# On the Interplay Between Property (A) and Ideal Avoidance in Commutative Rings and Modules

**Ahmed Naithel Ashour Al-Shammari**
**Department of Mathematics, University of Maragha, Tabriz, Iran**
**ahmdashwralshmry7@gmail.com**
**https://orcid.org/0009-0002-0688-361x**

***DOI:*** **http://dx.doi.org/10.31642/JoKMC/2018/130211**
***Received Apr. 22 2026. Accepted for publication May. 25, 2026***

***Abstract—*** *This paper studies the relation between Property (A) and the ideal avoidance property in commutative rings and modules. Using the annihilator family $\boldsymbol{\mathcal{A}}(\boldsymbol{M}) = \{\boldsymbol{Ann_R}(\boldsymbol{m}) : \boldsymbol{0} \neq \boldsymbol{m} \in \boldsymbol{M}\}$, we reformulate Property (A) as a containment problem for finitely generated ideals inside unions of annihilator ideals. This point of view yields a finite-cover criterion: if a ring has the ideal avoidance property and the zero-divisor set of a module is a finite union of annihilator ideals, then the module has Property (A). We also prove a prime-annihilator criterion, showing that finite covers by prime annihilator ideals force Property (A); as corollaries, every Noetherian module has Property (A), and every reduced ring with finitely many minimal primes has Property (A). A radical version yields nilpotent annihilation from finite covers by prime radicals of annihilator ideals. In addition, we establish invariance under passage to the quotient by $\boldsymbol{Ann_R}(\boldsymbol{M})$ and characterize Property (A) for finite direct products of rings and modules. Examples are included to show that ideal avoidance is sufficient but not necessary, and that the finite-annihilator-cover hypothesis is only a sufficient device. The paper provides a unified framework that links contemporary work on annihilator conditions and ideal avoidance in commutative algebra.*

**Keywords: Property (A); ideal avoidance; prime avoidance; annihilator ideals; zero-divisors; modules.**

## I. INTRODUCTION

Property (A) occupies a natural position among annihilator conditions in commutative algebra. Roughly speaking, it requires that every finitely generated ideal lying inside the zero-divisor set admit a single nonzero annihilating element. For rings, this condition and its transfer to polynomial or extension constructions were studied in detail by Hong et al [1] Strong and proper strong variants were later developed by Mahdou and Rahmouni Hassani [2] and by Ait Ouahi et al. [3] On the module-theoretic side, annihilator conditions were systematically investigated by Anderson and Chun [4], and more recently by Mahdou et al.[5] . Relative forms of Property (A), especially versions taken along an ideal, were studied by Bouchiba and Arssi [6] and by Ait Ouahi et al [3].

A second development concerns avoidance phenomena. The classical prime avoidance lemma is one of the basic tools of commutative algebra, but recent work has pushed the idea beyond prime ideals. Azarang studied a prime avoidance property for subsets of $\mathrm{Spec}(R)$ , while Chen and Tarizadeh introduced and analyzed the ideal avoidance property, showing that it behaves well for several important classes of ideals and under suitable ring-theoretic constructions [7]. This naturally raises the following question: when a zero-divisor set is covered by finitely many annihilator-type ideals, can an avoidance principle select one ideal that annihilates the whole finitely generated ideal under consideration?

The present paper answers this question in a general and transparent way. Our starting point is the annihilator family

$$\mathcal{A}(M) = \{\mathrm{Ann}_R(m) : 0 \neq m \in M\},$$

which records the annihilator ideals of nonzero elements of the $R$-module $M$. We first show that Property (A) is equivalent to the statement that every finitely generated ideal contained in $Z_R(M)$ is contained in one member of $\mathcal{A}(M)$. This simple reformulation is conceptually useful because it turns Property (A) into a finite-union problem.

Once this perspective is adopted, ideal avoidance becomes a natural selection principle. If $R$ has ideal avoidance and $Z_R(M)$ is a finite union of annihilator ideals, then every finitely generated ideal inside $Z_R(M)$ must be contained in one of those annihilator ideals, hence $M$ has Property (A). When the covering ideals are prime annihilators, classical prime avoidance yields an even cleaner criterion. In particular, one recovers Property (A) for Noetherian modules from the standard description of zero-divisors as a finite union of associated primes.

The paper also develops two further directions. First, we prove a radical-annihilator criterion: if $Z_R(M)$ is contained in a finite union of prime radicals of annihilator ideals, then every finitely generated ideal inside $Z_R(M)$ kills a nonzero element after passing to a suitable power. If the annihilator ideals are radical, this again implies Property (A). Second, we establish two structural stability results: invariance under passage from $R$ to $R/\mathrm{Ann}_R(M)$ and a finite direct product criterion for modules over product rings.

Throughout the paper, all rings are commutative with identity and all modules are unital. For a module $M$, we write

$$Z_R(M) = \{r \in R: rm = 0 \text{ for some } 0 \neq m \in M\}$$

for the zero-divisor set of $R$ on $M$. When $M = R$, this agrees with the usual set of zero-divisors of the ring.

## II. PRELIMINARIES

**Definition 2.1.** Let $M$ be an $R$-module.

1. We say that $M$ has **Property (A)** if for every finitely generated ideal $I \subseteq Z_R(M)$ there exists a nonzero element $m \in M$ such that $Im = 0$.
2. When $M = R$, we simply say that the ring $R$ has Property (A).

This is the module-theoretic form of the annihilator condition used in the literature on rings and modules with Property (A) [4].

- **Definition 2.2.** Let $I$ be an ideal of $R$. We say that $I$ has the **ideal avoidance property** if for every finite family of ideals $J_1, \dots, J_n$ of $R$,
- $I \subseteq \bigcup_{k=1}^{n} J_k$

implies $I \subseteq J_t$ for some $t$. We say that $R$ is an **avoidance ring** if every ideal of $R$ has the ideal avoidance property [8]

When all the ideals $J_k$ are prime, the previous definition specializes to the classical prime avoidance principle; the recent point is that no primeness assumption is imposed in the ideal avoidance setting [6].

**Lemma 2.3.** For every $R$-module $M$,

$$Z_R(M) = \bigcup_{0 \neq m \in M} \mathrm{Ann}_R(m).$$

*Proof.* If $r \in Z_R(M)$, then $rm = 0$ for some $0 \neq m \in M$, so $r \in \mathrm{Ann}_R(m)$. Conversely, if $r \in \mathrm{Ann}_R(m)$ for some $0 \neq m \in M$, then $r$ is a zero-divisor on $M$.

The next observation is elementary but useful. It shows that over rings whose finitely generated ideals are principal, Property (A) is automatic for every module.

**Proposition 2.4.** Assume that every finitely generated ideal of $R$ is principal; in particular, the conclusion holds when $R$ is a Bézout ring or a principal ideal ring. Then every $R$-module has Property (A).

*Proof.* Let $I \subseteq Z_R(M)$ be a finitely generated ideal. By assumption, $I = (a)$ for some $a \in R$. Since $a \in I \subseteq Z_R(M)$, there exists $0 \neq m \in M$ such that $am = 0$. Hence $Im = 0$, so $M$ has Property (A).

**Proposition 2.5.** Every principal ideal ring is an avoidance ring.

*Proof.* Let $R$ be a principal ideal ring and let $I = (a)$ be an ideal of $R$ such that $I \subseteq \bigcup_{k=1}^{n} J_k$. Since $a \in I$, we have $a \in J_t$ for some $t$, hence $(a) \subseteq J_t$. Thus $I$ has the ideal avoidance property. As every ideal is principal, $R$ is an avoidance ring.

## III. AN ANNIHILATOR-FAMILY VIEWPOINT

**An annihilator-family viewpoint**

For a nonzero $R$-module $M$, define

$$\mathcal{A}(M) = \{\mathrm{Ann}_R(m): 0 \neq m \in M\}.$$

This family controls the geometry of $Z_R(M)$ by Lemma 2.3.

**Theorem 3.1.** Let $M$ be an $R$-module. The following statements are equivalent.

3. $M$ has Property (A).
4. For every finitely generated ideal $I \subseteq Z_R(M)$ there exists $0 \neq m \in M$ such that $I \subseteq \mathrm{Ann}_R(m)$.
5. Every finitely generated ideal contained in $\bigcup \mathcal{A}(M)$ is contained in one member of $\mathcal{A}(M)$.

*Proof.* The equivalence of (1) and (2) is immediate from the identity $Im = 0$ if and only if $I \subseteq \mathrm{Ann}_R(m)$. By Lemma 2.3, we have $Z_R(M) = \bigcup \mathcal{A}(M)$, so (2) and (3) are equivalent as well.

The previous theorem shows that Property (A) amounts to a finite containment principle inside the annihilator family. This motivates the following notion.

**Definition 3.2.** We say that $M$ has a **finite annihilator cover** if there exist nonzero elements $m_1, \dots, m_n \in M$ such that

$$Z_R(M) = \bigcup_{i=1}^{n} \mathrm{Ann}_R(m_i).$$

**Theorem 3.3.** Let $M$ be an $R$-module. If $R$ is an avoidance ring and $M$ has a finite annihilator cover, then $M$ has Property (A).

*Proof.* Suppose

$$Z_R(M) = \bigcup_{i=1}^{n} \mathrm{Ann}_R(m_i)$$

for some nonzero $m_1, \dots, m_n \in M$. Let $I \subseteq Z_R(M)$ be a finitely generated ideal. Then

$$I \subseteq \bigcup_{i=1}^{n} \mathrm{Ann}_R(m_i).$$

Since $R$ is an avoidance ring, $I$ is contained in one of these annihilator ideals, say $I \subseteq \mathrm{Ann}_R(m_j)$. Therefore $Im_j = 0$ with $m_j \neq 0$, and $M$ has Property (A).

Theorem 3.3 gives a clean mechanism: ideal avoidance selects one annihilator ideal out of a finite cover of the zero-divisor set.

A useful special case occurs when the annihilator ideals in the cover are prime, because prime avoidance can be invoked directly.

**Theorem 3.4.** Let $M$ be an $R$-module. Assume that

$$Z_R(M) \subseteq \bigcup_{i=1}^{n} \mathfrak{p}_i,$$

where each $\mathfrak{p}_i$ is a prime ideal and there exists $0 \neq m_i \in M$ such that $\mathfrak{p}_i = \mathrm{Ann}_R(m_i)$. Then $M$ has Property (A).

*Proof.* Let $I \subseteq Z_R(M)$ be a finitely generated ideal. Then

$$I \subseteq \bigcup_{i=1}^{n} \mathfrak{p}_i.$$

By the classical prime avoidance lemma, $I \subseteq \mathfrak{p}_j$ for some $j$. Since $\mathfrak{p}_j = \mathrm{Ann}_R(m_j)$, we obtain $Im_j = 0$ with $m_j \neq 0$. Hence $M$ has Property (A).

**Corollary 3.5.** If

$$Z_R(M) \subseteq \bigcup_{i=1}^{n} \mathfrak{p}_i$$

for associated primes $\mathfrak{p}_1, \dots, \mathfrak{p}_n \in \mathrm{Ass}_R(M)$, then $M$ has Property (A).

*Proof.* By definition of associated prime, each $\mathfrak{p}_i$ is the annihilator of a nonzero element of $M$. Apply Theorem 3.4.

**Corollary 3.6.** Every Noetherian $R$-module has Property (A).

*Proof.* If $M$ is Noetherian, then $\mathrm{Ass}_R(M)$ is finite and

$$Z_R(M) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M)} \mathfrak{p}.$$

Now apply Corollary 3.5.

The next proposition gives a ring-theoretic application in the reduced case.

**Proposition 3.7.** Let $R$ be a reduced ring with finitely many minimal prime ideals

$$\mathrm{Min}(R) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}.$$

Then $R$ has Property (A).

*Proof.* For each fixed $i$, choose $a_j \in \mathfrak{p}_j \backslash \mathfrak{p}_i$ for every $j \neq i$, and put

$$x_i = \prod_{j \neq i} a_j.$$

Then $x_i \in \bigcap_{j \neq i} \mathfrak{p}_j$ and $x_i \notin \mathfrak{p}_i$. Since $R$ is reduced, we have

$$\bigcap_{k=1}^{n} \mathfrak{p}_k = 0.$$

We claim that $\mathrm{Ann}_R(x_i) = \mathfrak{p}_i$. Indeed, if $r \in \mathfrak{p}_i$, then for every $k$ we have $rx_i \in \mathfrak{p}_k$: this is clear for $k = i$ because $r \in \mathfrak{p}_i$, and for $k \neq i$ because $x_i \in \mathfrak{p}_k$. Hence $rx_i \in \bigcap_k \mathfrak{p}_k = 0$, so $r \in \mathrm{Ann}_R(x_i)$. Conversely, if $rx_i = 0$, then $rx_i \in \mathfrak{p}_i$, and since $\mathfrak{p}_i$ is prime while $x_i \notin \mathfrak{p}_i$, it follows that $r \in \mathfrak{p}_i$. Thus $\mathrm{Ann}_R(x_i) = \mathfrak{p}_i$.

It remains to show that

$$Z(R) = \bigcup_{i=1}^{n} \mathfrak{p}_i.$$

If $r \in \mathfrak{p}_i$, then $rx_i = 0$ and $r$ is a zero-divisor. Conversely, if $r \in Z(R)$, choose $0 \neq b \in R$ such that $rb = 0$. Since $b \neq 0$ and $\bigcap_i \mathfrak{p}_i = 0$, there exists $k$ with $b \notin \mathfrak{p}_k$. As $0 = rb \in \mathfrak{p}_k$ and $\mathfrak{p}_k$ is prime, we obtain $r \in \mathfrak{p}_k$. Hence $Z(R)$ is a finite union of prime annihilator ideals, so Theorem 3.4 applies with $M = R$.

## IV. RADICAL COVERS AND STRUCTURAL CONSEQUENCES

The prime-annihilator criterion can be weakened to a radical statement. In that form one no longer obtains immediate annihilation by $I$, but one does get annihilation by a suitable power of $I$.

**Theorem 4.1.** Let $M$ be an $R$-module. Assume that

$$Z_R(M) \subseteq \bigcup_{i=1}^{n} \mathfrak{p}_i,$$

where each $\mathfrak{p}_i$ is prime and satisfies

$$\mathfrak{p}_i = \sqrt{\mathrm{Ann}_R(m_i)}$$

for some nonzero $m_i \in M$. Then for every finitely generated ideal $I \subseteq Z_R(M)$ there exist an index $i$ and an integer $N \geq 1$ such that

$$I^N m_i = 0.$$

In particular, if every $\mathrm{Ann}_R(m_i)$ is radical, then $M$ has Property (A).

*Proof.* Let $I = (a_1, \dots, a_t) \subseteq Z_R(M)$. Since $I \subseteq \bigcup_{i=1}^{n} \mathfrak{p}_i$, prime avoidance yields $I \subseteq \mathfrak{p}_i$ for some $i$. Because $\mathfrak{p}_i =$

$\sqrt{\mathrm{Ann}_R(m_i)}$, for each generator $a_j$ there exists $e_j \geq 1$ such that $a_j^{e_j} \in \mathrm{Ann}_R(m_i)$. Set

$$N = (e_1 - 1) + \cdots + (e_t - 1) + 1.$$

Every monomial of total degree $N$ in the generators $a_1, \ldots, a_t$ has some exponent at least $e_j$; therefore each such monomial belongs to $\mathrm{Ann}_R(m_i)$. It follows that $I^N \subseteq \mathrm{Ann}_R(m_i)$, hence $I^N m_i = 0$.

If every $\mathrm{Ann}_R(m_i)$ is radical, then

$$\sqrt{\mathrm{Ann}_R(m_i)} = \mathrm{Ann}_R(m_i),$$

so $I \subseteq \mathrm{Ann}_R(m_i)$ and therefore $Im_i = 0$. Thus $M$ has Property (A).

The next result shows that Property (A) depends only on the faithful quotient of the ring acting on the module.

**Proposition 4.2.** Let $M$ be an $R$-module and put $A = \mathrm{Ann}_R(M)$. Then $M$ has Property (A) as an $R$-module if and only if it has Property (A) as an $R/A$-module.

*Proof.* Ideals of $R/A$ are precisely the ideals $\bar{I} = I/A$ with $A \subseteq I \subseteq R$. We first note that

$$\bar{I} \subseteq Z_{R/A}(M) \quad \Leftrightarrow \quad I \subseteq Z_R(M).$$

Indeed, if $\bar{r} \in Z_{R/A}(M)$, then $\bar{r}m = 0$ for some $0 \neq m \in M$, which means $rm = 0$ because every element of $A$ annihilates $M$; thus $r \in Z_R(M)$. The converse is immediate.

Assume now that $M$ has Property (A) over $R$. Let $\bar{I} \subseteq Z_{R/A}(M)$ be a finitely generated ideal. Then the corresponding ideal $I$ of $R$ satisfies $I \subseteq Z_R(M)$, so there exists $0 \neq m \in M$ with $Im = 0$. Hence $\bar{I}m = 0$, and $M$ has Property (A) over $R/A$.

Conversely, assume that $M$ has Property (A) over $R/A$. Let $I \subseteq Z_R(M)$ be a finitely generated ideal of $R$. Then $\bar{I} \subseteq Z_{R/A}(M)$, so there exists $0 \neq m \in M$ with $\bar{I}m = 0$. Equivalently, $Im = 0$. Thus $M$ has Property (A) over $R$.

We next examine direct products.

**Proposition 4.3.** Let

$$R = R_1 \times \cdots \times R_n$$

and

$$M = M_1 \times \cdots \times M_n,$$

where each $M_i$ is an $R_i$-module. Then $M$ has Property (A) as an $R$-module if and only if each $M_i$ has Property (A) as an $R_i$-module.

*Proof.* It suffices to treat the case $n = 2$, the general case following by induction. Put $R = R_1 \times R_2$ and $M = M_1 \times M_2$. One checks directly that

$$Z_R(M) = \left(Z_{R_1}(M_1) \times R_2\right) \cup \left(R_1 \times Z_{R_2}(M_2)\right).$$

Assume first that $M_1$ and $M_2$ have Property (A). Let $I \subseteq Z_R(M)$ be a finitely generated ideal of $R$. Since every ideal of $R$ has the form $I_1 \times I_2$, write $I = I_1 \times I_2$. If neither $I_1 \subseteq Z_{R_1}(M_1)$ nor $I_2 \subseteq Z_{R_2}(M_2)$ were true, we could choose $a_1 \in I_1 \setminus Z_{R_1}(M_1)$ and $a_2 \in I_2 \setminus Z_{R_2}(M_2)$, so $(a_1, a_2) \in I$ but $(a_1, a_2) \notin Z_R(M)$, a contradiction. Hence one of the containments holds. Suppose $I_1 \subseteq Z_{R_1}(M_1)$. Since $M_1$ has Property (A), there exists $0 \neq m_1 \in M_1$ with $I_1 m_1 = 0$. Then

$$I(m_1, 0) = 0,$$

and $(m_1, 0) \neq 0$. Thus $M$ has Property (A). The case $I_2 \subseteq Z_{R_2}(M_2)$ is symmetric.

Conversely, assume that $M$ has Property (A). Let $I_1 \subseteq Z_{R_1}(M_1)$ be a finitely generated ideal. Consider the ideal

$$I = I_1 \times R_2$$

of $R$. For every $(a_1, a_2) \in I$, we have $a_1 \in Z_{R_1}(M_1)$, so there exists $0 \neq m_1 \in M_1$ with $a_1 m_1 = 0$. Hence

$$(a_1, a_2)(m_1, 0) = 0,$$

which shows that $I \subseteq Z_R(M)$. By Property (A) for $M$, there exists $(m_1, m_2) \neq (0,0)$ such that

$$I(m_1, m_2) = 0.$$

Taking $(0,1) \in I$ yields $(0, m_2) = 0$, so $m_2 = 0$ and therefore $m_1 \neq 0$. Taking $(a_1, 0) \in I$ for $a_1 \in I_1$ then gives $a_1 m_1 = 0$. Hence $I_1 m_1 = 0$, and $M_1$ has Property (A). The same argument works for $M_2$.

## V. EXAMPLES AND LIMITATIONS

The previous results identify natural sufficient conditions for Property (A), but those conditions are not necessary. The following examples clarify the range of the theory.

**Example 5.1.** Property (A) does not imply ideal avoidance.

Let $k = \mathbb{F}_q$ be a finite field and consider

$$R = k[x, y]/(x, y)^2.$$

Write $\mathfrak{m} = (x, y)/(x, y)^2$. Then $R$ is Artinian, hence Noetherian, so $R$ has Property (A) by Corollary 3.6. On the other hand, $\mathfrak{m}^2 = 0$, so $\mathfrak{m}$ is a two-dimensional $k$-vector space and every one-dimensional $k$-subspace is an ideal of $R$. Therefore

$$\mathfrak{m} = \bigcup_{(a:b) \in \mathbf{P}^1(k)} (ax + by),$$

a finite union of proper ideals. Since no one of these one-dimensional ideals contains all of $\mathfrak{m}$, the ideal $\mathfrak{m}$ does not have the ideal avoidance property. Hence $R$ is not an avoidance ring. This shows that ideal avoidance is sufficient for Theorem 3.3 but not necessary for Property (A).

**Example 5.2.** The finite-annihilator-cover hypothesis in Theorem 3.3 is only sufficient.

Let $\mathcal{P}$ be an infinite set of prime numbers and set

$$M = \oplus_{p\in\mathcal{P}}\mathbb{Z}/p\mathbb{Z}$$

as a module over $R = \mathbb{Z}$. Since every finitely generated ideal of $\mathbb{Z}$ is principal, Proposition 2.4 shows that $M$ has Property (A). We claim that $M$ does not have a finite annihilator cover.

Indeed, every nonzero element of $M$ has finite support, say on the distinct primes $p_1, \dots, p_t$, and its annihilator is the principal ideal $(p_1 \cdots p_t)$. Suppose, by contradiction, that

$$Z_{\mathbb{Z}}(M) = \bigcup_{j=1}^{n}(d_j)$$

for finitely many annihilator ideals $(d_j)$. Only finitely many prime numbers divide the integers $d_1, \dots, d_n$. Choose $p \in \mathcal{P}$ not dividing any $d_j$. Then $p \in Z_{\mathbb{Z}}(M)$ because it annihilates the $\mathbb{Z}/p\mathbb{Z}$-summand, but $p \notin (d_j)$ for every $j$, contradiction. Hence $M$ has Property (A) without admitting a finite annihilator cover.

**Example 5.3.** A finite annihilator cover over $\mathbb{Z}$.

Let

$$M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$$

as a $\mathbb{Z}$-module. Then

$$Z_{\mathbb{Z}}(M) = (2) \cup (3).$$

Moreover,

$$(2) = \mathrm{Ann}_{\mathbb{Z}}(\bar{1}, \bar{0}), \qquad (3) = \mathrm{Ann}_{\mathbb{Z}}(\bar{0}, \bar{1}).$$

Thus $M$ has a finite annihilator cover. Since $\mathbb{Z}$ is a principal ideal domain, it is both an avoidance ring by Proposition 2.5 and a ring over which every module has Property (A) by Proposition 2.4. In this simple example, Theorem 3.3 recovers the conclusion directly from the finite cover.

## VI. CONCLUSION

The main theme of this paper is that Property (A) can be understood through finite containment inside annihilator families, while ideal avoidance and prime avoidance provide the selection mechanism that turns those containments into genuine annihilation statements. This perspective yields three useful criteria: a finite-annihilator-cover criterion over avoidance rings, a prime-annihilator criterion, and a radical-annihilator criterion. The framework also gives short proofs of Property (A) for Noetherian modules and for reduced rings with finitely many minimal primes, and it remains stable under passage to $R/\mathrm{Ann}_R(M)$ and under finite direct products.

Several directions remain open. It would be natural to extend the present approach to the relative versions of Property (A) studied along an ideal, to localization questions, and to stronger variants such as proper strong Property (A). Another promising direction is to investigate whether ideal avoidance can be characterized in terms of canonical covers of the zero-divisor set by annihilator-type or associated-prime-type ideals. These questions may further clarify the structural role of avoidance phenomena in commutative module theory.

## REFERENCES


[1]. C. Y. Hong, N. K. Kim, Y. Lee, and S. J. Ryu, “Rings with Property (A) and their extensions,” *J. Algebra*, vol. 315, no. 2, pp. 612–628, Sep. 2007, doi: 10.1016/j.jalgebra.2007.01.042.

[2]. N. Mahdou, S. Koç, E. Yıldız, and Ü. Tekir, “Annihilator Condition on Modules,” *Iranian Journal of Science*, vol. 47, no. 5–6, pp. 1713–1721, Dec. 2023, doi: 10.1007/s40995-023-01533-4.

[3]. A. Ait Ouahi, S. Bouchiba, and M. El-Arabi, “On proper strong Property (𝒜) for rings and modules,” *J. Algebra Appl.*, vol. 19, no. 12, p. 2050239, Dec. 2020, doi: 10.1142/S0219498820502394.

[4]. D. D. Anderson and S. Chun, “Annihilator conditions on modules over commutative rings,” *J. Algebra Appl.*, vol. 16, no. 08, p. 1750143, Aug. 2017, doi: 10.1142/S0219498817501432.

[5]. N. Mahdou and A. R. Hassani, “On Strong (A)-Rings,” *Mediterranean Journal of Mathematics*, vol. 9, no. 2, pp. 393–402, May 2012, doi: 10.1007/s00009-010-0106-4.

[6]. S. Bouchiba and Y. Arssi, “ON PROPERTY (A) OF RINGS AND MODULES ALONG AN IDEAL,” 2020.

[7]. J. Chen and A. Tarizadeh, “On the ideal avoidance property,” *J. Pure Appl. Algebra*, vol. 228, no. 3, p. 107500, Mar. 2024, doi: 10.1016/j.jpaa.2023.107500.

[8]. J. Chen and A. Tarizadeh, “On the ideal avoidance property,” *J. Pure Appl. Algebra*, vol. 228, no. 3, p. 107500, Mar. 2024, doi: 10.1016/j.jpaa.2023.107500.