Transform of Ramadan Group to Solve Generalized Weakly Singular Volterra Integral Equations

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Abstract—*Many differential equations with beginning or boundary conditions represented as integral equations can be solved using the integral transform method. In this research. We investigated the basic characteristics of the Transform of Ramadan Group (ToRG) and applied it to solve generalized Volterra integral equations (WSVIEs) with weakly singular kernels integral of the type* $(\boldsymbol{x}-\boldsymbol{t})^{-\boldsymbol{\alpha}}$ *for a constant* $\boldsymbol{\alpha} \in (\mathbf{0},\mathbf{1})$. *Also, a connection between the ToRG and some useful integral transforms thereby making the equation simpler to solve theoretically and numerically. This improves the accuracy and efficiency of solving equations with weak singularities*

Keywords— Integral transforms, Transform of Ramadan Group (ToRG), Gamma function, Singular Integral Equations)

I. Introduction

Many scientific fields have studied the theoretical and numerical aspects of (WSVIEs) in mathematical physics, scattering theory, electrochemistry, semiconductors, heat conduction, and population dynamics [6]. Most research on this topic concentrates on (WSVIEs) kernels of the type $(x-t)^{-\alpha}$, with $0<\alpha<1$ [1,7] One type of mathematical operator is an integral transform [3].

New techniques for analytically solving fractional problems have emerged recently. Integral transformations have been widely used to solve this type of equation. Algebraic equations can be created from fractional differential equations using these methods. The main objective of the integral transform is to reduce complex fractional differential equations, along with geometric and physical equations, to get exact solutions[4,5].

Recently, in 2016 Eshaghi et al. discussed a new method for converting weakly singular nonlinear VIEs into singularity-free equations using partial order Legendre functions [7]. The Transform of Ramadan Group (ToRG), Ramadan et al. introduced a revolutionary integrative transform in 2016, integrates the characteristics of the Sumudu and Laplace transforms into a single generic formula [4].

In 2018, Ramadan and Mesrega presented using the convolution of ToRG to solve partial and Integro-differential equations [8].

In 2024, Zheng and Martin presented Volterra integral equations with a weak singularity kernel that can be solved using a Nestrom method based on product integration and stepwise slicing with the variable exponent $\alpha(t)$ between 0 and 1 [2].

The generalized WSVIEs of the second kind are given by

$$\nu(\xi)=f(\xi)+\int_0^{\xi}\frac{\beta}{(\xi-\tau)^{\alpha}}\nu(\tau)d\tau, 0<\alpha<1, \xi\in[0,l] \qquad (1)$$

where β is a constant [1,10].

To make the equation easier to solve theoretically, research is being conducted on the ToRG for weak VIEs with weak singularities to diminish or eliminate the singularity. This enhances the precision and effectiveness of solving weak singularity equations.

II. Gamma Function

The Gamma function or second order Euler (1707-1783) denoted by $\Gamma(\cdot)$ is defined as the improper integral [9,11].

$$\Gamma(\alpha)=\int_0^{\infty} e^{-\tau}\tau^{\alpha-1}d\tau,\ \ \alpha>0 \qquad (2)$$

This diverges for $\alpha \leq 0$ and converges for all positive values of α. The Gamma function's fundamental characteristics are:

- A simple integration by parts establishes the fundamental identity

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \qquad \alpha > 0 \tag{3}$$

- The Gamma function's reflection property is provided by

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{sin(\pi\alpha)}, 0 \leq \alpha \leq 1 \tag{4}$$

- For computational reasons, the following specific Gamma function values may be helpful.

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{5}$$

Proof. Based on the Gamma function definition (2), we have

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-\tau}\tau^{-\frac{1}{2}}d\tau,$$

making the substitution $\tau = \nu^2, d\tau = 2\nu d\nu$, can be written as

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-\nu^2}d\nu.$$

Since $\int_0^\infty e^{-\nu^2}d\nu = \int_0^\infty e^{-\nu^2}d\nu$, then

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \left(2\int_0^\infty e^{-u^2}du\right)\left(2\int_0^\infty e^{-v^2}dv\right)$$

$$= 2^2\int_0^\infty\int_0^\infty e^{-(\nu^2+\nu^2)}d\nu d\nu.$$

By using $u = r\cos\theta$ and $v = r\sin\theta$, we can calculate the double integral:

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 2^2\int_0^{\pi/2}\int_0^\infty e^{-r^2}rdrd\theta,$$

$$= -2\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}(-2r)drd\theta = \pi.$$

Therefore,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \quad \blacksquare$$

III. Transform of Ramadan Group

For functions of exponential order, a new integral ToRG is defined. Functions in the set A are taken into consideration, defined by [3]:

$$A = \left\{f(\xi): \exists M, \xi_1, \xi_2 > 0 \text{ s.t. } |f(\xi)| < Me^{\frac{|\xi|}{\xi_n}}, \text{if } \xi \in (-1)^n \times [[0,\infty)]\right\}$$

The ToRG is defined by

$$\mathbb{RG}[f(\xi);(s,\nu)] = K(s,\nu)$$

$$= \begin{cases} \int_0^\infty e^{-s\xi}f(\nu\xi)d\xi, & -\xi_1 < \nu \leq 0 \\ \int_0^\infty e^{-s\xi}f(\nu\xi)d\xi, & 0 \leq \nu < \xi_2 \end{cases} \tag{6}$$

where $\mathbb{RG}$ is the ToRG operator.

TABLE I. **ToRG of some basic functions**

$\boldsymbol{f(x)}$	$\mathbb{RG}[\boldsymbol{f(x)};(\boldsymbol{s},\boldsymbol{u})] = \boldsymbol{K(s,u)}$
$k,$	$\mathbb{RG}[k;(s,\nu)] = \frac{k}{c}, k \in R$
$\xi,$	$\mathbb{RG}[\xi;(s,\nu)] = \frac{\frac{s}{\nu}}{s^2},$
$\xi^n,$	$\mathbb{RG}[\xi^n;(s,\nu)] = \frac{n!\,\nu^n}{s^{n+1}}, n \in N$
$\xi^\alpha,$	$\mathbb{RG}[\xi^\alpha;(s,\nu)] = \frac{\Gamma(\alpha+1)\nu^\alpha}{s^{\alpha+1}},$ $s > 0, \alpha > -1$
$e^{a\xi},$	$\mathbb{RG}\left[e^{a\xi};(s,\nu)\right] = \frac{1^{s^{\alpha+1}}}{s-a\nu},$
$\sin(a\xi),$	$\mathbb{RG}[\sin(a\xi);(s,\nu)] = \frac{a\nu}{s^2+(a\nu)^2},$
$\cos(a\xi),$	$\mathbb{RG}[\cos(a\xi);(s,\nu)] = \frac{s}{s^2+(a\nu)^2},$
$\sinh(a\xi),$	$\mathbb{RG}[\sinh(a\xi);(s,\nu)] = \frac{a\nu}{s^2-(a\nu)^2},$
$\cosh(a\xi),$	$\mathbb{RG}[\cosh(a\xi);(s,\nu)] = \frac{s}{s^2-(a\nu)^2},$

The inverse ToRG is

$$\mathbb{R}G^{-1}[K(s,\nu);\xi] = f(\xi) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{-\frac{s}{\nu}\xi}K(s,\nu)d\xi.$$

where $\mathbb{RG}^{-1}$ is the Inverse ToRG operator. The following are some basic properties of the ToRG:

- Linearity property:

$$\mathbb{RG}[af(\xi) + bg(\xi)] = a\mathbb{RG}[f(\xi)] + b\mathbb{RG}[g(\xi)],$$
$$= aF(s,\nu) + bG(s,\nu).$$

where a, b are constants.

- Convolution property:

$$\mathbb{RG}[(f * g)(\xi);(s,\nu)] = \mathbb{RG}[f(\xi) * g(\xi)],$$
$$= \nu\mathbb{RG}\{f(\xi)\}\mathbb{RG}\{g(\xi)\},$$
$$= \nu F(s,\nu)G(s,\nu),$$

where $f(\xi) * g(\xi)$ is defined by

$$f(\xi) * g(\xi) = \int_0^\xi f(\xi-\tau)g(\tau)d\tau = \int_0^\xi f(\xi)g(\xi-\tau)d\tau.$$

IV. Examples

In this section, three different examples, $\alpha = \frac{1}{2}, \frac{3}{4}, \frac{1}{3}$, respectively, are used to solve the generalized WSVIEs by ToRG to obtain the exact solution.

Example 1. Solve the following generalized WSVIE by ToRG

$$\nu(\xi) = \left(1-\frac{\pi}{2}\right)\xi + \sqrt{\xi}\left(1-\frac{4}{3}\xi\right) + \int_0^\xi \frac{1}{\sqrt{\xi-\tau}}\nu(\tau)d\tau \# (7)$$

Taking the ToRG on both sides of Eq. (7) and using the properties of ToRG, as follows:

$$\mathbb{RG}\{\nu(\xi)\}(s,\nu) = \mathbb{RG}\left\{\left(1-\frac{\pi}{2}\right)\xi + \sqrt{\xi}\left(1-\frac{4}{3}\xi\right)\right\}(s,\nu)$$

$$+\mathbb{RG}\left\{\int_0^{\xi}\frac{1}{(\xi-\tau)^{\frac{1}{2}}}v(\tau)d\tau\right\}(s,v),$$

$$U(s,v)=\left(1-\frac{\pi}{2}\right)\mathbb{RG}\{\xi\}+\mathbb{RG}\{\xi^{1/2}\}-\frac{4}{3}\mathbb{RG}\{\xi^{3/2}\}$$
$$+\mathbb{RG}\left\{\xi^{-\frac{1}{2}}*v(\xi)\right\},$$

$$U(s,v)=\left(1-\frac{\pi}{2}\right)\frac{v}{s^2}+\frac{\Gamma\left(\frac{3}{2}\right)v^{\frac{1}{2}}}{s^{\frac{3}{2}}}-\frac{4}{3}\frac{\Gamma\left(\frac{5}{2}\right)v^{\frac{3}{2}}}{s^{\frac{5}{2}}}$$
$$+v\mathbb{RG}\left\{\xi^{-\frac{1}{2}}\right\}\cdot\mathbb{RG}\{v(\xi)\},$$

$$U(s,v)=\left(1-\frac{\pi}{2}\right)\frac{v}{s^2}+\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)v^{\frac{1}{2}}}{s^{\frac{3}{2}}}-\frac{4}{3}\frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)v^{\frac{3}{2}}}{s^{\frac{5}{2}}}$$
$$+v\frac{\Gamma\left(\frac{1}{2}\right)v^{-\frac{1}{2}}}{s^{-\frac{1}{2}+1}},$$

$$U(s,v)=\left(1-\frac{\pi}{2}\right)\frac{v}{s^2}+\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)v^{\frac{1}{2}}}{s^{\frac{3}{2}}}-2^{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)v^{\frac{3}{2}}}$$
$$+\frac{\Gamma\left(\frac{1}{2}\right)v^{\frac{1}{2}}}{s^{\frac{5}{2}}}U(s,v), \tag{8}$$

Operating and simplifying of Eq. (8) gives

$$U(s,v)=\frac{vs^{\frac{1}{2}}-\frac{\pi}{2}vs^{\frac{1}{2}}+\frac{1}{2}\Gamma\left(\frac{1}{2}\right)v^{\frac{1}{2}}s-\Gamma\left(\frac{1}{2}\right)v^{\frac{3}{2}}}{s^2\left[s^{\frac{1}{2}}-\Gamma\left(\frac{1}{2}\right)v^{\frac{1}{2}}\right]},$$

$$U(s,v)=\frac{v}{s^2}+\frac{\Gamma\left(\frac{3}{2}\right)v^{\frac{1}{2}}}{s^{\frac{3}{2}}}, \tag{9}$$

and taking the inverse ToRG on both sides of Eq. (9), we get

$$\mathbb{RGG}^{-1}\{U(s,v)\}(\xi)=\mathbb{RG}^{-1}\left\{\frac{v}{s^2}+\frac{\Gamma\left(\frac{3}{2}\right)v^{\frac{1}{2}}}{s^{\frac{3}{2}}}\right\}(\xi),$$

$$v(\xi)=\mathrm{x}+\sqrt{\xi}.$$

which is the exact solution of the given Eq. (7).

Example 2. Solve the following generalized WSVIE by ToRG

$$v(\xi)=\xi^2-\frac{128}{45}\xi^{\frac{9}{4}}+\int_0^{\xi}\frac{1}{(\xi-\tau)^{\frac{3}{4}}}v(\tau)d\tau. \tag{10}$$

Taking the ToRG on both sides of Eq. (10) and using the properties of ToRG, as follows:

$$\mathbb{RG}\{v(\xi)\}(s,v)=\mathbb{RG}\left\{\xi^2-\frac{128}{45}\xi^{\frac{9}{4}}\right\}(s,v)$$
$$+\mathbb{RG}\left\{\int_0^{\xi}\frac{1}{(\xi-\tau)^{\frac{3}{4}}}v(\tau)d\tau\right\}(s,v),$$

$$U(s,v)=\mathbb{RG}\{\xi^2\}-\frac{128}{45}\mathbb{RG}\left\{\xi^{\frac{9}{4}}\right\}+\mathbb{RG}\left\{\xi^{-\frac{3}{4}}*v(\xi)\right\},$$

$$U(s,v)=\frac{2!\,v^2}{s^3}-\frac{128}{45}\frac{\Gamma\left(\frac{13}{4}\right)v^{\frac{9}{4}}}{s^{\frac{13}{4}}}+v\mathbb{RG}\left\{\xi^{-\frac{3}{4}}\right\}\cdot\mathbb{RG}\{v(\xi)\},$$

$$U(s,v)=\frac{2v^2}{s^3}-\frac{128}{45}\frac{\frac{9}{4}\Gamma\left(\frac{9}{4}\right)v^{\frac{9}{4}}}{s^{\frac{13}{4}}}+v\frac{\Gamma\left(\frac{1}{4}\right)v^{-\frac{3}{4}}}{s^{-\frac{3}{4}+1}}U(s,v),$$

$$U(s,v)=\frac{2v^2}{s^3}-2\frac{\Gamma\left(\frac{1}{4}\right)v^{\frac{9}{4}}}{s^{\frac{13}{4}}}+\frac{\Gamma\left(\frac{1}{4}\right)v^{\frac{1}{4}}}{s^{\frac{1}{4}}}U(s,v). \tag{11}$$

Operating and simplifying of Eq. (11) gives

$$U(s,v)=\frac{2v^2-2s^{-\frac{1}{4}}\Gamma\left(\frac{1}{4}\right)v^{\frac{9}{4}}}{s^{\frac{11}{4}}\left[s^{\frac{1}{4}}-\Gamma\left(\frac{1}{4}\right)v^{\frac{1}{4}}\right]},$$

$$U(s,v)=\frac{2v^2}{s^3}, \tag{12}$$

and taking the inverse ToRG on both sides of Eq. (12) we get

$$\mathbb{RG}G^{-1}\{U(s,v)\}(\xi)=\mathbb{R}G^{-1}\left\{\frac{2v^2}{s^3}\right\}(\xi),$$

$$v(\xi)=\xi^2.$$

which is the exact solution of the given Eq. (10).

Example 3. Solve the following generalized WSVIE by ToRG

$$v(\xi)=\xi^3-\frac{243}{440}\xi^{\frac{11}{3}}+\int_0^{\xi}\frac{1}{(\xi-\tau)^{\frac{1}{3}}}v(\tau)d\tau, \tag{13}$$

Taking the ToRG on both sides of Eq. (13) and using the properties of ToRG, as follows:

$$\mathbb{RG}\{v(\xi)\}(s,v)=\mathbb{RG}\left\{\xi^3-\frac{243}{440}\xi^{\frac{11}{3}}\right\}(s,v)$$
$$+\mathbb{RG}\left\{\int_0^{\xi}\frac{1}{(\xi-\tau)^{\frac{1}{3}}}v(\tau)d\tau\right\}(s,v),$$

$$U(s,v)=\mathbb{RG}\{\xi^3\}-\frac{243}{440}\mathbb{RG}\left\{\xi^{\frac{11}{3}}\right\}+\mathbb{RG}\left\{\xi^{-\frac{1}{3}}*v(\xi)\right\},$$

$$U(s,v)=\frac{3!\,v^3}{s^4}-\frac{243}{440}\frac{\Gamma\left(\frac{14}{3}\right)v^{\frac{11}{3}}}{s^{\frac{14}{3}}}+v\mathbb{RG}\left\{\xi^{-\frac{1}{3}}\right\}\cdot\mathbb{RG}\{v(\xi)\},$$

$$U(s,v)=\frac{6v^3}{s^4}-\frac{243}{440}\frac{\frac{11}{3}\Gamma\left(\frac{11}{3}\right)v^{\frac{11}{3}}}{s^{\frac{14}{3}}}+v\frac{\Gamma\left(\frac{2}{3}\right)v^{-\frac{1}{3}}}{s^{\frac{2}{3}}}U(s,v),$$

$$U(s,v)=\frac{6v^3}{s^4}-6\frac{\Gamma\left(\frac{2}{3}\right)v^{\frac{11}{3}}}{s^{\frac{14}{3}}}+\frac{\Gamma\left(\frac{2}{3}\right)v^{\frac{2}{3}}}{s^{\frac{2}{3}}}U(s,v). \tag{14}$$

Operating and simplifying of Eq. (14) gives

$$U(s,v)=\frac{6v^3s^{\frac{2}{3}}-6\Gamma\left(\frac{2}{3}\right)v^{\frac{11}{3}}}{s^4\left[s^{\frac{2}{3}}-\Gamma\left(\frac{2}{3}\right)v^{\frac{2}{3}}\right]},$$

$$U(s,v)=\frac{6v^3}{s^4}, \tag{15}$$

and taking the inverse ToRG on both sides of Eq. (15), we get

$$\mathbb{RG}G^{-1}\{U(s,\nu)\}(\xi) = \mathbb{R}G^{-1}\left\{\frac{6\nu^3}{s^4}\right\}(\xi),$$

$$\nu(\xi) = \xi^3.$$

It is the exact solution of the given Eq. (13).

V. Conclusions

This paper's primary outcome is an examination of the ToRG's basic characteristics. Moreover, generalized WSVIEs can be solved using ToRG. The ToRG has a deeper connection to integral transforms and is a useful tool for solving generalized WSVIEs.. The integral transform is useful for resolving a variety of scientific and technological issues, according to the results. In this manuscript, ToRG and integral transforms are closely related to one another. In this regard, generalized WSVIEs have been effectively solved using the ToRG. These duality relations will be a very useful tool for many future applications.

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