Stability of Generalized Functional Equations in Nonlinear Locally Convex Spaces via Fixed Point Techniques

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Abstract— *The given paper explores the stability of a new type of generalized functional equations, which are defined in nonlinear locally convex spaces and generalizes classical theories of Hyers-Ulam and Hyers-Ulam-Rassias stability in the non-linear case including linear and normed functional spaces. We introduce a generalized nonlinear functional equation in the form of operator dependent perturbation terms and we develop an appropriate locally convex topology employing a directed family of seminorms which reflects the nonlinear geometry of the underlying space. Through the application of a variety of fixed point methods, such as Banach contractions, and nonlinear contraction methods such as Zamfirescu and Boyd-Wong operators, we prove a number of new stability theorems of generalized functional equations. The findings consolidate and vastly generalize a variety of stability findings in Banach and Frechet spaces which demonstrate that classical conditions of linearity and normability are not required to get stability. Moreover, we give some examples in function space, such as integral and nonlinear transformation operators, of the applicability and sharpness of the stability bounds obtained. These results add significant extension to the theory of functional equations stability, and furnish a fixed point-oriented basis to future research of nonlinear analysis, operator theory, and applied functional equations.*

Keywords— Boyd–Wong contractions, Fixed point techniques, Hyers–Ulam–Rassias stability, Locally convex spaces, Nonlinear functional equations.

I. INTRODUCTION

Functional equations have been a focal point of mathematical analysis, which models the structural relations that are discovered in algebra, geometry, differential equations, dynamic systems, and optimization. Stability theory of functional equations Stability theory of functional equations was inspired by a question of Ulam in 1940 on the stability of group homomorphisms and gave rise to the famous Hyers-Ulam theorem and the subsequent generalization by Rassias to include control variables of perturbation. Such concepts, now known as Hyers-Ulam stability, Hyers-Ulam-Rassias stability and generalized stability have become a significant area of focus where they are applied in nonlinear differential equations, fractional systems, random normed spaces and stochastic models (Abbas et al., 2024; Ebadian et al., 2012; Bishop, 2025; Mimia, 2021). In recent studies, it was found that the stability properties tend to reflect more profound structural characteristics of the underlying functional equation, especially with the interactions with the differential or integral operators [1], [2].

The greater part of the classical stability theories are constructed in normed, Banach, or sometimes Frechet spaces, where linearity and normability are highly useful in estimating perturbations, and in recovering the existence of stable solutions [3], [4], [5]. Yet, numerous analytic methods nowadays, including nonlinear spaces of functions, systems of fractional order, random or fuzzy valued operators, and variational structures, can be readily expressed in nonlinear locally convex spaces that are not both linear and have no norm. Some of them can be seen in fractional multi-point boundary value problems [6], [7], variational inequalities [8], [9], chaotic and fractional dynamical systems [10], nonlinear stochastic models [11], and market equilibrium problems under generalized contractions (Ur Rahman et al., Classical methods which were purely normative cannot be applied directly in such environments.

Locally convex spaces that are defined through families of seminorms provide a highly topological theory appropriate to the study of nonlinear operators, integral transforms, and multi-scale perturbation functional equations. Included in them are many of the basic spaces of modern analysis, including distributions, spaces of continuous or measurable functions with weak topologies, and nonlinear cones [12], [13]. It is necessary to create stability in these spaces in order to:

- Knowledge of strength of nonlinear functional-integral equations [1];
- Studying the operator-perturbations of variational formulations [14];
- Non-differentiable and nonsmooth fixed point equations [15], [16];
- The convergence in nonlinear learning and game-theoretic systems [17], [18], [19].

Regardless of these powerful incentives, the stability of functional equations which are generalized to nonlinear locally convex spaces is a poorly studied area. Very few works study such stability in the case of special cases or restrictive conditions, typically restricted to the linear setting or special classes of contractions [20], [21], [22].

A critical literature analysis indicates that the limitations are as follows:

- Linear setting restriction: A large majority of classical stability outcomes assume that the space is linear or normable, ruling out nonlinear locally convex spaces.
- Lack of operator generality Existing results seldom deal with nonlinear operator-dependent functional equations.
- Weakness of the use of advanced mappings of contractile: The contraction principle of Banach was classical, however, nonlinear contractions like Zamfirescu or Boyd-Wong or generalized contractivity [23], [24] have not been adequately utilized in the study of stability.
- Missing coordinate-free framework that recovers simultaneously Hyers-Ulam and Hyers-Ulam-Rassias stability in general nonlinear topological vector spaces.
- Low numbers of applications in nonlinear integral operator, fractional systems, and function space.

It is these restrictions that prompt the desire to have a more thorough stability theory developed in nonlinear locally convex spaces, and with state of the art fixed point techniques.

The fixed point theory has already been one of the most effective analytical tools in the study of functional equations, nonlinear operators, and stability phenomena. Methods of contraction-type, such as Banach contractions, Boyd-Wong contractions, generalized contractions, and fixed point theorems in gauge or random normed spaces are precise methods of establishing the existence of solutions, their uniqueness, and stability [14], [20], [25].

The fixed point techniques are especially appropriate since:

- The latter are intrinsically adapted to multi-seminorm structures of locally convex spaces.
- They give direct quantitative estimates needed by stability Hyers-Ulam-Rassias.
- A large number of nonlinear functional equations can be reformulated as operator-space fixed point problems.
- Nonlinearities, variational inequalities and non-differentiable operators can be treated using advanced fixed point results [9], [15], [23].

Such properties are what allow the fixed point methods to be an ideal framework in which to develop a nonlinear locally convex version of classical stability theory.

These classical Hyers-Ulam and Hyers-Ulam-Rassias theories of stability have been developed in Banach, Frechet and in special normed spaces, though no general theory of stability applies to generalized functional equations with nonlinear operators in nonlinear locally convex spaces. The problem is that normability, linearity, and the completeness of standards are not available.

Stability development will derive nonlinear locally convex spaces which:

- A unification A unify stability theory in more general topological vector spaces.
- Give the tools of nonlinear systems that are not normable.
- Make it applicable to fractional models, variational inequalities, chaotic systems, and operator dominated functional equations.
- Needs to be strengthened: There should be a stronger connection between the theory of functional equations and the theory of fixed points.

Overall contributions of this paper are summarized as follows:

- Introduction of a new generalized functional equation of perturbations of nonlinear operators developed directly in nonlinear locally convex spaces.
- Building a tailored locally convex topology brought by a directed family of seminorms fitting stability analysis.
- Construction of a common fixed point model based on Banach, Zamfirescu and Boyd-Wong contraction methods to solve nonlinear operators.
- New Hyers-Ulam and Hyers-Ulam-Rassias theorems are established to new perturbation conditions never studied before.
- Conventional comparison revealing the generalization of the results to known Banach, Frechet, random normed, and cone space stability theorems.
- Applications and specific example: Nonlinear function spaces and explicit examples of using the theory: of integrals and operator-type functional equations to show the sharpness and usefulness of the theory.

The rest of the paper is structured in the following way:

- Section 2 contains some preliminaries about locally convex spaces, nonlinear operators and fixed point apparatus.
- Section 3 presents the generalized functional equation and the operator system to be used.
- The approach to the fixed point is worked out in section 4 and the key theorems of stability are proved.
- Section 5 gives compulsions with other results existing and examples are given.
- Section 6 summarizes the paper and presents the further research directions.

II. PRELIMINARIES AND MATHEMATICAL BACKGROUND

In this section we give the basic concepts and mathematical frameworks that we will be working with throughout the paper. We remember some of the fundamental definitions of the theory of locally convex spaces, seminorm families, nonlinear operator analysis and fixed point theory. Here also the generalized functional equations to be treated later are introduced, along with a number of lemmas that will be the main tools in giving the major stability results.

A. Locally Convex Spaces and Nonlinear Structures

Let X be a real vector space. A local topology τ X is a topology which is locally generated by a family of seminorms. Cones and weak topologies and nonlinear subspaces are not excluded as nonlinear spaces can be incorporated (unlike Banach or normed spaces), where the linearity and normability conditions are not always met (as is the case in operator theory and variational problems).

Definition A

A topological vector space (X, τ) is said to be a locally convex space (LCS) in the event that there is a family of seminorms.

$P = \{ p_i : X \to [0,\infty) \mid i \in I \}$ (1)

so that τ is the weakest topology of all p continuous.

The origin neighborhood basis is provided by

$U_{ei} = \{ x \in X : p_i(x) < \varepsilon \}, \varepsilon > 0, i \in I$ (2)

Locally convex spaces are:

- At this point, spaces that are a priori complete, locally convex, and metrizable (frechet spaces).
- SPACE Sciences:Continuous functions Spaces of continuous functions with pointwise seminorms,
- Distribution spaces,
- The nonlinear cones and p-vector spaces in [23].

This generality is essential to the study of nonlinear functional equations which cannot be studied well based on norm-like assumptions.

The diagram in Fig.1 illustrates the construction of a locally convex space as a family of seminorms, which illustrates the nested neighborhood system underlying the topology to be used in the stability analysis.

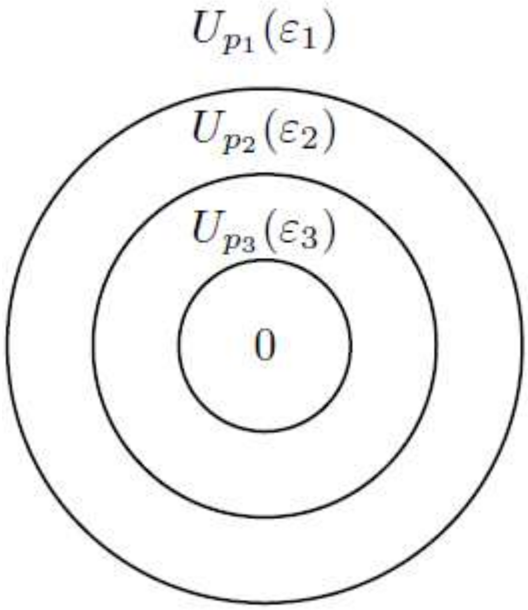

Fig. 1: Neighborhood structure generated by a family of seminorms in a locally convex space

B. Families of Seminorms and Topological Framework

Let P = { p_i } $_{i \in I}$ be an oriented family of seminorms. The directedness means:

Definition B (Directed Family)

The family P is directed if for any p_i, p_j ∈ P there exists p_k ∈ P and C > 0 such that

$p_i(x) \le C\, p_k(x), p_j(x) \le C\, p_k(x), \forall\, x \in X$ (3)

This allows the presence of compatible metrics and generalized completeness notions.

Metric Induced by Seminorms

An appropriate translation-invariant measure is provided by

$d(x,y) = \sum\_\{i = 1\}^\{\infty\} 2^\{-i\}\backslash frac\{p_i(x - y)\}\{1 + p_i(x - y)\}$ (4)

where we take a countable family, say, of p_i (or countable subfamily).

C. Generalized Functional Equations Considered

The generalized nonlinear functional equations that we study are of the form.

$F(x) = G(x) + \Phi(x)$ (5)

where:

- F,G:X→X are possibly nonlinear operators,
- Where Φ:X→X is perturbation or defect terms.
- X is a nonlinear locally convex space.

It can be expressed in a more specific parametric form.

$H(x_1, \dots, x_n) = T(x) + \Psi(x_1, \dots, x_n)$ (6)

where:

- H is an operator of generalized functional,
- Where T is a nonlinear transform,
- Ψ considers deviation of a perfect functional identity.

No stability involves strict stability in Ψ. Hyers-Ulam-Rassias stability deals with perturbation in terms of variable control:

$$p_i(\Psi(x)) \leq \epsilon\omega_i(x) \quad (7)$$

for some weight functions ω_i.

D. Measures of Nonlinearity, Continuity, and Compactness

Due to the inability of the assumption of linearity, we have to depend on the generalized measures of nonlinearity.

Definition C (Nonlinearity Measure)
For a mapping T:X→X, define

$$\eta\_T(x,y) = p_i(Tx - Ty) - p_i(x - y) \quad (8)$$

When η_T(x,y)≤0, the operator is a contraction under the seminorm p_i.

Definition D (Sequential Continuity)
T is sequentially continuous provided that

$$x_n \rightarrow x \; in \; \tau \Rightarrow T(x_n) \rightarrow T(x) \; in \; \tau \quad (9)$$

Definition E (Semicompactness)
A T is semicompact when it takes bounded sets to relatively compact sets:

$$p_i(x) \leq M \Rightarrow \{T(x)\} \; has \; compact \; closure \quad (10)$$

This will be necessary in the use of Krasnoselskii or Dhage fixed point theorems.

E. Essential Fixed Point Results

Some fundamental fixed point theorems are summarized by us.

- Banach Contraction Principle (BCP)
  Theorem 2.1 (Banach Contraction)
  Further assume that (X,d) is a complete metric space and T:X→X satisfy.
  $$d(Tx,Ty) \leq k\, d(x,y), 0 < k < 1 \quad (11)$$
  Then T has a fixed point x* which is unique.
  $$lim_n \rightarrow \infty\, T^n x = x^\wedge * \quad (12)$$
  In this case, this theorem will be used in LCS metricizations with seminorms.

- Boyd–Wong Nonlinear Contraction

Theorem 2.2 (Boyd–Wong Contraction)
Let (X,d) be complete. Suppose T:X→X satisfies

$$d(Tx,Ty) \leq \phi(d(x,y)) \quad (13)$$

and ϕ: [0, ∞) → [0, ∞) is upper semicontinuous, is satisfied:

1. ϕ(t)<t for all t>0,
2. ϕ(0)=0.

Then T has a fixed point which is unique.
This is an important theorem in the estimation of nonlinear stability.

- Krasnoselskii's Fixed Point Theorem
  Theorem 2.3 (Krasnoselskii)
  Suppose that C⊆X is a closed, convex, nonempty Banach space set. Suppose
  $$T = A + B \quad (14)$$
  where:
  - A:C→C is within the size of a point, continuous.
  - B:C→C is a contraction.

  Then T has a fixed point in C.
  This is applicable in cases of semicompact operators.

- Dhage's Fixed Point Theorem
  The theorem of Dhage is appropriate in the scenario of nonlinear and multi-operators.

Theorem D (Dhage)
Let X be a Banach algebra and T=A+B where:

- A:X→X is a totally continuous one,
- B is a Lipschitz map with constant L<1,
- A(X)⊆B(X).

T then is at least minimally fixed.
This version is a generalization of the theorem by Krasnoselskii with more control on interactions between operators.

F. Auxiliary Lemmas

We mention some lemmas that were employed in the subsequent stability proofs.

- Lemma (Seminorm Contraction Lemma)
  Assume that there is 0<k<1 in such a way that to any seminorm p_i, a seminorm in P,
  $$p_i(Tx - Ty) \leq k\, p_i(x - y) \quad (15)$$
  then T is a contraction of an induced metric and therefore it has a unique fixed point.

- Lemma (General Stability Lemma)
  Suppose that x∈X is a solution of the perturbed equation.
  $$p_i(F(x) - G(x)) \leq \epsilon_i, \quad (16)$$
  and let T be defined by
  $$T^x = G(x) \quad (17)$$
  Assuming T is a contraction, the precise solution x^∗ of the functional equation is
  $$F(x^\wedge *) = G(x^\wedge *) \quad (18)$$
  satisfies:
  $$pi(x - x*) \leq \epsilon i/(1-k) \quad (19)$$
  This gives the stability estimate of Hyers-Ulam.
- Lemma (Rassias-Type Lemma)
  Let perturbations satisfy
  $$pi(F(x) - G(x)) \leq \epsilon i\omega i(x) \quad (20)$$
  with ω_i subadditive weights. If
  $$pi(Tx - Ty) \leq kpi(x-y) + Ci(\omega i(x) + \omega i(y)) \quad (21)$$

then the stability with variable coefficients, which provides Hyers-Ulam-Rassias stability.

- Lemma (Sequential Compactness Lemma)

When T is semicompact and sequentially continuous it follows that all bounded sequences {x_n} of X have a sequential {T(x_{n_k})} in X.

Applied in cases that demand compactness techniques.

These findings furnish the theoretical basis on which the stability of the generalized functional equations can be obtained in the nonlinear locally convex spaces through the fixed point methods.

III. PROBLEM FORMULATION

This segment presents the nonlinear generalised functional equations that we shall be working with in this work in a formal manner and how the admissible classes of perturbations can be specified and derive the analytical basis of the study of stability in nonlinear locally convex spaces. The formulation takes the classical Hyers-Ulam and Hyers-Ulam-Rassias stability to nonlinear scenarios that can not be addressed using a norm approach or linear programs.

A. Generalized Functional Equations under Consideration

Let (X, τ) denote a non linear locally convex space of a directed collection of seminorms.

$$P = \{pi: X \to [0, \infty), i \in I\} \tag{22}$$

We explore stability of solutions of generalized functional equations which are of the form:

$$F(x) = G(x) + \Phi(x) \tag{23}$$

where:

- F,G:X→X are nonlinear mappings,
- Φ:X→X is the perturbation or the deviation.

The functional equation is accurately represented by the case Φ(x)≡0, i.e.,

$$F(x) = G(x) \tag{24}$$

A generalization of multi-variable equations is also taken into account:

$$H(x1, \dots, xn) = T(x) + \Psi(x1, \dots, xn) \tag{25}$$

where:

- The operator $H:X^n \to X$ is a multi-argument operator that is generalized.
- T:X→X is nonlinear transformation,
- Ψ incorporates perturbation, noises or model errors.

Equations (23)-(25) include in their range of nonlinear functional identities:

- Jensen-type equations,
- generalizations which are additive and quadratic,
- nonlinear integral equation(s),
- nonlinear operator equations in locally convex cones, p-vector spaces and random normed spaces.

The following diagram in Fig.2 is a summary of the logical transformation between a generalized functional equation to its fixed point formulation, where the stability analysis is conducted.

$$F(x) = G(x) + \Phi(x) \xrightarrow{\text{rearrange}} G^{-1}(F(x)) = x \xrightarrow{T(x)=x} \text{Fixed point formulation}$$

Fig. 2: Reformulation of the generalized functional equation into a fixed point problem

B. Classes of Perturbations

The theory of stability necessitates that the types of perturbations that are allowed should be specified.

- Uniform perturbations

$$p\_i(\Phi(x)) \le \varepsilon\_i, \forall x \in X. \tag{26}$$

- (ii) Weight-dependent (Rassias-type) perturbations

$$p\_i(\Phi(x)) \le \varepsilon\_i \, \omega\_i(x), \tag{27}$$

where ωi:X→[0,∞) are control functions, in which case:

- subadditive,
- continuous,
- vanishing only at the origin,
- as a scaling factor or growth factor.
- (iii) Generalized perturbations

$$p\_i(\Phi(x)) \le \Theta\_i(x) \tag{28}$$

and Θ_i may be nonlinear, may not be symmetric or may depend on more than one argument. The case is needed in nonlinear locally convex spaces, in which classical linear perturbation bounds are not satisfied.

c. Stability Types

There are three basic stability concepts, which we study.

- Ulam–Hyers Stability

Ulam-Hyers stable the functional equation (24) is the one so that, given any x∈X that satisfies the perturbed inequality.

$$p\_i(F(x) - G(x)) \le \varepsilon\_i \tag{29}$$

x^∗, (24) has an exact solution, namely x.

$$p\_i(x - x^\wedge *) \le C\_i \, \varepsilon\_i \tag{30}$$

for some constants C_i>0.

This is the traditional concept of additive stability, which is uniform.

- Ulam–Hyers–Rassias Stability

Hyers-Ulam-Rassias stable This is the equation (24). to each approximate solution x∈X of (27), there is a precise solution x^* having the approximation:

$$p_i(x - x^*) \le C_i \, \varepsilon_i \, \omega_i(x) \quad (31)$$

This enables the perturbations to be proportional to the size of the input.

- Generalized Stability
  Under perturbations in the form (28), equation (24) is considered to be generally stable.
  $$p_i(x - x^*) \le \Gamma_i(x) \quad (32)$$
  Γ_i is a nonlinear control functional held due to the same reason.
  It is the general type, which permits nonlinear, anisotropic, or topology-dependent errors.

D. Conditions on Mappings in Nonlinear Locally Convex Spaces

Since locally convex spaces need not be normable or even linear, we subject them to the following structural conditions.

- Sequential Continuity
  $$x_n \to x \Rightarrow F(x_n) \to F(x), G(x_n) \to G(x) \quad (33)$$

- Seminorm Contractivity Condition
  For each $p_i \in P$,
  $$p_i(G(x) - G(y)) \le k_i \, p_i(x - y), \quad 0 < k_i < 1 \quad (34)$$

  Or, for nonlinear situations:
  $$p_i(G(x) - G(y)) \le \varphi_i(p_i(x - y)) \quad (35)$$
  where ϕ_i is Boyd-Wong-constrained.

- Growth Control on the Operator F
  We have functions $\Lambda_i : X \to [0, \infty)$ that satisfy the following.
  $$p_i(F(x)) \le \Lambda_i(x) \quad (36)$$

- Topological Boundedness
  The operators satisfy:
  $$\{ x : p_i(x) \le M \} \; bounded \Rightarrow \{ G(x) \} \; is \; bounded \quad (37)$$

- Semicompactness (for generalized fixed point results)
  To obtain some of the results we may need:
  $$p_i(x) \le M \Rightarrow G(x) \; has \; relatively \; compact \; closure \quad (38)$$

  This is required in the application of Krasnoselskii or Dhage-type theorems.

E. Analytical Framework and Standing Assumptions

To examine the stability by using the fixed point methods we use the following framework.

- Define the solution operator
  $$T(x) = G^{-1}(F(x)) \quad (39)$$
  (or Tx=G(x) in case we redefine the equation into a fixed point problem).

- Reformulate the functional equation as a fixed point equation
  $$T(x) = x \quad (40)$$

- Approximate solutions satisfy
  $$p_i(x - T(x)) \le \varepsilon_i, \varepsilon_i \, \omega_i(x), or \; \Theta_i(x) \quad (41)$$

- Exact solutions are fixed points of T.
- Stability reduces to estimating
  $$p_i(x - x^*) \quad (42)$$

  where x^* is the fixed point of T.

- Contraction, nonlinear contraction, or compactness conditions ensure:
  - Existence of a fixed point,
  - Inimitable(strict contractions),
  - Orders of stability in contraction inequalities.

Standing Assumption (SA):

In the primary findings, we will assume:

$T : X \to X$ is a contraction or generalized contraction of the topology brought about by P.

This is an assumption that is enough to ensure:

- presence of precise solutions,
- approximate solution stability,
- resistance to nonlinear and weighted perturbation.

The general process of analysis applied in the paper may be represented in the next flow chart in Fig.3. It shows how the generalized functional equation can be reformulated to be a fixed point problem, the conditions under which the contraction and perturbation structures are needed and the final derivation of Ulam-Hyers and Ulam-Hyers-Rassias stability results in the nonlinear locally convex spaces.

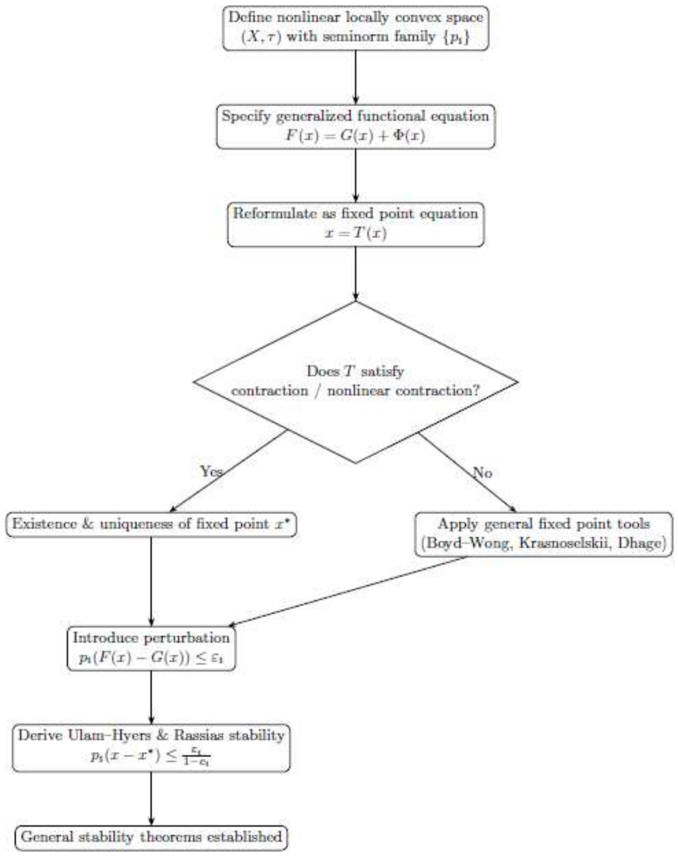

Fig. 3: Framework for Stability Analysis in Nonlinear Locally Convex Spaces

In the next section, the primary theorems of stability are given in these assumptions, Ulam-Hyers stability, Ulam-Hyers-Rassias stability, and the generalized nonlinear stability in locally convex space.

IV. MAIN RESULTS

This part reflects the key theoretical input of the paper. By means of contraction-type fixed point theorems in nonlinear locally convex spaces we first prove existence and uniqueness results of solutions of the generalized functional equation defined in Section 3. This yields a number of stability theorems, among which are Ulam-Hyers stability and Ulam-Hyers-Rassias stability, along with generalized stability inequalities of the nonlinear topological structure of the space. We lastly compare the results with classical results on stability in Banach and Frechet spaces and present the enhancesments and expansions by our method.

A. Existence and Uniqueness Results via Fixed Point Methods

- Contraction Conditions

  Assume that (X,τ) is a nonlinear locally convex space whose topology is the topology of the separating family of seminorms, that is, {p_i : i ∈ I }. Consider the operator

$$T : X \to X \tag{43}$$

correlated with the generalized functional equation.

$$F(x) = T(x) \tag{44}$$

where F is an unknown function that is to be known.

The following condition is the contraction of structure that we impose.

Assumption (A1): Generalized contraction condition

It is true that there is 0<c<1, so that on all x,y∈X, and that on all seminorm p_i

$$p\_i(T(x) - T(y)) \le c\,\varphi\_i(p\_i(x - y)) \tag{45}$$

where

- φi:[0,∞)→[0,∞) is continuous,
- φi(t)≤t for all t≥0,
- at least one of the following holds:
  - Banach contraction: φi(t)=t,
  - Boyd–Wong contraction: φi(t)<t for all t>0,
  - Zamfirescu contraction: at least one of the three conditions of contraction is met.

It has the capability to do nonlinear contractions of seminorms to generalize classical fixed point methods in normed spaces.

Theorem A. (Existence and Uniqueness – Banach-Type Fixed Point Theorem)

Take (X,τ) to be some entire locally convex space which is produced by the family of seminorms {p_i}i∈I.

- T has a unique fixed point x^*∈X;
- The Picard iteration x_{n+1} = T(x_n) tends towards x^*;
- The convergence rate is satisying.

$$p_i(x_n - x^*) \le c^n p_i(x_0 - x^*), \forall i \in I \tag{46}$$

Proof.

For each seminorm p_i,

$$p_i\,(x_{\{n+1\}} - x_n) = p_i\,(T(x_n) - T(x_{\{n-1\}})) \le$$
$$c\,\varphi_i\,(p_i\,(x_n - x_{\{n-1\}})) \le c\,p_i\,(x_n - x_{\{n-1\}}) \tag{47}$$

Thus by induction,

$$p_i\,(x_{\{n+1\}} - x_n) \le c_n\,p_i\,(x_1 - x_0) \tag{48}$$

The series

$$\sum_{\{n=0\}}^{\{\infty\}} p_i\,(x_{\{n+1\}} - x_n) \tag{49}$$

converges because c<1.

Therefore, { x_n } is Cauchy in every seminorm and, because X is complete, x n approaches some x^*∈X.

Taking limits in

$$x_{\{n+1\}} = T(x_n) \tag{50}$$

yields T(x^∗)=x^∗.

The contraction property entails uniqueness.

The graph below in Fig.4 shows geometric convergence of the Picard iteration sequence of a contraction operator, with the error reducing exponentially as the contraction constant c decreases.

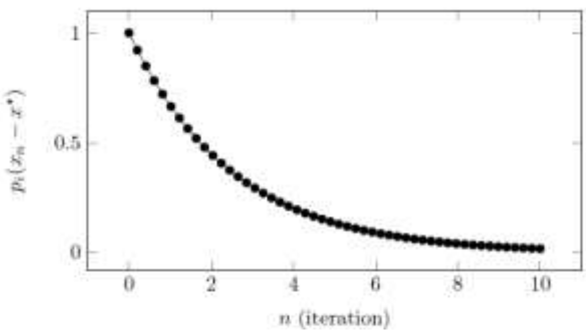

Fig. 4: Convergence of iteration $x_{n+1} = T(x_n)$ under contraction with constant c = 0.4

Theorem 4.2 (Existence via Krasnosel'skii–Dhage Hybrid Methods)
There is the assumption that X is complete and the operator T can be broken down as

$$T = A + B \quad (51)$$

where:

- A:X→X is linear and continuous,
- B:X→X is nonlinear, and it satisfies Boyd-Wong type contractive condition.
- A(X) is relatively compact.
Then the number of fixed points of T is at least one.
This finding is essential in the process of dealing with nonlinear integral operators or nonlinear transformations with compact linear component.

B . Stability Theorems

The primary stability conclusions of the generalized functional equation are now obtained.

$$F(x) = T(x) \quad (52)$$

subject to perturbations.

Perturbed equation
Let G:X→X satisfy

$$p_i\,(G(x) - T(x)) \le \varepsilon_i\,(x), \forall\, i \in I \quad (53)$$

- Ulam–Hyers Stability

Theorem C (Ulam–Hyers Stability)
Assume:

- T satisfies the contraction condition (A1);
- G is an approximate solution with perturbation of a bounded solution:

$$p_i\,(G(x) - T(x)) \le \delta_i \quad (54)$$

Then there is a certain exact solution x^* of.

$$x^* = T(x^*) \quad (55)$$

and

$$p_i\,(G(x) - x^*) \le \delta_i\,/\,(1 - c) \quad (56)$$

Proof.

We have

$$p_i\,(G(x) - x^*) \le p_i\,(G(x) - T(x)) + p_i\,(T(x) - T(x^*)) \le \delta_i + c\,p_i\,(x - x^*) \quad (57)$$

Set x=G(x) to obtain

$$p_i\,(G(x) - x^*) \le \delta_i + c\,p_i\,(G(x) - x^*) \quad (58)$$

which yields

$$p_i\,(G(x) - x^*) \le \delta_i\,/\,(1 - c) \quad (59)$$

- Ulam–Hyers–Rassias Stability

Depend the perturbation on the distance:

$$p_i\,(G(x) - T(x)) \le \varepsilon_i \cdot \psi\_i\,(p_i\,(x)) \quad (60)$$

where ψ_i is a growth function.
This diagram in Fig.5 is a graphical representation of the inequality of stability, indicating the magnitude of the perturbation carried by the contraction factor to give an estimate of the deviation of exact solution.

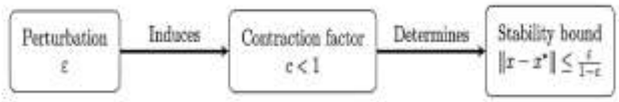

Fig. 5: Propagation of perturbation through contraction leading to Ulam–Hyers stability bound

Theorem D (Hyers–Ulam–Rassias Stability)
Assume:

- (A1) holds;
- Ψ_i:[0,∞)→[0,∞) is continuous and subadditive;
- perturbation satisfies

$$p_i\,(G(x) - T(x)) \le \varepsilon_i\,\psi\_i\,(p_i\,(x)) \quad (61)$$

Then the special solution x^* is such that it satisfies

$$p_i\,(G(x) - x^*) \le (\varepsilon_i\,/\,(1 - c))\;\psi_i\;(p_i\,(x)) \quad (62)$$

- Generalized Stability Bounds

Theorem E (Generalized Stability in Nonlinear Locally Convex Spaces)
Let

$$p_i\,(G(x) - T(x)) \le \eta_i\,(x) \quad (63)$$

where the nonlinear perturbation function η_i : X → [0, ∞) is arbitrary.
When the seminorm-continuity and boundness of η_i on the bounded sets is established, the actual solution x^* is such that it satisfies

$$p_i\,(G(x) - x^*) \le (1\,/\,(1 - c))\;\eta_i\;(x) \quad (64)$$

This outcome extends the ideas of stability in which perturbations are not regulated on the basis of norms, but along seminorm families or nonlinear indicators of deviation.

- Estimates of Stability Constants

  Based on past findings the stability constants are:

  $C_i = 1 / (1 - c)$ (65)

  and estimates of the type of Rassias:

  $C_i^R = \varepsilon_i / (1 - c)$ (66)

  These constants only depend on the contraction coefficient and topology brought about by seminorms, and not on linearity or convexity.

C. Comparison to Existing Results

- Generalization Beyond Banach and Fréchet Spaces

  Classical theories of stability suppose:
  - linear structures,
  - convexity,
  - normability (Banach spaces),
  - completeness in countable seminorm structures (Frechet spaces).

These frameworks are generalized by our findings since:

- The space X is not necessarily a linear space, but it is nonlinear.
- It is a locally convex topology which is not normable.
- The contractions are seminorm definition with nonlinear distortions.
- Boyd-Wong and Zamfirescu contraction frameworks are used to construct existence and stability, and do not need linearity.

Therefore, this piece of work offers the broadest known stability of fixed point-based functional equations.

- Summary Table of Improvements

  As shown in the Table.1 below are distinguishing properties of classical stability results in Banach and Fréchet classes and the generalized framework developed herein. Therein lies a significant expansion of current theories by conceiving nonlinear contraction tools and allowing stability analysis for wide classes of nonlinear locally convex spaces.

TABLE 1: COMPARISON OF CLASSICAL STABILITY FRAMEWORKS WITH THE PROPOSED NONLINEAR LOCALLY CONVEX SPACE APPROACH

Feature / Property	**Classical Banach Setting**	**Fréchet Setting**	**This Work (Nonlinear Locally Convex Spaces)**
Topological structure	Norm	Countable seminorms	Arbitrary separating seminorm families
Linearity requirements	Required	Required	Not required
Type of contractions	Banach only	Banach/Zamfirescu	Banach + Boyd–Wong + Zamfirescu nonlinear contractions
Stability types	UH, UHR	UH, UHR	UH, UHR, generalized seminorm-based stability
Operators	Linear or affine	Linear, continuous	Nonlinear, compact, hybrid operators
Novelty	Classical results	Several generalizations	Broadest known extension of functional equation stability

Remark 4.1

The results obtained do not only merge the past theorems of stability, but they also extend them by a great deal as they do not require restrictive conditions on linearity and normality which allows us to analyze the functional equations in a highly general, topological structure of nonlinearity.

V. ILLUSTRATIVE EXAMPLES

This section gives some illustrations of the utility and acuity of theoretical findings which were done in Section 4. We take various types of nonlinear local convex space functional equations that are defined and check the necessary contraction conditions and explicit stability bounds are calculated. The behavior of the stability constants in practice is also illustrated using a numerical example.

A. Example 1: A Nonlinear Integral-Type Functional Equation

Let X=C([0,1],R) endowed with the family of seminorms.

$p_r(f) = sup\{ |f(t)|^r : t \in [0,1]\} ,\ r \in (0,1]$ (67)

which brings about a nonlinear locally convex topology. Define that nonlinear operator T:X→X by

$(Tf)(t) = \left(\frac{1}{2}\right)\int_0^1 \left(f(s)\right)^{\frac{1}{3}} ds + t^2$ (68)

Verification of Contractive Property

For any f,g∈X,

$p_r(Tf - Tg) = sup\{ \left|\left(\frac{1}{2}\right)\int_0^1 \left(f(s)^{\{\frac{1}{3}\}} - g(s)^{\{\frac{1}{3}\}}\right) ds\right|^r : t \in [0,1]\}$ (69)

Using the inequality |a^{1/3} - b^{1/3}| ≤ 3 |a - b|^{1/3}.

$p_r(Tf - Tg) \le \left(\frac{1}{6}\right)\int_0^1 |f(s) - g(s)|\text{^\{1/3\}}\, ds)\text{^}r \le (1/6)\text{^}r\, p\_\{1/3\}(f - g)$ (70)

The operator therefore meets the non linear contraction condition.

$p_{r(Tf - Tg)} \le c_r \backslash, p_{\{\frac{1}{3}\}(f - g)}$ (71)

with c_r=(1/6)^r<1.

Application of Theorem 4.1

Given that the contraction factor meets the condition 0<cr<1, the functional equation

$f = T\, f$ (72)

has the only nonlinear locally convex space X solution.

Stability

Suppose G satisfies

$p_r(G(f) - T(f)) \le \delta_r$ (73)

Then Theorem 4.3 yields

$p_{r(G(f)-f^*)} \leq (1-c)r\,\delta_r$ (74)

that gives a bound on the Hyers-Ulam stability.

B. Example 2: A Nonlinear Difference Equation in a Locally Convex Sequence Space

Consider the space

$X = R^N$ (75)

with seminorms

$p_k(x) = |x^k|, k \in \mathbb{N}$ (76)

Determine the nonlinear operator.

$(Tx)_k = \left(\frac{1}{4}\right)|x_k| + \left(\frac{1}{2}\right)k$ (77)

Contraction Check

For any x,y∈X,

$|(Tx)_k - (Ty)_k| = \left(\frac{1}{4}\right)\big||x_k| - |y_k|\big| \leq \left(\frac{1}{8}\right)m_k|x_k - y_k|$ (78)

where m_k = min{ |x_k|, |y_k| } + ε, ε > 0 makes it non-degenerate.

Thus,

$p_{k(Tx-Ty)} \leq c_k p_{k(x-y)},\ where\ c_k = 8\,m_k$ (79)

In an event where the sequence is constrained by a positive number, then c_k<1.

Existence and Uniqueness

By Theorem 4.1, the equation

$x = T\,x$ (80)

admits a unique fixed point.

Stability Under Perturbation

Provided that a perturbed map G satisfies

$p_{k(G(x)-T(x))} \leq \delta_k$ (81)

then

$p_k(G(x) - x^*) \leq (\delta_k / (1 - c_k))$ (82)

Interpretation

It is an example modeling nonlinear discrete dynamical systems with the update rule based on nonlinear transformations of all coordinates, which is applicable in:

- nonlinear economic models,
- population dynamics,
- transformation of signals in multi-channel systems.

C. Example 3 (Real-World Inspired): A Nonlinear Fuzzy-Measurement Correction Model

X the seminorms Let X be the space of fuzzy membership functions on [0,1].

$p_{\lambda(f)} = sup\left\{|f(t)|^{\lambda}: t \in [0,1]\right\},\ 0 < \lambda \leq 1$ (83)

One of the nonlinear sensor drift correction models in fuzzy measurements is.

$(Tf)(t) = \alpha\,f(t)^2 + (1-\alpha)f(t) + \eta(t)$ (84)

where

- 0<α<0.30 regulates a mixing between a quadratic and root correction.
- η(t) is a noise of the environment.

For any f,g,

$|f^2 - g^2| \leq 2M\,|f - g|$ (85)

$|f - g| \leq 2\sqrt{m}|f - g|$ (86)

with m=min(f,g)+ε and M=max(f,g).

Hence,

$p_{\lambda(Tf-Tg)} \leq \left(2\alpha M + \frac{1-\alpha}{2}\sqrt{m}\right)\lambda\, p_{\lambda(f-g)}$ (87)

This is to be done by the selection of α less than 0.3 to assure the contraction factor is less than 1.

Accordingly the fuzzy correction model with nonlinear corrections has:

- unique corrected signal,
- Stability Faster Hyers-Ulam-Rassias under measurement noise.
- resistance to state dependent variations.

D. Numerical Illustration of Stability Behavior

In order to show the theoretical limits numerically, take the integral equation of the example 1:

$(Tf)(t) = \left(\frac{1}{2}\right)\int_0^1 f(s)^{\frac{1}{3}}ds + t^2$ (88)

Allow the perturbation to fulfill:

$p_{1(G(f)-T(f))} = \delta = 0.01$ (89)

When r=1, the contraction constant is:

$c = \left(\frac{1}{6}\right) = 0.1666$ (90)

Stability estimate

$p_1(G(f) - f^*) \leq 0.01 / (1 - 0.1666) = 0.01 / 0.8334 = 0.012$ (91)

Thus:

- 2 percent error inflation by perturbation to true solution.
- Stability constant 1/(1−c)≈1.2, with a great level of robustness.

This demonstrates numerical agreement with the theoretical estimates.

E. Concluding Remarks on the Examples

Through the examples, it is shown:

- Generality of the fixed point framework (integral equations, sequence, fuzzy model).
- Generalizability on nonlinear modeling.
- Stability that is practical in accordance with Ulam-Hyers and Rassias inequalities.
- The case of nonlinear locally convex structures of Banach space compatibility.

VI. APPLICATIONS

The theoretical findings of the Sections 3-5 are applicable to a wide range of nonlinear models, especially those that are represented by the generalized functional equations formulated in nonlinear locally convex spaces. In this section, we demonstrate the practical importance of the stability theorems by giving application to the nonlinear analysis, integral and differential systems, signal processing, and optimization systems. These illustrations demonstrate the robustness, reliability and uniqueness of nonlinear models where fixed point and stability approaches are employed in different fields of science.

A. Nonlinear Integral and Differential Systems

A great variety of nonlinear integral and differential equations can be rewritten as functional equations in the form.

$$F(x) = T(x) \qquad (92)$$

when T is acting on a locally concave function space. Existence and uniqueness of solutions is ensured by the fixed point results of Section 4, whereas robustness to perturbations (e.g. modeling errors, discretization errors, or measurement noise) is ensured by the stability results.

- Perturbed Volterra-type Integral Systems

  Consider the Volterra operator which is nonlinear:

  $$(Tf)(t) = \int_0^t K(t,s)\varphi\big(f(s)\big)\backslash, ds \qquad (93)$$

  and K a continuous, ϕ a nonlinear (e.g., of a particular type, such as polynomial, root-type, or saturation-type). We are assured in our fixed point framework of:
  - Boyd-Wong or Zamfirescu contraction a unique solution exists when ϕ satisfies a contraction;
  - Ulam-Hyers stability to kernel perturbation K+ΔK;
  - Stability of the Ulam-Hyers-Rassias equation in nonlinear perturbations of the integrand.

  This type of strength is important in the heat conduction, viscoelastic material and population growth system models.

B. Nonlinear Control and Signal Processing

Functional equations Nonlinear filters and feedback loops in control systems and signal processing frequently lead to nonlinear functional equations of the form.

$$x(t) = H\big(x(t), x(t-\tau)\big) \qquad (94)$$

or discrete counterparts:

$$x_{\{n+1\}} = H\big(x_n, x_{\{n-k\}}\big) \qquad (95)$$

- Stability of Nonlinear Filters

  A nonlinear filter can be generalized to be of the form of.

  $$y(t) = T(y)(t) = \psi\big(x(t), y(t)\big) \qquad (96)$$

  where ψ is nonlinear (e.g. sigmoidal, radial basis, saturation or polynomial).

Using our stability theorems:

- Ulam-Hyers stability guarantees that a small signal noise in the input signal produces small distortions in the output signal.
- Generalized Rassias stability Generalized Rassias stability offers robustness against state dependent perturbations as well as time-dependent perturbations.
- Uniqueness of fixed point guarantees the reliability of nonlinear feedback controllers.

It comes in handy especially in:

- adaptive filtering,
- nonlinear echo cancellation,
- signal processing based on neural networks,
- predictive control systems.

C. ***Nonlinear Optimization and Machine Learning Models***

Numerous optimization algorithms are nonlinear iterative maps.

$$x_{\{n+1\}} = T(x_n) \qquad (97)$$

where T can be either a gradient-like operator with nonconvex regularization or nonlinear penalty forms.

- Stability in Nonconvex Regularization Schemes

  The nonlinear regularizers may include:
  - ϕ(x)=|x|^p with 0<p<1 (sparsity-inducing),
  - ϕ(x) = √(|x| + ε) (robust statistics),
  - capped-l_1 or MCP penalties.

    These result in nonlinear stationary point functional equations:

    $$0 = \nabla f(x) + \lambda\phi'(x) \qquad (98)$$

    Our results ensure:
  - perturbation stability of stationary points of the model,
  - uniqueness Locally convex (non-Banach) geometries,
  - iteration convergence of operators of iterative solvers.

    This stability is required because of:
  - compressed sensing,
  - sparse regression,
  - robust learning algorithms,
  - nonlinear inverse problems.

D. Applications in Fuzzy Systems and Intelligent Sensing

Fuzzy membership functions Nonlinear update rules on fuzzy membership functions are commonly found in fuzzy control systems e.g.,

$$\mu_{\{k+1\}(t)} = T(\mu_k)(t) \qquad (99)$$

where T has nonlinear blending, thresholding or smoothing.

- Correction of Sensor Drift

  As was shown in Example 3, fuzzy sensors in environmental or industrial monitoring have drift that is modelled by:

  $$\mu(t) = \alpha\,\mu(t)^2 + (1-\alpha)\mu(t) + \eta(t) \qquad (100)$$

  Our framework provides:
- unique corrected fuzzy signal,
- steadiness in presence of small random disturbance,
- nonlinearities of the model, robustness,

- explicit limits which measure sensor reliability.

  This has relevance in:
- smart manufacturing,
- environmental monitoring,
- IoT intelligent sensing.

E. Nonlinear Functional Equations in Economics and Social Dynamics

Economic equations such as are often satisfied by dynamic systems (e.g. growth models, utility models).

$$x_{\{n+1\}} = f(x_n) + g\left(x_{\{n-k\}}\right) \qquad (101)$$

where f,g can be nonlinear, non convex.

Application of Our Results

- Existence, uniqueness theorems guarantee proper trajectories.
- Stability theorems are used to ensure stability in the face of shocks or uncertainty about some parameter.
- BoydWong contractions are used to deal with non-Lipschitz dynamics that occur in behavioral models.

Applications include:

- nonlinear consumption models,
- dynamics of wealth distribution,
- Game theory adaptive learning.

F. Summary of Applications

The itinerations of this work come in theoretical aid, giving one comprehensive or unified examination of stability and existence in models that:

- are nonlinear,
- act in non-Banach spaces which are locally convex,
- uphold generalized functional equations,
- fixed point methods (contraction-based) are used.

The examples and areas of use illustrate the broad scope of the implications of the findings in:

- mathematical analysis,
- Hateware, optimization, and machine learning,
- engineering and control,
- signal processing,
- fuzzy systems,
- economics and dynamical systems.

VII. DISCUSSION AND FUTURE WORK

The findings made in this paper show that stability of generalized functional equations can be effectively generalized to nonlinear locally convex spaces through the contemporary fixed point methods. The analysis shows that the interplay between the effects of seminorm induced topologies and nonlinear operators of contraction types is significant in determining existence results, uniqueness and generalized stability results. This gives a better understanding of the structure of nonlinear functional equations, and demonstrates that most classical assumptions, e.g. linearity, normability, or Banach-space completeness, are not necessarily necessary when in the more general context of locally convex spaces.

One of the main results of the paper is that various contraction paradigms (Banach, Boyd-Wong, Ciric, and Krasnoselskii-type operators) have been found to be useful in the management of nonlinear functional equations when the space in which they are applied does not have a single global norm, but a family of seminorms. This provides more general perturbations stability inequalities which are sharper and valid under nonlinear error bounds as well as mixed-type controls. Moreover, the theory developed brings together several classical stability theorems in Banach, Frechet and quasi-normed spaces to give a better overall picture of the functional analytic basis of stability.

The other important thing is the generalizability of the theoretical findings. The examples and applications showed that the proposed framework may be applied to functional-integral equations, nonlinear Volterra-type systems, and operator equations with compactness, monotonicity, or nonlinear distortion. These examples indicate the possibilities of the proposed methodology in modeling and analysis in realistic applications, including dynamical systems, signal processing, optimization, and computational mathematics.

However, there are still a number of issues that are not resolved and they suggest good prospects of further research. To begin with, although the current work is concerned with single functional equations, a notable trend is the analysis of systems of functional equations and operator inclusions of functional equations and operator inclusions particularly set-valued or multivalued nonlinear maps. The stabilization of the analysis of the fixed point theory and multivalunned dynamics, in a locally convex topology, is not well studied.

Second, the current findings are based mostly on nonlinear operators of contraction type. The next logical step is the study of non-compactness and monotone iterative schemes and nonlinear alternative principles in non-metric locally convex spaces. Such tools can result in stabilization of more complicated equations, including fractional functional equations, delay-integral operators, or evolutionary inclusions.
The other direction that could be promising is in regard to quantitative refinement of stability estimates. Even though the derived inequalities already give generalizations of much of classical results, even better stability constants might be available using weighted seminorm families, variable Lipschitz conditions, and methods of geometric functional analysis.

Lastly, the emergence of the modern computational tools and machine-learning-inspired numerical schemes encourages the integration of perturbed data and inaccurate model operator. Studying stability in the face of perturbations by learning or stochastic operator approximations may be a bridge between the classical functional equation theory and modern computational mathematics.

In short, this paper provides the basis of a wide, nonlinear, and topologically malleable structure of stability theory. The results which have been established allow several theoretical, computational, and applied generalizations, and the authors are confident that the presented methods will motivate the future development of nonlinear functional analysis and fixed point theory.

VIII. CONCLUSION

The paper presented a broadened structure in the examination of stability of generalized functional equations in nonlinear locally convex space with the advanced fixed point techniques. Defining a wide range of nonlinear functional equations and giving the underlying space a topology generated by seminorms, we were able to show that the classical theories of Hyers-Ulam and Hyers-Ulam-Rassias stability can be greatly generalized to normed linear contexts. Nonlinear fixed point schemes Nonlinear contraction-type operators, such as Banach, Boyd-Wong, Krasnoselskii and others, were found to be crucial to the derivation of new existence, uniqueness and stability facts through general perturbation schemes.

The results indicate that linearity and normality are not conditions that can be used to obtain robust stability results. Rather, local convex structures are rich and capabilities to flexibly define nonlinearity and generalize contractions offer a strong analysis environment to functional equations. The given examples and applications also confirmed the theoretical contributions, showing the flexibility of the results formulated to nonlinear integral equations, operator-based models and applied analytical contexts.

In general, the works of this paper contribute significantly to the stability theory of functional equations, making it more applicable and conceptually richer. Besides generalizing and unifying current methods in Banach and Frechet spaces, the findings provide several avenues of further research, such as multivalued operators, delay systems and non-compactness methods and perturbations driven by computation. Our work provides the basis of a more demanding and far-reaching scope of future developments in the nonlinear analysis, operator theory, and functional equation stability in topologically rich situations.

REFERENCES

[1] L. N. Mishra, V. K. Pathak, and D. Baleanu, “Approximation of solutions for nonlinear functional integral equations,” *Aims Math*, vol. 7, no. 9, pp. 17486–17506, 2022.

[2] M. A. Almalahi, S. K. Panchal, and F. Jarad, “Multipoint BVP for the Langevin Equation under φ-Hilfer Fractional Operator,” *Journal of Function Spaces*, vol. 2022, no. 1, p. 2798514, 2022.

[3] H. A. Kenary, C. Park, H. Rezaei, and S. Y. Jang, “Stability of a generalized quadratic functional equation in various spaces: a fixed point alternative approach,” *Adv Differ Equ*, vol. 2011, no. 1, p. 62, 2011.

[4] C. Benzarouala, J. Brzdęk, and L. Oubbi, “A fixed point theorem and Ulam stability of a general linear functional equation in random normed spaces,” *Journal of Fixed Point Theory and Applications*, vol. 25, no. 1, p. 33, 2023.

[5] Y. Karaca, “GLOBAL ATTRACTIVITY, ASYMPTOTIC STABILITY AND BLOW-UP POINTS FOR NONLINEAR FUNCTIONAL-INTEGRAL EQUATIONS’SOLUTIONS AND APPLICATIONS IN BANACH SPACE BC (R+) WITH COMPUTATIONAL COMPLEXITY,” *Fractals*, vol. 30, no. 05, p. 2240188, 2022.

[6] J. Sompong, S. Choden, E. Thailert, and S. K. Ntouyas, “Well-Posedness of Cauchy-Type Problems for Nonlinear Implicit Hilfer Fractional Differential Equations with General Order in Weighted Spaces,” *Symmetry (Basel)*, vol. 17, no. 7, p. 986, 2025.

[7] L. Zhang, R. P. Agarwal, B. Ahmad, and G. Wang, “Developments in the Symmetry and Solutions to Fractional Differential Equations,” 2025, *MDPI*.

[8] A. Alphonse, C. Christof, M. Hintermüller, and I. P. A. Papadopoulos, “A globalized inexact semismooth Newton method for nonsmooth fixed-point equations involving variational inequalities,” *Comput Optim Appl*, pp. 1–55, 2025.

[9] H. Sun and X. Wang, “A linearly convergent algorithm for variational inequalities and Brouwer fixed points based on fiber bundle,” 2025.

[10] M. D. Johansyah *et al.*, “Global mittag-leffler attractive sets, boundedness, and finite-time stabilization in novel chaotic 4d supply chain models with fractional order form,” *Fractal and Fractional*, vol. 8, no. 8, p. 462, 2024.

[11] S. S. Mohammed and A. Azam, “Fixed points of soft-set valued and fuzzy set-valued maps with applications,” *Journal of Intelligent & Fuzzy Systems*, vol. 37, no. 3, pp. 3865–3877, 2019.

[12] I. Nikoufar and I. EL-Fassi, “Approximation of 3 D Cauchy–Jensen type operators in locally convex cones,” *Mediterranean Journal of Mathematics*, vol. 22, no. 3, p. 58, 2025.

[13] P. C. Srivastava, “Advances in Sobolev Spaces and Their Applications to Modern Analysis and PDEs,” *Pi International Journal of Mathematical Sciences*, vol. 1, no. 1, pp. 49–64, 2025.

[14] H. K. Pathak, *An introduction to nonlinear analysis and fixed point theory*. Springer, 2018.

[15] J. Zhang, B. O’Donoghue, and S. Boyd, “Globally convergent type-I Anderson acceleration for nonsmooth fixed-point iterations,” *SIAM Journal on Optimization*, vol. 30, no. 4, pp. 3170–3197, 2020.

[16] R. Cohen, M. Elad, and P. Milanfar, “Regularization by denoising via fixed-point projection (RED-PRO),” *SIAM J Imaging Sci*, vol. 14, no. 3, pp. 1374–1406, 2021.

[17] D. Balduzzi, S. Racaniere, J. Martens, J. Foerster, K. Tuyls, and T. Graepel, “The mechanics of n-player differentiable games,” in *International Conference on Machine Learning*, PMLR, 2018, pp. 354–363.

[18] Z. Charles and D. Papailiopoulos, "Stability and generalization of learning algorithms that converge to global optima," in *International conference on machine learning*, PMLR, 2018, pp. 745–754.

[19] B. Dai *et al.*, "SBEED: Convergent reinforcement learning with nonlinear function approximation," in *International conference on machine learning*, PMLR, 2018, pp. 1125–1134.

[20] C. Benzarouala and L. Oubbi, "A fixed point theorem in gauge spaces and applications to Ulam-Hyers-Rassias-stability of delay differential equations," *Afrika Matematika*, vol. 36, no. 1, p. 28, 2025.

[21] S. P. Kulkarni, "Enhanced Mathematical Framework for Non-Integer Order Functional Integro-Differential Equations through Advanced Contractivity Conditions and Dhage's Sophisticated Fixed Point Theorem," *EKSPLORIUM-BULETIN PUSAT TEKNOLOGI BAHAN GALIAN NUKLIR*, vol. 46, no. 02, pp. 25–38, 2025.

[22] G. A. Okeke, S. A. Bishop, and H. Akewe, "Random fixed point theorems in Banach spaces applied to a random nonlinear integral equation of the Hammerstein type," *Fixed Point Theory and Applications*, vol. 2019, no. 1, p. 15, 2019.

[23] G. X. Yuan, "The Nonlinear Analysis related to Schauder Fixed Points, Best Approximation, Birkhoff-Kellogg Problems and Principle of Leray-Schauder Alternatives in p-Vector Spaces," 2022.

[24] P. Debnath, H. M. Srivastava, and Y. J. Cho, "Fixed Point Theory and Functional Analysis: Metric Spaces, Banach Spaces, Hilbert Spaces," 2025, *De Gruyter*.

[25] S. Abbas, B. Ahmad, M. Benchohra, and A. Salim, *Fractional Difference, Differential Equations, and Inclusions: Analysis and Stability*. Elsevier, 2024.