Diametral Trees Decomposition of Helm Graph

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Abstract- *Because it helps solve numerous issues and offers insights into the graph's structure, studying the diametral path is crucial for network research and design. Graph decomposition, which splits complex graphs into smaller components, is a crucial area of research in graph theory. It helps facilitate the study and understanding of the graph's features. A set of discrete subgraphs covering the set of edges in a graph G is called a partition. Decomposition using trees to link every vertex or diametral paths that can exist between any two vertices in a graph is still widely studied. As a result, many computationally challenging graph problems become straightforward if the graph is a tree. In this paper, we introduce novel concepts in graph decomposition like diametral tree decomposition Dtd, diametral tree decomposition number Dt_n and diametral tree decomposition index Dt_i of the helm graph.*

Keywords— Graph decomposition, Diametral path, Diametral tree decomposition, Helm graph.

I. INTRODUCTION

In the last few decades, graph theory has experienced significant development. In the majority of studies, a graph usually refers to a single mathematical model, but a network frequently refers to a real system. A graph serves as a mathematical representation of the network. Graph theoretic techniques offer a robust foundation for understanding and resolving problems in network systems [1]. A collection of vertices represents the items in a graph, and a set of edges created between pairs of vertices provides the interactions between the objects. Any problem we encounter may be seen, analyzed, and generalized with the use of graphs [2]. In a graph, the diametral path depicts the path between these vertices, and the diameter gives crucial information about the distance between vertices. Dividing a complicated object into smaller components with a certain structure is a very typical process in many mathematical fields. In reality, different graph theory challenges may be expressed as decomposition issues [3]. A decomposition of a graph G is a set of edge-disjoint subgraphs covering its edge set. As a result, many computationally difficult graph problems become straightforward if the graph is a tree. It seems helpful to use a structural feature of a graph that is a tree in order to solve a computer issue more efficiently [2], [3]. This motivated academics to research this subject further. Since at least 1960, research has been conducted on graph division and decomposition issues [4]. In 1968, Gallai proposed that every linked simple graph on n vertices can have a decomposition with at most $\left\lfloor\frac{n}{2}\right\rfloor$ elements and only paths [5]. Abueida and Daven examined variants on the covering, factorization problems, and subgraph packing to find an effective way to break down a large complete graph into well-behaved subgraphs [6]. Yamamoto et al. examined the difficulty of decomposing a whole graph into a union of stars or line-disjoint claws [7]. Kumar looked into the P_4 −decomposition of various graph classes and specifically demonstrated that a complete r-partite graph is P_4 −decomposed [8]. Chung proved that any graph with n vertices can be decomposition into trees with a maximum size of $\left\lfloor\frac{n}{2}\right\rfloor$. Mangam et al. investigated the quantity of diametral paths, identifying the pathways, cycles, and stars [9]. The diametral path decomposition index and diametral path decomposition position number in various graph types were investigated by Tabitha et al. [2]. Jasim and Hashim used the diametral path to study the decomposition in Qn $(n \geq 2)$ [10]. Miriam investigated the use of tree-decomposition to describe graphs in a tree-like form [2]. The concepts of fuzzy route decomposition and fuzzy tree decomposition were first presented by Takaaki Fujita, and they analyze the traits and implications of these extended concepts [11]. Sethuraman and Murugan's innovative hypothesis decomposed a whole graph into copies of two arbitrary trees. Antika et al. decomposition an undirected, unweighted complete graph (K_n) into a minimum number of edge disjoint trees [4]. Fábio Botler studied that every graph with n vertices permits a decomposition into trees with a maximum degree of at most $\left\lfloor \frac{n}{2} \right\rfloor$ and a maximum size of at least $\left\lceil \frac{n}{2} \right\rceil$ [8].

In this paper we have addressed a new concept in decomposition. Which is called the diametral tree decomposition also discuss the diametral tree decomposition number and the diametral tree decomposition index of the helm graph.

II. PRELIMINARIES

Let G be an undirected graph with vertex set $V(G)$ and edge set $E(G)$. If there is an edge $ab \in E(G)$, we say that a vertex $a \in V(G)$ *is* adjacent to another vertex $b \in V(G)$. The degree of a vertex v, represented as $d(v)$, is the number of vertices that are adjacent to it. A connected graph T is called a tree if it is connected and has no cycles. A vertex in a tree is called a leaf if its degree is 1 [12] , [13].

A spanning tree of connected graph G is a subgraph of contains all the vertices and some edge, such that $V(T) = V(G)$ and $E(T) \leq E(G)$. The two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ are isomorphic (written $G \cong H$) If a bijection is present $f: V_G \longrightarrow V_H$ such that for any tow vertices $u, v \in V_G$, the edge (u, v) is in E_G if and only if the edge $\big(f(u), f(v)\big)$ is in E_H, the bijection f is called an isomorphism. The distance between any two vertices in a graph is shown by the length of the shortest path connecting them and denoted by $Dis(a, b)$ where $a, b \in V(G)$, A graph's diameter, represented by $Dia(G)$, is the maximum distance between any two of its vertices. $Dia(G) = \max\{Dis(a, b), a, b \in V(G)\}$ [13], [14],[15].

Let W_n represent the wheel on $n \geq 4$ vertices that was created by attaching a universal vertex v to the cycle C_{n-1}. Keep in mind that n even corresponds to an odd rim C_{n-1}, whereas n odd corresponds to an even rim C_{n-1}. The size of a wheel graph is given by $|E(W_n)| = 2n - 2$, and the order of a wheel graph is given by $|V(W_n)| = n$ [16].

III. DECOMPOSITION OF GRAPHS

The decomposition of graphs is one of the most well-known graph theory problems. It involves splitting an input graph into subgraphs that satisfy specific criteria. A decomposition of a graph G, $\{G_1, G_2, \ldots, G_k\}$, is a set of edge-disjoint subgraphs of G, such that $E(G) = \bigcup_{i=1}^{k} E(G_i)$. It involves breaking up an input graph into subgraphs that satisfy specific properties. These problems fall into two general categories: The first, referred to as simple decompositions, divides the input graph into edge-disjoint subgraphs of the same sort. The input graph decomposes into at least two distinct types of edge-disjoint subgraphs using the second approach, it's called "multiple decomposition" [17].

IV. DIAMETRAL PATH DECOMPOSITION

Diametral path decomposition, represented as by Dp, is a set of edge disjoint diametral path of a graph in which each edge of the graph appears in precisely one diametral path. The cardinality of such a collection is represented by the diametral path decomposition number, or dc_e. The number of these decompositions is known as the diameter path decomposition index, or Dc_e . We recall the following results for dc_e and Dc_e for specific types of graphs, which were displayed in previous research [17].

- $dc_e(K_n) = n$, and $Dc_e(K_n) = 1$, where $n \geq 2$.
- $dc_e(P_n) = 1$, and $Dc_e(P_n) = 1$, where $n \geq 2$.
- $dc_e(C_n) = 2$, and $Dc_e(C_n) = \frac{n}{2}$,where $n \geq 2$.
- $dc_e(Q_n) = n$, and $Dc_e(Q_n) = \frac{2^n}{2}$, where $n \geq 2$.
- $dc_e(W_n) = n - 1$, where $n \geq 5$.
- $dc_e\big(K_{m,n}\big) = \frac{mn}{2}$, where $m, n \geq 2$ and m or n is even.
- $dc_e(S_n) = \frac{n}{2}$, and $Dc_e(S_n) = (n-1)(n-3)(n-5) \ldots 1$, where $n \geq 2$ and n is even.

V. DIAMETRAL TREE DECOMPOSITION

In this section, we introduce a new decomposition concept. The diametral tree decomposition is the name given to this idea. The following is an explanation of the concept:

Definition 1.

Let $G = (V, E)$ be a connected graph. A triple (Tv, Te, D) is called a diametral tree decomposition of G, denoted by Dtd, if the following condition holds:

- $Tv = \{v_i : 1 \leq i \leq 2n,\ n \geq 4\}$ and $\bigcup v_i = V$.
- $Te = \{e \colon |e| = 2n - 1, n \geq 4\}$ and $e \in E$.
- Every decomposition tree, Dtd, is connected and has a diameter denoted by D, that is measured from the root to the furthest point of the tree and equals to the diameter of the original graph. That is,

$$Dtd\ (G) = Dia(G) = D$$

Definition 2. The cardinality of such a set of decompositions is known as the diametral tree decomposition number, denoted by Dt_n.

Definition 3. The diametral tree decomposition index is the number of Dt_n and denoted by Dt_i.

VI. MAIN RESULTS

The helm graph $H_n (n \geq 4)$ is determined by the wheel graph W_n a pendant edge to each rim vertex. The size of a helm graph is $|E(H_n)| = 3n$, and the order of a helm graph is $|V(H_n)| = 2n + 1$. Vertices in a graph are divided into three types: central vertex c, middle vertex mv, $mv = \{1, 2, 3, \dots, n\}$, and peripheral vertex pv, $pv = \{n+1, n+2, \dots, 2n\}$. Fig. 1 shows the helm graph's graphical representation H_n [16],[18].

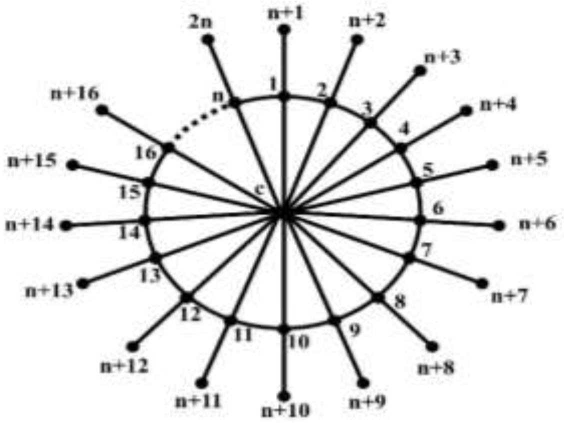

Fig. 1. Helm graph H_n with $n \geq 4$.

To talk about the Helm graph's decomposition into diametral trees and how to determine $Dt_n(H_n)$ and $Dt_i(H_n)$. We introduce Theorem 1.

Theorem 1. If $n \geq 7$, then the diametral tree decomposition number of the helm graph is given by:

$$Dt_n(H_n) = \begin{cases} 1 & if\ root\ \text{c} \\ \left\lfloor \frac{n-1}{3} \right\rfloor + \left\lfloor \frac{n-4}{3} \right\rfloor + \left\lfloor \frac{n-7}{3} \right\rfloor + 3, & if\ root\ \text{mv} \\ 3 & if\ root\ \text{pv} \end{cases}$$

And the diametral tree decomposition index of the helm graph is given by:

$$Dt_i(H_n) = \begin{cases} 1 & if\ root\ \text{c} \\ n(\left\lfloor \frac{n-1}{3} \right\rfloor + \left\lfloor \frac{n-4}{3} \right\rfloor + \left\lfloor \frac{n-7}{3} \right\rfloor + 3) & if\ root\ \text{mv} \\ 3n & if\ root\ \text{pv} \end{cases}$$

Proof.

Assume that $H_n, n \geq 7$ is a helm graph with $2n$ vertices, $3n$ edges, and $Dia(H_n) = 4$. Next, we are trying to figure out how many maximum diametral tree decompositions there can be in H_n. Taking into account the helm graph in Fig. 1, we divide the proof into cases.

Case 1: The tree root that emerges from the diametral tree decomposition is located at central vertex c in H_n , we get the maximum number of diametral tree decompositions. Consequently, depending on the value of n, the diametral tree decompositions seen in Fig. 2 will be the result. Hence, we get only one decomposition from the root c, such that such that $Dt_n(H_n) = 1$ and $Dt_i(H_n) = 1$.

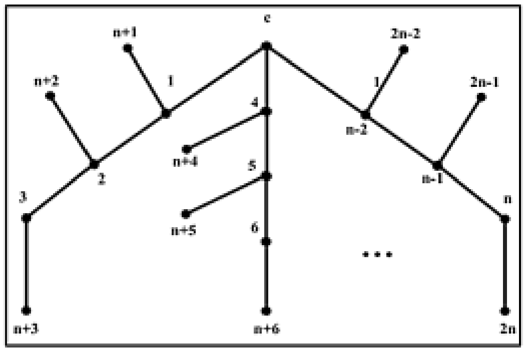

Fig. 2. The vertex c is the root

Case 2: The tree root that emerges from the diametral tree decomposition is located at a peripheral vertex $pv = \{n+1, n+2, \dots, 2n\}$ of H_n. Consequently, employing the isomorphism idea yields the diametral tree decompositions shown in Fig. 3a, 3b, and 3c. Hence, we get the number decomposition from the root $pv = \{n+1, n+2, \dots, 2n\}$ such that $Dt_n(H_n) = 3$ and $Dt_i(H_n) = 3n$.

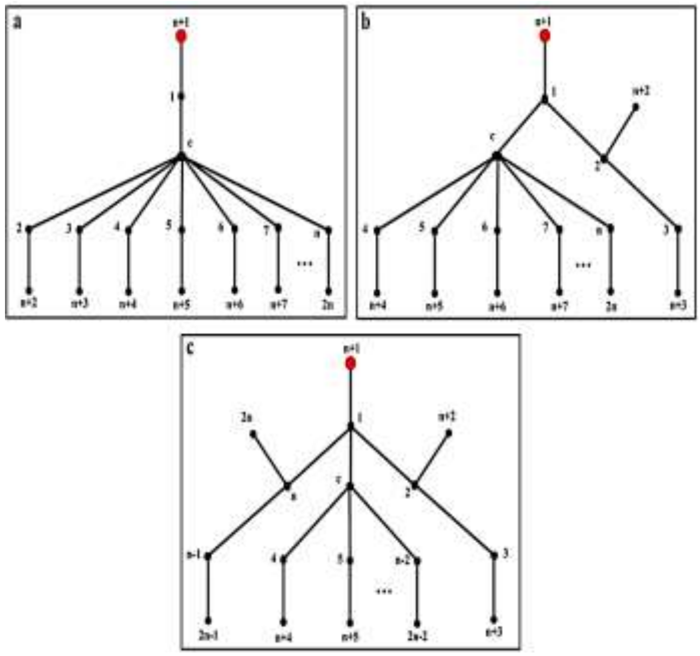

Fig. 3. The vertex pv is the root

Case 3: The tree root that emerges from the diametral tree decomposition is located at a middle vertex mv, $mv = \{1,2,3, \dots, n\}$ of H_n. The diametral tree decomposition output is made up of the total of several subtrees denoted by S_t . Each st starts at the root mv; on one side, it connects to the center vertex and passes via middle vertices and ends at the peripheral vertices, and we divide the proof of this case into the following:

1- The degree 2 tree root is adjacent to a central and peripheral vertex. Each S_t should contain two or three middle vertices. We will rely on the three-vertex indication in the proof because it provides the biggest benefit. We refer to the number of S_{t3} in S_t by $|S_{t3}|$, which may be found as follows:

a. If $|S_{t3}| = 0$, there are only two middle vertices in each st, as illustrated in Fig. 4a.
b. If $|St_3| = 1$, there is only one S_t with three middle vertices, while the other S_t has two middle vertices, as shown in Fig. 4b.
c. If $|S_{t3}| = 2$, there are only two S_t with three middle vertices, while the other S_t has two middle vertices, as shown in Fig. 4c.
d. Similarly, depending on the choice of n, this technique can be expanded to obtain all S_t with three middle vertices for the graph, so that $\left\lfloor\frac{n-1}{3}\right\rfloor$ gives the number of S_t as illustrated in Fig. 4d.

Thus, one of the diametral tree decompositions depicted in Fig. 4a, 4b, 4c, and 4d is obtained by applying the isomorphism concept with (a) to (d). Hence, we get the diametral tree decomposition number from the root mv such that $Dt_n(H_{n(\mathrm{mv})}) = \left\lfloor\frac{n-1}{3}\right\rfloor + 1$. Since root mv can generate many diametral tree decompositions that resemble Fig. 4a, 4b, 4c, and 6d the diametral tree decomposition index equals $n\left\lfloor\frac{n-1}{3}\right\rfloor + n$; hence $Dt_i(H_{n(mv)}) = n(\left\lfloor\frac{n-1}{3}\right\rfloor + 1)$.

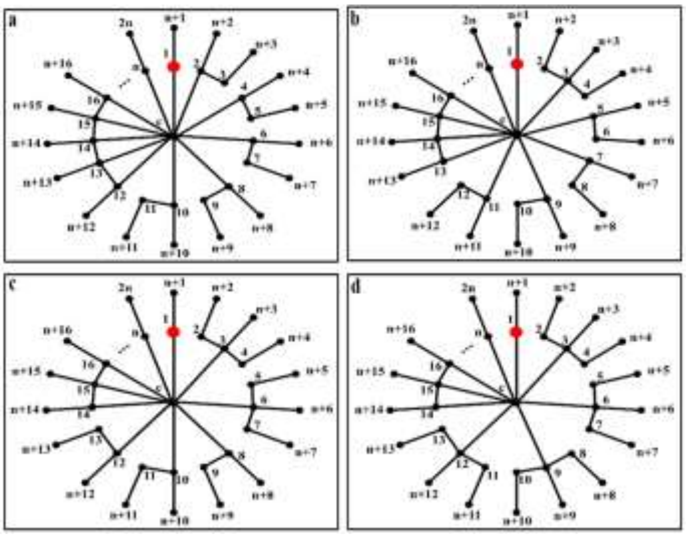

Fig. 4. The vertex $mv = 1$ is the root

2- The degree 3 tree root is adjacent to a peripheral vertex, a middle vertex, and a central vertex. Each S_t should contain two or three middle vertices. With the exception of the three middle vertices that are attached to the root. We will rely on the three-vertex indication in the proof because it provides the biggest benefit, which may be found as follows:

e. If $|S_{t3}| = 0$, there are only two middle vertices in each st, as illustrated in Fig. 5a.
f. If $|S_{t3}| = 1$, there is only one S_t with three middle vertices, and the remaining S_t contains two middle vertices, as shown in Fig. 5b.
g. If $|S_{t3}| = 2$, there are only two S_t's with three middle vertices, and the remaining S_t contains two middle vertices, as shown in Fig. 5c.
h. Similarly, depending on the choice of n, this technique can be expanded to obtain all S_t with three middle vertices for the graph, so that $\left\lfloor\frac{n-4}{3}\right\rfloor$ gives the number of S_t as illustrated in Fig. 5d.

Thus, one of the diametral tree decompositions depicted in Fig. 5a, 5b, 5c, and 5d is obtained by applying the isomorphism concept with (e) to (h). Hence, we get the diametral tree decomposition number from the root mv such that $Dt_n(H_{n(\mathrm{mv})}) = \left\lfloor\frac{n-4}{3}\right\rfloor + 1$. Since root mv can generate many diametral tree decompositions that resemble Fig. 5a, 5b, 5c, and 5d the diametral tree decomposition index equals $n\left\lfloor\frac{n-4}{3}\right\rfloor + n$; hence $Dt_i(H_{n(mv)}) = n(\left\lfloor\frac{n-4}{3}\right\rfloor + 1)$.

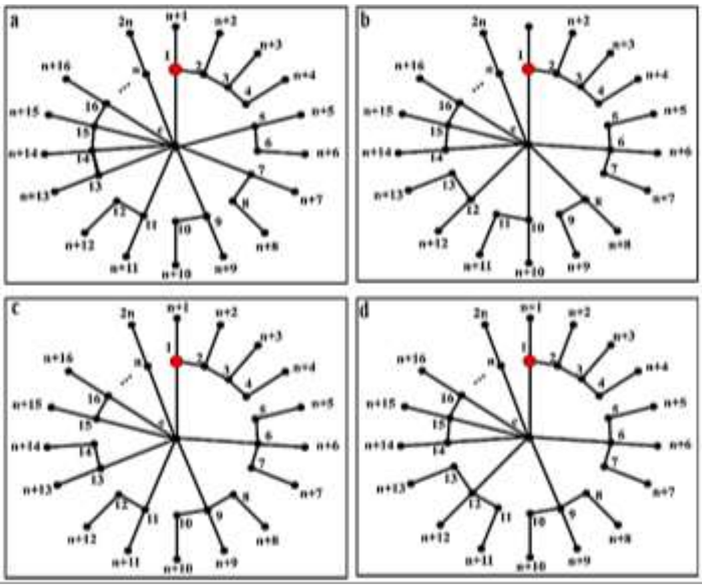

Fig. 5. The vertex $mv = 1$ is the root

3- The degree 4 tree root is adjacent to a peripheral vertex, a central vertex and two middle vertices. Each S_t should contain two or three middle vertices. With the exception of the six middle vertices that are attached to the root. We will rely on the three-vertex indication in the proof because it provides the biggest benefit, which may be found as follows:

i. If $|S_{t3}| = 0$, there are only two middle vertices in each S_t, as illustrated in Fig. 6a.

j. If $|S_{t3}| = 1$, there is only one S_t with three middle vertices, and the remaining S_t contains two middle vertices, as shown in Fig. 6b.

k. If $|S_{t3}| = 2$, there are only two S_t's with three middle vertices, and the remaining S_t contains two middle vertices, as shown in Fig. 6c.

l. Similarly, depending on the choice of n, this technique can be expanded to obtain all S_t with three middle vertices for the graph, so that $\left\lfloor\frac{n-7}{3}\right\rfloor$ gives the number of S_t as illustrated in Fig. 6d.

Thus, one of the diametral tree decompositions depicted in Fig. 6a, 6b, 6c, and 6d is obtained by applying the isomorphism concept with (i) to (l). Hence, we get the diametral tree decomposition number from the root mv such that $Dt_n(H_{n(\text{mv})}) = \left\lfloor\frac{n-7}{3}\right\rfloor + 1$. Since root mv can generate many diametral tree decompositions that resemble Fig. 6a, 6b, 6c, and 6d the diametral tree decomposition index equals $n\left\lfloor\frac{n-7}{3}\right\rfloor + n$; hence $Dt_i(H_{n(mv)}) = n\left\lfloor\frac{n-7}{3}\right\rfloor + n$.

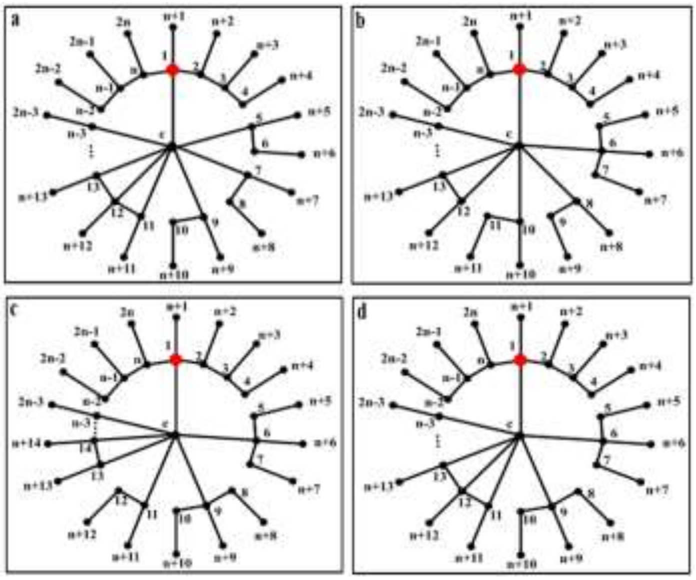

Fig. 6. The vertex $mv = 1$ is the root

The following is an expression for the $Dt_n(H_n)$ and DTE final outcomes from compiling the results of the above cases:

$$Dt_n(H_n) = \left\lfloor\frac{n-1}{3}\right\rfloor + \left\lfloor\frac{n-4}{3}\right\rfloor + \left\lfloor\frac{n-7}{3}\right\rfloor + 7 \quad \text{and}$$

$$Dt_i(H_n) = n\left(\left\lfloor\frac{n-1}{3}\right\rfloor + \left\lfloor\frac{n-4}{3}\right\rfloor + \left\lfloor\frac{n-7}{3}\right\rfloor + 6\right) + 1,$$ and that proves Theorem 1.

CONCLUSION

Graph decomposition is an important topic of graph theory research that breaks down complicated networks into smaller components. It is still very common to investigate decomposition using trees and diametral pathways that can exist between any two vertices in a graph. In this study, we present a new idea in graph decomposition, diametral tree decomposition (Dtd) of the helm graph. Such that every decomposition tree result is connected and has a diameter denoted by D, which is measured from the root to the furthest point of the tree and equals the diameter of the original graph. The following table presents the results of $Dt_n(H_n)$ and $Dt_i(H_n)$, with $7 \leq n \leq 18$, obtained using Theorem 1.

TABLE 1. THE RESULTS OF $Dt_n(H_n)$ AND $Dt_i(H_n)$, WITH $7 \leq n \leq 18$

H_n	$Dt_n(H_n)$	$Dt_i(H_n)$
H_7	10	64
H_8	10	73
H_9	10	82
H_{10}	13	121
H_{11}	13	133
H_{12}	13	145
H_{13}	16	196
H_{14}	16	211
H_{15}	16	226
H_{16}	19	289
H_{17}	19	307
H_{18}	19	325

REFERENCES

[1] A. N. Jasim and A. A. Najim, "Solving Edges Deletion Problem of Complete Graphs," Baghdad Science Journal, vol. 21, no. 12, pp. 4073-4082, 2024, doi: https://doi.org/10.21123/bsj.2024.10128.

[2] A. B. Adcock, B. D. Sullivan and Michael W., " Mahoney, "Tree decompositions and social graphs," Internet Math., vol. 12, no. 5, pp. 315-361, 2016, doi: https://doi.org/10.1080/15427951.2016.1182952.

[3] J. K. Fichte, M. Hecher, M. Morak and Stefan Woltran, " Dynamic Programming on Tree Decompositions in Action, International Symposium on Parameterized and Exact Computation (IPEC)" vol.- 89, no.-17, 2017, pp 17:1-17:13, doi: https://doi.org/10.4230/LIPIcs.IPEC.2017.17.

[4] A. Sinha, S. K.Saha and P. Basuchowdhuri, "An algorithm for the decomposition of complete graph into minimum number of edge disjoint trees," arXiv preprint arXiv:2405.18506, 2024, doi: https://doi.org/10.48550/arXiv.2405.18506.

[5] F. Botler, "Decomposition of graphs into trees with bounded maximum degree," Matemática Contemporânea, vol. 46, pp. 94-102, 2019, doi: http://doi.org/10.21711/231766362020/rmc469.

[6] P. Lakshmanan, I. Maniprakasam and M. Appu, "Multidecomposition of complete graphs into cycles and claws," Journal of Algebra Combinatorics Discrete Structures and Applications, vol.13, no.2, pp.131-141, 2026, doi: https://doi.org/10.13069/jacodesmath.v13i2.271.

[7] C. S. Kumar, "on P4-decomposition of graphs ," Taiwanese Journal Mathematical , vol.7, no. 4, pp. 657

-
664, 2003,doi: https://doi.org/10.11650/twjm/1500407584
.

[8] S. Yamamoto, H. Ikeda, S. Shige-eda, K. Ushio and N. Hamada, "On claw-decomposition of complete graphs and complete bigraphs, " Hiroshima Mathematical Journal, vol.5 ,no. 1 , pp.33-42, 1975, doi: https://doi.org/10.32917/hmj/1206136782.

[9] M. M. Belavadi and T. A. Mangam, "Platt number of total graphs," International Journal of Applied Mathematics, vol. 31, no.5 , pp. 593-602, 2018, doi: http://dx.doi.org/10.12732/ijam.v31i5.5 .

[10] A. N. Jasim and H. R. Hashim, "Decompositions of hypercube graphs into diametral paths and cycle decompositions," Journal of Mathematics and Statistics Studies, vol. 6, no. 1, pp. 13-17, 2025. doi: https://doi.org/10.32996/jmss.2025.6.1.2.

[11] T. Fujita, "Tree-decomposition on fuzzy graph," preprint (ResearchGate), 2024, doi: https://doi.org/10.19139/soic-2310-5070-3117.

[12] A.N. Jasim, "On the maximum diameter of graphs," Passer Journal of Basic and Applied Sciences, vol. 7, no. 2, pp. 729-732, 2025, doi: https://doi.org/10.24271/psr.2025.499398.1897.

[13] R. Diestel, "Graph Theory," Fifth Edition, Springer, Berlin, Heidelberg, Graduate Texts in Mathematics, vol. 173, pp. 1-428, 2017, doi: http://doi:10.1007/978-3-662-53622-3.

[14] A. N. Jasim and A. A. Najim, "Edges deletion problem of hypercube graphs for some n," Discrete Mathematics, Algorithms and Applications, vol. 17, no. 3, pp. 2450045, 2025, doi: https://doi.org/10.1142/S1793830924500459.

[15] B. Freyberg and R. Peters, "Decomposition of complete graphs into forests with six edges," Discuss. Mathematical. Graph Theory, vol. 45, no. 2, p. 677, 2025, doi: https://doi.org/10.7151/dmgt.2554.

[16] J. Allagan and K. Pereyra, "Hall numbers of complete multipartite graphs, 2-trees, and wheels, " *Utilitas Mathematica*, vol. 127, pp. 129-139, 2026, doi: https://doi.org/10.61091/um127-09.

[17] M. T. Agnes, L. S. Reddy and J. Varghese and J. S. Mangam, "Decomposition of graph into diametral paths," Bulletin of International Mathematical Virtual Institute, vol. 6, no. 2, pp. 209-216, 2016.

[18] K. Karunanayake, A. Perera, K. Dhananjaya and A. Perera, "Reconstruction of helm graph and web graph," Journal of Sci. of the Uni. of Kelaniya, vol. 13, no. 1, pp. 31-39 , 2022, doi: http://doi.org/10.4038/jsc.v13i1.41.