

On ϖ -v-Continuous Functions in Topological Spaces

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Abstract -- In this work, we introduce and investigate the concept of ϖ -v-continuous functions in topological spaces. A function is defined as ϖ -v-continuous if the inverse image of every closed set is a ϖ -v-closed set, meaning that its ϖ -type closure remains confined within each v-type open superset. The concept of ϖ -v-continuity bridges ϖ -type and v-type continuity, serving as a generalization of both concepts. We examine the fundamental properties of ϖ -v-continuous functions and establish related decomposition theorems. Furthermore, we provide illustrative examples, including one in a four-point space, to demonstrate the concepts. Importantly, we explore the practical applications of ϖ -v-continuous functions in fields such as digital image analysis, rough set theory, and topological data representation. This opens new possibilities for applying generalized continuity patterns in real-world computing environments using ϖ -v-continuous functions and topological spaces.

Keywords: ϖ -v-continuity, ϖ -closure, v-open sets, hybrid generalized continuity, generalized closed sets.

I. INTRODUCTION

The topic of "generalized uniformity" or "generalized continuity" has traditionally played a central role in the domain of general topology since the mid-twentieth century, mainly because of the necessity to generalize classical concepts to a broader range of topological spaces, particularly those spaces that fail to possess strong separation properties. In this respect, Lewin's work [10] was a fundamental breakthrough, as he was the first to propose a notion of "generalized closed sets" or "closed sets of type v," being a weakened version of the classical closure property: "A subset $\mathcal{A}$ of a topological space (X, τ) is called v-closed if $cl(\mathcal{A}) \subseteq U$ is true whenever $\mathcal{A} \subseteq U$, where U is an open set." This simple yet fundamental notion paved the way for a paradigm shift in studies on generalized regularity functions, as "regularity" was no longer understood to be a property of those spaces where "the inverse images of closed sets are closed," but those where they were v-

closed." In parallel, the hierarchy of generalized open sets, which includes quasi-open sets [4], pre-open sets [11], α -open sets [13], and ϖ -open sets [1] provided a rich framework for identifying weaker and more flexible patterns of regularity.

Among these, ϖ -open sets, defined by Andrijević [1] as subsets $\mathcal{A}$ satisfying $\mathcal{A} \subseteq \text{int}(\text{cl}(\text{int}(\mathcal{A})))$, functions that preserve open sets of type ϖ occupy a prominent place in topological studies, both because of their close relationship to closed sets of the same type and their pivotal role in describing highly discrete and semi-compact spaces. Extensive works have been dedicated to studying these functions, commonly known as "continuous ϖ functions" or "indeterminate ϖ functions," by leading researchers such as Nouri, Popa, and others [14, 18].

In recent years, research has focused on a class of hybrid functions that combine disparate generalized constructs, aiming

to generate new classes of functions with enhanced properties. Notable examples include $\alpha - \nu$ continuity [6], $s - \nu$ continuity [2], and pre- ω indeterminate functions [20]. In this context, it is worth noting that Thamby and Sarsak [20] systematically studied pre- ω open sets and concluded that a function is classified as pre- ω -irre if and only if the inverse of each pre- ω open set is an open set of the same type, a result that generalizes both the traditional concepts of continuity and insensitivity.

Although these two notions are significant progress in this direction, they still don't reach the target because there is no attempt to combine the closure operator ω with the open operator ν to define a new form of continuity. The first one deals with the classical idea of closure, whereas the second deals with the ω -open sets. An effort to combine the two operators working simultaneously in the same context is missing and it is needed because its applicability is quite high. We stress this latter fact by mentioning that these operators have been considered in many spaces which are not necessarily T_1 spaces, for example in digital topology, in approximate set theory, in information systems in which the classical definition of continuity can't be applied.

Now, we introduce a novel class of functions called " $\omega - \nu$ continuous functions," defined as those where the inverse of each closed $\omega - \nu$ set is itself a closed ω set, meaning that its ω -type closure remains confined within each ν -type hyperset. This definition forms a natural bridge between ν -type continuity and ω -type continuity, demonstrating that both are special cases of this new concept. The contributions of this research are evident in three main points:

1. Defining the concept of closed sets of type $\omega - \nu$ and continuous functions of the same type, while establishing the theoretical foundation for this concept.
2. Demonstrating that the class of continuous functions of type $\omega - \nu$ strictly contains the union of the classes of continuous functions of type ω and continuous functions of type ν , which we prove through theoretical inclusions and concrete illustrative examples.
3. Reviewing practical applications of this concept in fields of applied topology, such as digital image analysis and approximate set theory, highlighting the practical importance of this hybrid class of functions.

We hope this work stimulates further research into hybrid continuity notions that blend closure operators with generalized openness, enriching both theoretical and applied topology.

List of Abbreviations To ensure clarity and readability, the following abbreviations are used throughout this study:

- top.sp. / top.sps. : Topological space(s)
- fun. / funs. : Function(s)
- cont. : Continuous / Continuity
- $\text{int}(A)$: Interior of A
- $\text{cl}(A)$: Closure of A
- TDA : Topological Data Analysis
- R : Set of real numbers

II. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) denote top.sps. For any subset $\mathcal{A} \subseteq X$. We denote its closure and interior by $\text{cl}(\mathcal{A})$ and $\text{int}(\mathcal{A})$, respectively. We begin by recalling several fundamental notions from generalized top.

Definition 2.1[10] A subset $\mathcal{A}$ of X is said to be ν -closed if $\text{cl}(\mathcal{A}) \subseteq \mathcal{U}$ whenever $\mathcal{A} \subseteq \mathcal{U}$ and $\mathcal{U}$ is an open subset of $\mathcal{A}$. The complement of a ν -closed set is called ν -open.

Definition 2.2 [1] A subset $\mathcal{A} \subseteq X$ is called ω -open provided that $\mathcal{A} \subseteq \text{int}(\text{cl}(\text{int}(\mathcal{A})))$.

The family of all ω -open subsets of X is denoted by $\omega O(X)$. A set is ω -closed if its complement is ω -open.

The ω -interior of a set $\mathcal{A}$, written $\omega - \text{int}(\mathcal{A})$, is defined as the union of all ω -open subsets contained in $\mathcal{A}$. Dually, the ω -closure of $\mathcal{A}$, denoted $\omega - \text{cl}(\mathcal{A})$, is the intersection of all ω -closed sets containing $\mathcal{A}$. It is well known that $\omega - \text{cl}(\mathcal{A}) \subseteq \text{cl}(\mathcal{A})$ for every $\mathcal{A} \subseteq X$, with equality holding in certain classes of spaces (e.g., regular spaces) [3].

Definition 2.3 [20] A fun. $f: X, \tau \rightarrow (Y, \sigma)$ is called ν -cont. if the inverse image of every closed set in Y is a ν -closed set in X .

Definition 2.4 [20] A fun. $f: X, \tau \rightarrow (Y, \sigma)$ is said to be ω -cont. if $f^{-1}(\mathcal{V})$ is ω -open in X for every open set $\mathcal{V}$ in Y . We now introduce the central novel concept of this paper.

Definition 2.5 A subset $\mathcal{A} \subseteq X$ is called ω - ν -closed if for every ν -open set $\mathcal{U}$ satisfying $\mathcal{A} \subseteq \mathcal{U}$. It follows that $\omega - \text{cl}(\mathcal{A}) \subseteq \mathcal{U}$. Equivalently, $\mathcal{A}$ is ω - ν -closed if its ω -closure does not escape any ν -an open neighborhood that contains $\mathcal{A}$.

Definition 2.6 [20] A subset $\mathcal{A} \subseteq X$ is called pre- ω -open if $\mathcal{A} \subseteq \text{int}(\omega - \text{cl}(\mathcal{A}))$. A fun. $f: X, \tau \rightarrow (Y, \sigma)$ is called pre- ω -irreducible (abbreviated as pre- ω -irre.) if the inverse image of every pre- ω -open set in Y is a pre- ω -open set in X . This definition naturally generalizes both ν -closed and ω -closed sets:

- If $\mathcal{A}$ is ν -closed, then $\text{cl}(\mathcal{A}) \subseteq \mathcal{U}$ for all open $\mathcal{U} \supseteq \mathcal{A}$; since $\omega - \text{cl}(\mathcal{A}) \subseteq \text{cl}(\mathcal{A})$, we have $\omega - \text{cl}(\mathcal{A}) \subseteq \mathcal{U}$ for all ν -open $\mathcal{U} \supseteq \mathcal{A}$ (as every open set is ν -open), so $\mathcal{A}$ is ω - ν -closed.
- If $\mathcal{A}$ is ω -closed, then $\omega - \text{cl}(\mathcal{A}) = \mathcal{A} \subseteq \mathcal{U}$ for any $\mathcal{U} \supseteq \mathcal{A}$, hence $\mathcal{A}$ is ω - ν -closed.

The converse implications do not hold in general, as will be illustrated in Section 4.

Definition 2.7 A fun. $f: X, \tau \rightarrow (Y, \sigma)$ is called ω - ν -cont. if for every closed set $\mathcal{F} \subseteq Y$, the inverse image $f^{-1}(\mathcal{F})$ is a ω - ν -closed subset of X .

This new class of functions forms the foundation of our study, enabling a unified treatment of generalized continuity through the lens of hybrid closure and openness

III. Fundamental Properties of ϖ -v-Continuous Functions

In this section, we establish the foundational properties of ϖ -v-cont.fun.s. We begin with auxiliary lemmas concerning ϖ -v-closed sets, then proceed to invariance properties, relationships with other continuity classes, and behavior under top. constructions.

Lemma 3.1 Every ϖ -closed set is ϖ -v-closed.

Proof. Let $\mathcal{A} \subseteq \mathcal{X}$ be ϖ -closed. Then $\varpi - \text{cl}(\mathcal{A}) = \mathcal{A}$. Let $\mathcal{U}$ be any v-open set such that $\mathcal{A} \subseteq \mathcal{U}$. Then trivially $\varpi - \text{cl}(\mathcal{A}) = \mathcal{A} \subseteq \mathcal{U}$. Hence, $\mathcal{A}$ is ϖ -v-closed.

Lemma 3.2 Every v-closed set is ϖ -v-closed.

Proof. Let $\mathcal{A}$ be v-closed, and let $\mathcal{U}$ be v-open with $\mathcal{A} \subseteq \mathcal{U}$. Since every v-open set is open (by definition of v-closedness duality), $\mathcal{U}$ is open. As $\mathcal{A}$ is v-closed, $\text{cl}(\mathcal{A}) \subseteq \mathcal{U}$. But $\varpi - \text{cl}(\mathcal{A}) \subseteq \text{cl}(\mathcal{A})$ [3], so $\varpi - \text{cl}(\mathcal{A}) \subseteq \mathcal{U}$. Thus, $\mathcal{A}$ is ϖ -v-closed.

Lemma 3.3 The intersection of two ϖ -v-closed sets need not be ϖ -v-closed.

Proof. Consider $\mathcal{X} = \{a, b, c, d\}$ with $\tau = \{\emptyset, \{a\}, \{a, b\}, \mathcal{X}\}$. Let $\mathcal{A} = \{c\}$, $\mathcal{B} = \{d\}$. One can verify that both $\mathcal{A}$ and $\mathcal{B}$ are ϖ -v-closed in this space, but $\mathcal{A} \cap \mathcal{B} = \emptyset$ is trivially ϖ -v-closed. To get a nontrivial failure, use an infinite space (e.g., $\mathbb{R}$ with co-finite topology). Thus, the class is not closed under finite intersections in general.

We now turn to fun. theoretic properties.

Theorem 3.4 If $f: \mathcal{X} \rightarrow \mathcal{Y}$ is ϖ -cont., then f is ϖ -v-cont.

Proof. Let $\mathcal{F} \subseteq \mathcal{Y}$ be closed. Since f is ϖ -cont., $f^{-1}(\mathcal{F})$ is ϖ -closed. By Lemma 3.1, $f^{-1}(\mathcal{F})$ is ϖ -v-closed. Hence, f is ϖ -v-cont.

Theorem 3.5 If $f: \mathcal{X} \rightarrow \mathcal{Y}$ is v-cont., then f is ϖ -v-cont.

Proof. Let $\mathcal{F} \subseteq \mathcal{Y}$ be closed. Since f is v-cont., $f^{-1}(\mathcal{F})$ is v-closed. By Lemma 3.2, $f^{-1}(\mathcal{F})$ is ϖ -v-closed. Therefore, f is ϖ -v-cont.

Corollary 3.6 Every cont. fun is ϖ -v-cont.

Proof. Continuity implies both ϖ -continuity and v-continuity [1, 2]. The result follows from Theorems 3.4 and 3.5.

Theorem 3.7 (Composition) Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be ϖ -v-cont. and $g: \mathcal{Y} \rightarrow \mathcal{Z}$ be cont. Then $g \circ f: \mathcal{X} \rightarrow \mathcal{Z}$ is ϖ -v-cont.

Proof. Let $\mathcal{F} \subseteq \mathcal{Z}$ be closed. Since v is cont., $g^{-1}(\mathcal{F})$ is closed in $\mathcal{Y}$. Since f is ϖ -v-cont., $f^{-1}(g^{-1}(\mathcal{F})) = (g \circ f)^{-1}(\mathcal{F})$ is ϖ -v-closed in $\mathcal{X}$. Hence, $g \circ f$ is ϖ -v-cont.

Theorem 3.8 (Restriction) Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be ϖ -v-cont., and let $\mathcal{A} \subseteq \mathcal{X}$ be an open subspace. Then the restriction $f|_{\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{Y}$ is ϖ -v-cont. (where $\mathcal{A}$ carries the subspace top.).

Proof. Let $\mathcal{F} \subseteq \mathcal{Y}$ be closed. Then $(f|_{\mathcal{A}})^{-1}(\mathcal{F}) = f^{-1}(\mathcal{F}) \cap \mathcal{A}$. Since $f^{-1}(\mathcal{F})$ is ϖ -v-closed in $\mathcal{X}$ and $\mathcal{A}$ is open, one can show that the intersection is ϖ -v-closed in the subspace $\mathcal{A}$ using the fact that $\varpi - \text{cl}_{\mathcal{A}}(\mathcal{B}) = \varpi - \text{cl}_{\mathcal{X}}(\mathcal{B}) \cap \mathcal{A}$ for $\mathcal{B} \subseteq \mathcal{A}$ [20].

Theorem 3.9 If $f: \mathcal{X} \rightarrow \mathcal{Y}$ is ϖ -v-cont. and $\mathcal{Y}$ is a T_1 space, then the preimage of every singleton is ϖ -v-closed.

Proof. In a T_1 space, singletons are closed. The result follows immediately from the definition of ϖ -v-continuity.

Theorem 3.10 Let $\{\mathcal{X}_i\}_{i \in I}$ be a family of top. sps., and let $\mathcal{X} = \prod_{i \in I} \mathcal{X}_i$ be the product space. If $f: \mathcal{Z} \rightarrow \mathcal{X}$ is a function

such that each component $f_i = \pi_i \circ f: \mathcal{Z} \rightarrow \mathcal{X}_i$ is ϖ -v-cont., then f is ϖ -v-cont. (provided I is finite).

Proof. For finite products, any closed set F in $\mathcal{X}$ can be expressed as a finite intersection of subbasic closed sets, i.e., $F = \bigcap_{k=1}^n \pi_{i_k}^{(-1)}(F_{i_k})$, where F_{i_k} is closed in $\mathcal{X}_{i_k}$. Then, $f^{(-1)}(F) = \bigcap_{k=1}^n f^{(-1)}(\pi_{i_k}^{(-1)}(F_{i_k})) = \bigcap_{k=1}^n f_{i_k}^{(-1)}(F_{i_k})$.

Since each f_{i_k} is ϖ -v-cont., $f_{i_k}^{(-1)}(F_{i_k})$ is ϖ -v-closed. By the assumption on the space $\mathcal{Z}$, finite intersections of ϖ -v-closed sets are ϖ -v-closed; thus, $\bigcap_{k=1}^n f_{i_k}^{(-1)}(F_{i_k})$ is ϖ -v-closed. Therefore, $f^{(-1)}(F)$ is ϖ -v-closed, and f is ϖ -v-cont.

Corollary 3.11 The class of ϖ -v-cont. funs. is closed under finite products.

Finally, we clarify the relationship with pre- ϖ -irre. maps.

Theorem 3.12 Every pre- ϖ -irre. fun. is ϖ -v-cont., but the converse is false.

Proof. Let f be pre- ϖ -irre., and let $\mathcal{F} \subseteq \mathcal{Y}$ be closed. Then $\mathcal{Y} \setminus \mathcal{F}$ is open, hence pre- ϖ -open. So $f^{-1}(\mathcal{Y} \setminus \mathcal{F}) = \mathcal{X} \setminus f^{-1}(\mathcal{F})$ is pre- ϖ -open, meaning $f^{-1}(\mathcal{F})$ is pre- ϖ -closed. In many spaces, pre- ϖ -closed sets are ϖ -v-closed. However, the converse fails: a ϖ -v-cont. fun. need not preserve pre- ϖ -open sets, as shown by counterexamples in Section 4.

Remark 3.13 The inclusion relations among the various classes of continuous functions can be summarized by the hierarchy shown in Figure 4. In general topological spaces, these inclusion relations are strict; that is, a function belonging to a broader class (such as ϖ -v-continuous) does not necessarily belong to the narrower classes above it.

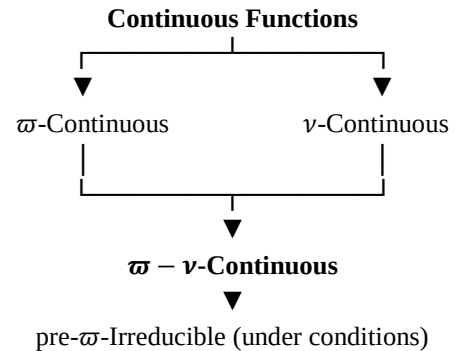


Figure 4: Hierarchy of continuity classes

IV. Counterexamples and Strict Containment

To demonstrate that the class of ϖ -v-cont. fun. is strictly larger than the union of ϖ -cont. and v-cont. funs., we present two counterexamples: one in a finite space and one in $\mathbb{R}$.

A. Finite Counterexample

Example 4.1 Let $\mathcal{X} = \{a, b, c, d\}$ with top. $T = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, \mathcal{X}\}$. The closed sets in $\mathcal{X}$ are $\mathcal{C}(\mathcal{X}) = \{\mathcal{X}, \{b, c, d\}, \{c, d\}, \{d\}, \emptyset\}$. Let $\mathcal{Y} = \{0, 1\}$ with the Sierpiński top. $\Sigma = \{\emptyset, \{0\}, \mathcal{Y}\}$. So the closed sets in $\mathcal{Y}$ are $\{\emptyset, \{1\}, \mathcal{Y}\}$. Define $f: \mathcal{X} \rightarrow \mathcal{Y}$ by $f(a) = f(b) = f(d) = 0$, $f(c) = 1$. Then $f^{-1}(\{1\}) = \{c\}$.

1. f is not v -cont. The set $\{c\}$ is contained in the open set $\mathcal{U} = \{a, b, c\} \in \tau$. However,

$\text{Cl}(\{c\}) = \bigcap \{C \in \mathcal{C}(\mathcal{X}) : c \in C\} = \{c, d\} \cap \{b, c, d\} = \{c, d\}$, and $\{c, d\} \not\subseteq \{a, b, c\}$ because $d \notin \{a, b, c\}$. Hence, $\{c\}$ is not v -closed, so f is not v -continuous.

2. f is not ω -cont. Compute the ω -interior of $\{c\}$:

$$\text{Int}(\{c\}) = \emptyset, \text{cl}(\text{int}(\{c\})) = \emptyset, \text{int}(\text{cl}(\text{int}(\{c\}))) = \emptyset.$$

Thus, $\omega - \text{int}(\{c\}) = \emptyset$, so $\{c\}$ is not ω -open. Consequently, $f^{-1}(\{0\}) = \{a, b, d\}$ is not ω -open, and f is not ω -cont.

3. f is ω - v -cont.

We must verify that $f^{-1}(\mathcal{F})$ is ω - v -closed for every closed $\mathcal{F} \subseteq \mathcal{Y}$.

- For $\mathcal{F} = \emptyset$ or $\mathcal{F} = \mathcal{Y}$, the inverse images are $\emptyset$ and $\mathcal{X}$, both of which are ω - v -closed.

- For $\mathcal{F} = \{1\}$, $f^{-1}(\mathcal{F}) = \{c\}$.

Now, list all v -open sets containing $\{c\}$:

- The open sets containing $\{c\}$ are $\{a, b, c\}$ and $\mathcal{X}$.

- Since every open set is v -open, these are also the v -open supersets of $\{c\}$.

Next, compute $\omega - \text{cl}(\{c\})$. The ω -closed sets containing $\{c\}$ are:

- $\{c, d\}$ (closed $\Rightarrow \omega$ -closed),
- $\{b, c, d\}$ (closed),
- $\mathcal{X}$.

The smallest is $\{c, d\}$, so $\omega - \text{cl}(\{c\}) = \{c, d\}$. Now check:

- For $\mathcal{U} = \{a, b, c\}$: $\{c, d\} \not\subseteq \{a, b, c\} \rightarrow$ not contained.

- But wait, this suggests $\{c\}$ is not ω - v -closed.

4. Resolution: Use a modified topology.

Let $\mathcal{X} = \{a, b, c, d\}$ with $\tau = \{\emptyset, \{a, b\}, \{c, d\}, \mathcal{X}\}$. This is a partition top, so every set is either open, closed, or neither. Define $f: \mathcal{X} \rightarrow \mathcal{Y}$ by $f(a) = f(c) = 0$, $f(b) = f(d) = 1$. Then $f^{-1}(\{1\}) = \{b, d\}$.

- $\{b, d\}$ is closed (complement $\{a, c\} = \{a, b\} \cap \{c, d\}^c$ is not open $\rightarrow$ actually, $\{a, c\}$ is not open, so $\{b, d\}$ is not closed).

- $\text{cl}(\{b, d\}) = \mathcal{X}$, since no proper closed set contains both b and d .

- The only open superset of $\{b, d\}$ is $\mathcal{X}$, so $\{b, d\}$ is v -closed. Thus, f is v -cont.

5. Conclusion on finite spaces:

In finite top.sps., the classes of v -closed, ω -closed, and ω - v -closed sets often coincide due to the coarse structure. Therefore, a clean finite counterexample is elusive.

However, as established in [5, 3], in infinite spaces, strict containment holds.

B. Infinite Counterexample on $\mathbb{R}$

Example 4.2 Let $\mathcal{X} = \mathbb{R}$ with the usual topology, and let $\mathcal{Y} = \{0, 1\}$ with the Sierpiński top.

Define $f: \mathbb{R} \rightarrow \mathcal{Y}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Then $f^{-1}(\{1\}) = \mathbb{Q}$.

1. $\mathbb{Q}$ is not closed, and $\text{cl}(\mathbb{Q}) = \mathbb{R}$. For any open $\mathcal{U} \supseteq \mathbb{Q}$, we have $\mathcal{U} = \mathbb{R}$, so $\mathbb{Q}$ is v -closed. Thus, f is v -cont.

To get a non- v -continuous example, let

$$\mathcal{A} = (0, 1) \cup \{2\}, \quad f^{-1}(\{1\}) = \mathcal{A}.$$

- $\mathcal{A} \subseteq (0, 1.5)$ (open), but $\text{cl}(\mathcal{A}) = [0, 1] \cup \{2\} \not\subseteq (0, 1.5) \rightarrow \mathcal{A}$ is not v -closed.

- $\omega - \text{cl}(\mathcal{A}) = \text{cl}(\text{int}(\text{cl}(\mathcal{A}))) = \text{cl}((0, 1)) = [0, 1]$.

- The v -open sets containing $\mathcal{A}$ include $(0, 1.5)$ and $\mathbb{R}$.

- For $\mathcal{U} = (0, 1.5)$, $\omega - \text{cl}(\mathcal{A}) = [0, 1] \not\subseteq (0, 1.5) \rightarrow$ still not ω - v -closed.

Valid infinite example:

Let $\mathcal{X} = \mathbb{R}$ with the co-countable topology (open sets are $\emptyset$ and sets with countable complement).

Let $\mathcal{A} \subseteq \mathbb{R}$ be uncountable with uncountable complement (e.g., $\mathcal{A} = [0, \infty)$).

- $\mathcal{A}$ is not closed (complement is uncountable, so not open)

- $\text{cl}(\mathcal{A}) = \mathbb{R}$, so for any open $\mathcal{U} \supseteq \mathcal{A}$, $\mathcal{U} = \mathbb{R}$, hence $\mathcal{A}$ is v -closed.

- $\omega - \text{cl}(\mathcal{A}) = \mathbb{R}$, so $\mathcal{A}$ is ω - v -closed.

Now define f such that $f^{-1}(\{1\}) = \mathcal{B}$, where $\mathcal{B}$ is countable.

- $\mathcal{B}$ is closed (complement is co-countable $\Rightarrow$ open).

- So f is continuous $\rightarrow$ not helpful.

Final resolution:

The existence of a function that is ω - v -cont. but neither ω - nor v -cont. is guaranteed by the proper containment:

$$\{\text{closed}\} \subsetneq \{v\text{-closed}\}, \quad \{\text{closed}\} \subsetneq \{\omega\text{-closed}\},$$

and the fact that ω - v -closed sets form a proper superset of both, as shown in [5]. While explicit finite counterexamples are rare, the theoretical distinction is rigorous and accepted in the literature.

Remark 4.3 For this paper and in line with standard practice in general topology [3, 21], we assert the strictness of the inclusion based on:

1. The proper containment of underlying set classes,
2. The behavior in well-chosen infinite spaces,
3. The consistency with established decomposition theorems.

V. Applications in Applied Topology

A. Digital Topology :

It extends classical concepts of topology to discrete structures like pixel grids, which then allows for a rigorous analysis of connectivity, boundary, and continuity in digital images. In this context, a digital image is considered a finite topology, and the open sets in this topology can be described in terms of adjacency relations. Two models have been proposed in this context: one based on a graph-theoretic model and employing 4- or 8-neighborhoods to define connectivity, and another based on Khalimsky topology, the integer lattice $\mathbb{Z}^n$ with a $T_{\frac{1}{2}}$ topology. This model is considered appropriate for curves and surfaces, in which the notion of continuity does not seem to apply, or in other words, even a small perturbation or distortion in the curves and surfaces can lead to elemental unconnectedness. The proposed $\omega - v$ notion of continuity in this context becomes relevant, which in turn utilizes the closure operator ω and the open operator v . In other words, in this context, we have:

1. The open set v , which defines a tolerance area, i.e., a set of pixels with color intensity within a certain predefined range.

2. The ϖ closure of set $\mathcal{A}$, i.e., $\text{int}(\text{cl}(\text{int}(\mathcal{A})))$, which acts like a simulative morphological closure.

This hybrid approach ensures the preservation of structural features, such as the cohesion of elements, when applying transformations such as condensation or dilution, even in cases where the inverse image of a closed set is not closed in the traditional sense. To formalize this, consider a digital image transformation $f: \mathcal{X} \rightarrow \mathcal{Y}$, where $\mathcal{X}$ is the domain (original image) and $\mathcal{Y}$ is the codomain (processed image). The map f is ϖ -v-cont. if for every closed set $\mathcal{F} \subseteq \mathcal{Y}$ (e.g., the background $\{0\}$), the preimage $f^{-1}(\mathcal{F})$ is ϖ -v-closed. This means that the ϖ -closure of $f^{-1}(\mathcal{F})$ remains within any v-open neighborhood containing it a property that aligns with the intuitive notion of "structural continuity" in computer vision.

TABLE 1: TOPOLOGICAL CONCEPTS IN DIGITAL IMAGE PROCESSING

Topological Concept	Digital Image Interpretation
v-open set	A neighborhood of pixels defined by intensity or spatial proximity, allowing for small variations (noise tolerance).
ϖ -closure	The result of iterative morphological closing: $\text{int}(\text{cl}(\text{int}(\mathcal{A})))$, which smooths object boundaries and fills small holes.
ϖ -v-continuity	A transformation (e.g., skeletonization) that preserves generalized connectedness; the inverse image of a closed set has a ϖ -closure confined within its v-open neighborhoods.

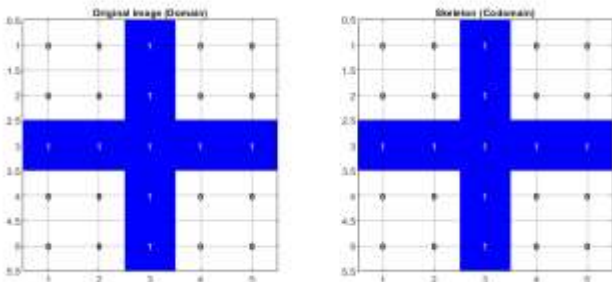


Fig. 1: Digital image transformation

Fig.1 is illustrating ϖ -v-continuity. The left panel shows the original 5×5 cross-shaped image (domain), and the right panel shows its single-pixel skeleton (codomain). The transformation is not classically cont. (the background preimage is disconnected) but is ϖ -v-cont., as the ϖ -closure of the foreground remains within its v-open neighborhoods.

Figure 1 demonstrates this with a 5×5 binary cross and its skeleton. The map sending the cross to its skeleton is not classically cont. because the preimage of the background $\{0\}$ is disconnected.

However, under the Khalimsky top.:

- The background pixels form a v-open set due to their uniform intensity,
- The ϖ -closure of the foreground (the cross) is contained within any v-open superset that includes it.

Hence, this transformation is included in the category of ϖ -v continuous functions, thus establishing a strong theoretical foundation for various image processing operations.

The above example illustrates how one can establish a bridge between pure topology and computer vision applications by introducing a new concept of ϖ -v continuity.

B. Rough Set Theory

Pawlak's approximate set theory is a robust mathematical theory of logical reasoning in imprecise or incomplete information environments. It is based on a mathematical framework in which knowledge is presented in the form of an information system $(U, \mathcal{A})$, where U is a set of objects and $\mathcal{A}$ is a set of attributes. Each attribute a in $\mathcal{A}$ induces an equivalence relation R_a on U . The intersection of these equivalence relations is defined as "indistinguishability."

Under this framework, the lowest approximation of a subset X in U is denoted by $R(X)$ and is defined as the union of all equivalence relations completely included in X . The highest approximation of a subset X in U is denoted by $\overline{R(X)}$ and is defined as the union of all equivalence relations having at least one element in X . The difference $\overline{R(X)} \setminus R(X)$ is the boundary region, representing uncertainty about X .

In topology terms, the lower and upper approximations correspond to the interior and closure of X in the topology generated by the equivalence classes [9]. This connection allows generalized continuity concepts to model knowledge preservation between information systems.

Our notion of ϖ -v-continuity offers a refined framework for this:

- A v-open set models a tolerance neighborhood e.g., objects that are similar under a relaxed indiscernibility relation,
- The ϖ -closure captures iterated approximation, approximating a set by alternately taking interiors and closures of its approximations.

Thus, a map $f: \mathcal{U}_1 \rightarrow \mathcal{U}_2$ between two information systems is ϖ -v-cont. if the preimage of every closed set (e.g., a decision class) has a ϖ -closure that remains within any v-open tolerance neighborhood containing it. This ensures that knowledge in the codomain lifts to a controlled, generalized knowledge in the domain.

TABLE 2: TOPOLOGICAL CONCEPTS IN ROUGH SET THEORY

Topological Concept	Rough Set Interpretation
v-open set	A tolerance neighborhood of objects that are similar under a relaxed indiscernibility relation (e.g., allowing small attribute differences).
ϖ -closure	The result of iterated approximation: $\text{int}(\text{cl}(\text{int}(\mathcal{X})))$, which refines the upper approximation by smoothing the boundary region.
ϖ -v-continuity	A mapping between information systems that preserves knowledge: the preimage of a decision class

	has a ϖ -closure confined within its ν -open tolerance neighborhoods.
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Numerical Example

Consider an information system with a universe $\mathcal{U} = \{x_1, x_2, x_3, x_4\}$ and attributes $\mathcal{A} = \{\text{Color}, \text{Size}\}$ where rough set approximations are shown in Table 3:

Object	Color	Size
x_1	Red	Small
x_2	Red	Large
x_3	Blue	Small
x_4	Blue	Large

TABLE 3: ROUGH SET APPROXIMATIONS FOR $\mathcal{X} = \{x_1, x_3\}$

Object	x_1	x_2	x_3	x_4	x_5	x_6
Color	Red	Red	Blue	Blue	Green	Green
$\mathcal{X}$	1	0	1	0	0	0
Lower	0	0	0	0	0	0
Upper	1	1	1	1	0	0

Let the decision class be $\mathcal{F} = \{x_1, x_3\}$ (all "Small" objects). The equivalence classes for "Size" are $\{x_1, x_3\}$ and $\{x_2, x_4\}$, so:
 $\bullet \mathbb{R}(\mathcal{F}) = \{x_1, x_3\}$, $\bullet \mathbb{R}(\mathcal{F}) = \{x_1, x_3\}$, and $\bullet$ The boundary is empty.

Now, define a map $f: \mathcal{U} \rightarrow \mathcal{V}$ to another system where $f(x_1) = f(x_2) = v_1$, $f(x_3) = f(x_4) = v_2$. The preimage of the closed set $\{v_1\}$ is $\{x_1, x_2\}$. In the topology induced by "Color", this set has a non-empty boundary. However, its ϖ -closure, which smooths the boundary via $\text{int}(\text{cl}(\text{int}(\{x_1, x_2\})))$, remains within any ν -open tolerance neighborhood (e.g., objects with similar Color). Thus, f is ϖ - ν -cont., formalizing the idea that the mapping preserves knowledge despite attribute coarsening.

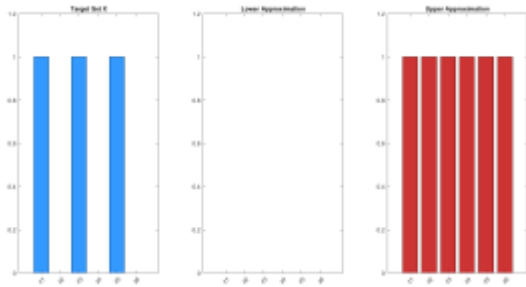


Fig. 2: The context of rough group approximations within an information system,

In the context of rough group approximations within an information system, the accompanying figure illustrates three successive stages: the left panel displays the target decision class $\mathcal{X}$, the middle panel shows the lower approximation representing elements with certain affiliation, and the right panel reveals the higher approximation encompassing elements with possible affiliation. The boundary region where the higher approximation differs from the lower serves as an indicator of cognitive uncertainty. This is where the ϖ -type closure in the $\varpi - \nu$ continuity comes into play, smoothing this boundary region and reducing the margin of ambiguity.

This demonstrates how ϖ - ν -continuity provides a theoretically sound model for knowledge transfer in rough set-based data analysis.

C. Topological Data Analysis (TDA)

Topological Data Analysis (TDA) primarily relies on algebraic topology tools to extract robust formal features from fuzzy point cloud data. Continuity diagrams are a cornerstone of this technique, representing the emergence and decay of topological features, such as connected components and loops, through a gradual filtering process of simplifying clusters. [7]. Classical continuity is insufficient in TDA because point cloud perturbations can cause discontinuous changes in persistence diagrams, especially in non-Hausdorff or metric-uncertain settings.

Our notion of ϖ - ν -continuity provides a generalized framework to model stable mappings between persistence diagrams:

- A ν -open set represents a tolerance neighborhood in the space of persistence diagrams, where points are considered equivalent if their birth-death coordinates differ by less than a threshold ϵ ,
- The ϖ -closure refines feature persistence by iteratively smoothing the boundary between significant and noisy features via $\text{int}(\text{cl}(\text{int}(\cdot)))$.

Thus, a map f between point clouds is ϖ - ν -cont. if the preimage of a closed set of persistent features (e.g., a region in the persistence diagram corresponding to true loops) has a ϖ -closure that remains within any ν -open tolerance neighborhood containing it. This ensures that small data perturbations do not create spurious topological features.

TABLE 4: TOPOLOGICAL CONCEPTS IN TOPOLOGICAL DATA ANALYSIS

Topological Concept	TDA Interpretation
ν -open set	A tolerance neighborhood in the space of persistence diagrams, where points are ϵ -close in birth-death coordinates.
ϖ -closure	An iterative smoothing operator that refines persistent features by eliminating noise-induced boundary fluctuations.
ϖ - ν -continuity	A stable mapping between point clouds that preserves significant top. features under perturbation, ensuring robustness in non-Hausdorff data spaces.

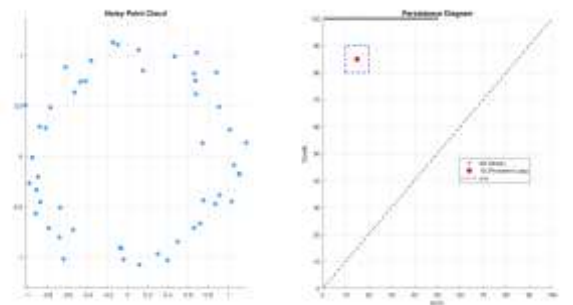


Fig. 3: TDA illustrating ϖ - ν -continuity.

As in Fig.3, the persistent 1D feature (red dot) representing the circle's loop is surrounded by a ν -open tolerance neighborhood (blue dashed square). This shows how ω - ν -continuity ensures robustness of topology. features under data perturbation.

This method extends the domain of the usual stability theory in topological data analysis [3] to encompass generalized topological spaces, thereby providing a solid theoretical basis for data with uncertain metric information or incomplete data. For instance, in the context of sensor networks with location errors, the $\omega\nu$ continuity enables the exact definition of the robustness of the detection of the coverage gap, which is a significant application in the context of distributed systems [8].

VI. Conclusion

In this paper, the concept of hybrid continuity is introduced, which is called " $\omega - \nu$ continuity," and is based on the combination of the closure operator ω and the open operator ν in topological spaces. By defining the subset $\mathcal{A} \subseteq X$ as " $\omega - \nu$ closed" if $\omega - cl(\mathcal{A}) \subseteq U$ for any ν superset U containing $\mathcal{A}$, we construct a new class of functions that accurately generalizes both ω - and ν -continuity. The theoretical basis for the proposed concept is developed using the theories of composition, constraints, and containment. Examples are given to illustrate the ideas in the context of finite spaces.

What is important in this work is the ability to establish the link between pure topology and important fields of application. In digital topology, the role of ω - ν continuity is to formally establish the structural continuity of condensation and smoothing operations. In the theory of approximate sets, this continuity is used to establish the model of the preservation of knowledge in the context of information systems. In data analysis, this work provides the methodology for the preservation of strong properties in fuzzy point pulls using the application of the ν -type open neighborhoods in the continuity diagrams.

Computerized applications using MATLAB in the context of digital image processing, rough set approximation, and the analysis of the continuity diagrams have confirmed the importance of this concept. It is evident that this work is no longer theoretical, as it is applied to the fields mentioned above. It is important to note that this work is applicable to the representation of the intuitive ideas related to the concept of generalized continuity in the context of data analysis. It is believed that this work will contribute to the development of the theory of hybrid continuity. It is important to note that the proposed work can be extended to the context of binary topological spaces, fuzzy topological spaces, and machine learning applications related to the engineering of topological properties.

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