaki-bounded if any entourage $U \in \mathcal{U}$, ther exict finite set $F \subset X$ and $m \in N$ s.t $B \subset U^m[F]$,. where$U^1 = U, U^2 = U \circ U$, & for $m \in N$ will be $U^m = U \circ (U^{m-1})$. In (X,d), B will be Bourbaki-bounded if any $\varepsilon > 0$ ther are a set $F = \{x_1, \dots, x_n\} \subset X$ & some $m \in N$ s.t

$$B \subset \bigcup_{i=1}^{n} B(x_i,\varepsilon)^m.$$

Here a point y belongs to $B(x_i,\varepsilon)^m$ means that there exists an $\varepsilon - chain$ of length m joining thepoints x & y, i.e., there exist points $a_0 = x, a_1, \dots, a_m = y\ in\ X$ with $d(a_i, a_{i+1}) < \varepsilon, for\ i = 0, \dots (m-1)$. We will denoted in the form $B\text{ß}_d(X)$ the bornology for Bourbaki-bounded set on the space (X,d).

It is easy to see that if in the definition of Bourbaki-bounded set, the number m is always 1 (or even $m \le m_0$, for somem_0), then we have precompactness. On different idea , any Bourbaki-bounded set is on particular bounded since $B(x,\varepsilon)^m \subset B(x, m.\varepsilon)$. Thus, in any (X,d) we will be have the follow,

$T\text{ß}_d(X) \subset B\text{ß}_d(X) \subset \text{ß}_d(X)$.

Next examples show that these bornologies can be different.

Example 6.1: If was $(X, d_{\|.\|})$ a not finite dimensional normed space.To be easy to check this every ball for center $x = 0$ is Bourbaki-bounded, and therefore the same is true for every bounded set. On different idea, to be well know that any ball in a normed space to be infinite dimensional its cannot be precompact. Moreover, since in this case $B\text{ß}_d(X) = \text{ß}_d$(X), clearly that $B\text{ß}_d(X)$ be in fact uniformly metrizable.

Example 6.2: Consider the space R withmetric $d^* = min\{1, du\}$, wherever du is usual metric. Then any subset on (R, d^*) to be bounded butnot Bourbakibounded. Indeed, since uniform equivalent metrics have the same Bourbakibounded sets, we have that $B\text{ß}_{d^*}(R) = B\text{ß}_{du}(R) = \text{ß}_{du}(R)$. And, in particular, it follows that$B\text{ß}_{d^*}(R)$is also uniformly metrizable.

Remark 6.3. Let (X, d) to be B-simple iff $B\text{ß}_d(X)$ is uniformly metrizable.

Corollary 6.4. Let a mba $(X, \text{ß}, d)$. The follow holds are equivalently:

1. $B\text{ß}_d(X)$ be Uniformalymetrizable.
2. (X,d)be B-simple.
3. (X,d) be σ-Bourbaki-bounded & B-conservative.

A more refined characterization is given by the follow theorem.

Theorem6.5. Let $the\ mba\ (X, \text{ß}, d)$ then The holds are equivalent:

1. $B\text{ß}_d(X)$be Uniformalymetrizable.
2. (X,d)be σ-Bourbaki-bounded and exists some$\delta > 0$ s.t $B(x,\delta)^m$is Bourbaki-bounded, for all $m \in N$and for every $x \in X$.

Proof. we need only to prove that condition (2) implies that (X,d) is conservative. Then, if $B \in B\text{ß}_d(X)$, there exist a finite set $F = \{x_1, \dots, x_n\} \subset X$, and some $m \in N$, s.t $B \subset \bigcup_{i=1}^{n} B(xi,\delta)^m$. And therefore the δ-enlargement

$$B^\delta \subset B(x_1,\delta)^{m+1} \cup \dots \cup B(x_n,\delta)^{m+1}$$

is in $B\text{ß}_d(X)$ since it is contained in a finite union of Bourbaki-bounded sets.

Definition6.6: The subset $D \subset X$ is Bourbaki-dense if of any $x \in$ and ε>0, ther exist $y \in D$ and $m \in N$ s.t $x \in B\ (\,y\,,\varepsilon\,)^m$. And The space be Bourbaki-separable if is have a countable Bourbaki-dense subset.

Theorem 6.7. If $(X, ß, d)$ will be mba. The follow holds are equivalently:

1. $Bß_d(X)$be uniformlymetrizable.
2. (X, d) be Bourbaki-separable and ther exists some $\delta > 0$ s.t $B(x, \delta)^m$is Bourbaki-bounded, for any $m \in N$and for every $x \in X$.

Proof.(1) ⇒ (2). By Theorem 6.5, since any σ-Bourbaki-bounded metric space is also Bourbaki-separable. Indeed, suppose $X = \bigcup B_n$where $B_n \in Bß_d(X)$, $for\ any\ n.$ It $can\ suppose\ without\ loss\ for\ generaly\ that$ B1⊂ $B_2 \subset \cdots \subset B_n \ldots, is\ a$ increasing sequence. for any n∈N, ther found the set $F_n = \{x_1^n, \ldots x_{kn}^n\} \subset X$ and some $m_n \in N$ s.t

$$B_n \subset B(x_1^n, 1/n)^{m_n} \cup \cdots \cup B(x_{k_n}^n, 1/n)^{m_n}.$$

If we take $D = \bigcup F_n$, it be easy for check ther D will be acountable Bourbaki-dense on X, and therefore X is Bourbaki-separable.

(2) ⇒ (1). Now, let $D = \{x_n : n \in N\}$ a countable Bourbaki-dense subset in X, & let $\delta > 0$ such that every consecutive δ-enlargement of any singleton is Bourbaki-bounded. Then, to be easy too seethat

$$X = \bigcup_{n,m \in N} B(x_n, \delta)^m.$$

We finish since X be the countable union for Bourbaki-bounded sets, & then (1) follows.

Anyway, the concept of “Bourbaki-connectedness” in metric spaces functions as a generalization of the classical notion of connectedness, establishing precise conditions under which the class of connected sets can be characterized by a single metric. This provides a significant contribution to the study of generating structures and to the analysis of the relationship between the properties of a space and the behavior of its subsets.

CONCLUSION

This paper has provided a unified framework for studying uniformly metrizable bornologies. The central characterization theorem serves as a powerful tool, enabling a clear and systematic analysis of several important bornologies:

- The finite bornology is uniformly metrizable precisely when the space is countable and uniformly locally finite.
- A compact bornology is uniformly metrizable iff the underlying space will be both Lindelöf & uniformly locallycompact. This condition is equivalently for the existence of a uniformly equivalent metric for which Heine-Borel property holds.
- The totally bounded bornology is uniformly metrizable iff the space is Lindelöf & uniformly locally totallybounded, a property determined by its completion.
- The Bourbaki-bounded bornology is uniformlymetrizable (i.e., a space be B-simple) iff it is σ-Bourbaki-bounded and B-conservative, with an equivalent formulation using Bourbaki-separability.

These results consolidate and extend existing literature, demonstrating the efficacy of the bornological approach in solving problems of uniform metrization.

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