[0,1] truncated lomax – lomax distribution with properties

Jumana A. Altawil
Ministry of Education ,Directorate General of Education in Holy Karbala,
Al Burhan School for Girls, Karbala, Iraq.
E-mail: jumana_abbas_alwan@karbalaedu.org

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Abstract — *We shall introduce a new continuous distribution dependent on [0,1] truncated lomax ([0,1] TLD) distribution , we will call it [0,1] truncated lomax – lomax ([0,1]TLD) distribution. The some properties of the ([0,1]TLLD) distribution will be derive , and also, the ([0,1]TLLD) stress-strength model with different parameters will be obtain. The purpose of this work is to develop a new composite distribution that is more flexible than other distributions..*

KEY WORDS — [0,1] TLD, [0,1]TLLD , Stress Strength Model of the [0,1]TLLD.

I. INTRODUCTION

In recent times , Researchers attempted to suggest a new classes of statistical distributions. Here, we created a distribution we hope to benefit from it .This work will be on the generalizations stimulated by Eugene et al. [1] ,so that, they provided beta generalization distribution from (CDF) , G(x) by.

$$F(x) = \left[\frac{1}{\beta(a,b)}\right] \int_0^{G(x)} Z^{a-1}[1-Z]^{b-1}\, dZ, \quad 0 < a,b < \infty \qquad (1.1)$$

where $\beta(a\,,b) = \int_0^1 Z^{a-1}(1-Z)^{b-1}\, dZ$.
Jones [1][2] ,who used $X = G^{-1}(N)$ the arbitrary variable N for $N \sim Beta(a,b)$and products X with cumulative distribution function (1.1) . Eugene et al , who knowns a Beta normal (BN) distribution through placing G(x) to be the cumulative distribution function of the normal distribution and also, inferred a few properties of this distribution while, Gupta and Nadarajah obtained formulas for the moments of above distribution [3]. Cordeiro and de Castro (2011) proposed Kumaraswamy generated family by exchanged the (BN) distribution with the (Kw) distribution.

The (PDF) relating to (1.1) is given by

$$f(x) = \frac{1}{\beta(a,b)}\, G(x)^{a-1} \left(1-G(x)\right)^{b-1} g(x) \qquad (1.2)$$

Where $g(x) = \frac{dG(x)}{dx}$ is (pdf)of the principle distribution.

The probability density and cumulative distribution functions of ([0,1] TLD) are one by one

$$v(x) = \frac{\eta\, \vartheta\, (1+\eta x)^{-(\vartheta+1)}}{(1-(1+\eta)^{-\vartheta})} \qquad 0 < x < 1 \qquad (1.3)$$

$$V(x) = \frac{\left(1-(1+\eta x)^{-\vartheta}\right)}{(1-(1+\eta)^{-\vartheta})} \qquad (1.4)$$

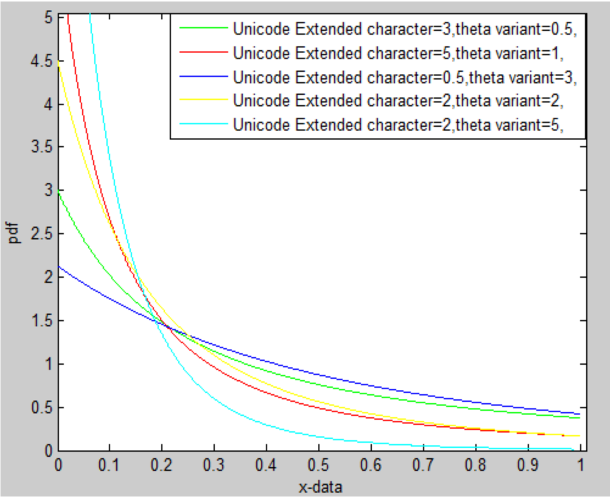

Figure 1: Plots PDF for ([0,1] TLD) distribution .

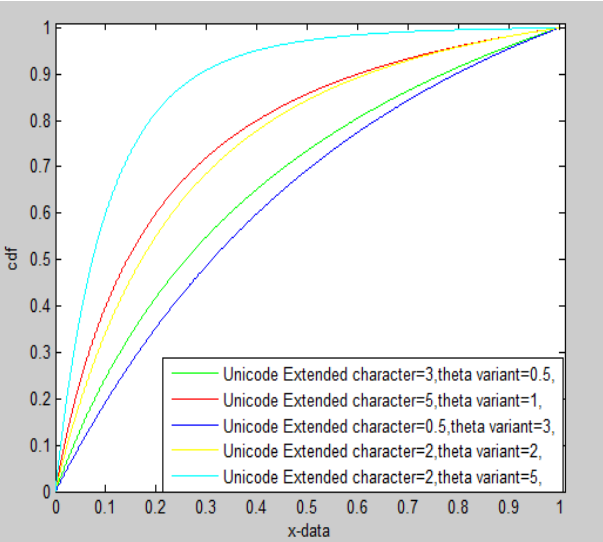

Figure 2: Plots CDF ([0,1] TLD) distribution

Presently , looking two totally cumulative distribution functions, W and G , our assume another distribution F by combination W with G , s.t $F(x) = W(G(x))$ be a cdf

$$\mathrm{F}(x) = \int_0^{G(x)} \frac{(\eta\vartheta(1+\eta t)^{-(\vartheta+1)})}{(1-(1+\eta)^{-\vartheta})}\, dt$$

$$F(x) = \frac{1-\{1+\eta G(x)\}^{-\vartheta}}{1-\{1+\eta\}^{-\vartheta}} \qquad (1.5)$$

and pdf :

$$fs(x) = \frac{\partial}{\partial x} F(x) = \frac{\partial}{\partial x}\left[\frac{1-\{1+\eta G(x)\}^{-\vartheta}}{1-\{1+\eta\}^{-\vartheta}}\right]$$

$$f(x) = \frac{\eta\vartheta\{1+\eta G(x)\}^{-(\vartheta+1)}}{\{1-\{1+\eta\}^{-\vartheta}\}} \,.\, g(x) \qquad (1.6)$$

Where $g(x) = \frac{\partial G(x)}{\partial x}$

We will consider Eq (1.5) and (1.6) as a generalized class of distributions, its name ([0,1] TLD – G) distribution. A suppose G be Lomax distribution.

II. THE [0,1] TLLD DISTRIBUTION

A suppose that $g(x) = \kappa\,\lambda(1+\kappa x)^{-(\lambda+1)}$ and $G(x) = 1-(1+\kappa x)^{-\lambda}$ referred as pdf and cdf for (r.v.) lomax ,are one by one .at that point ,through the application (1.5) and (1.6) ,get the cumulative distribution and probability distribution functions for (r.v.) of ([0,1] TLLD) .

$$.F(x) = \frac{1-(1+\eta\{1-(1+\kappa x)^{-\lambda}\})^{-\vartheta}}{(1-(1+\eta)^{-\vartheta})} \qquad (2.1)$$

$$f(x) = \frac{\eta\vartheta(\,1+\eta\{1-(1+\kappa x)^{-\lambda}\})^{-(\vartheta+1)}}{(1-(1+\eta)^{-\vartheta})}.\ \kappa\lambda(1+\kappa x)^{-(\lambda+1)}$$

$$f(x) = \frac{\vartheta\eta\kappa\lambda}{(1-(1+\eta)^{-\vartheta})}.(1+\kappa x)^{-(\lambda+1)}\,(1+\eta\{1-(1+\kappa x)^{-\lambda}\})^{-(\vartheta+1)} \qquad (2.2)$$

Furthermore, the reliability and the hazard rate functions are one by one

$$r(x) = 1-F(x)$$

$$= 1-\left[\frac{\left(1-(1+\eta\{1-(1+\kappa x)^{-\lambda}\})^{-\vartheta}\right)}{(1-(1+\eta)^{-\vartheta})}\right]$$

$$= \frac{\left(-\{1+\eta\}^{-\vartheta}+\left[1+\eta(1-\{1+\kappa x\}^{-\lambda})\right]^{-\vartheta}\right)}{(1-(1+\eta)^{-\vartheta})} \qquad (2.3)$$

$$\gamma(x) = \frac{f(x)}{r(x)}$$

$$= \frac{\left\{\vartheta\eta\kappa\lambda(1+\kappa x)^{-(\lambda+1)}(1+\eta[1-(1+\kappa x)^{-\lambda}])^{-(\vartheta+1)}\right\}}{\left\{-(1+\eta)^{-\vartheta}+(1+\eta[1-(1+\kappa x)^{-\lambda}])^{-\vartheta}\right\}} \qquad (2.4)$$

The r-th moment can be get by , [5] :

$$E(x^r) = \int_0^\infty x^r f(x)dx$$

$$E(x^r) = \int_0^\infty x^r \frac{\vartheta\eta\kappa\lambda}{(1-(1+\eta)^{-\vartheta})}.(1+\kappa x)^{-(\lambda+1)}(1+\eta\{1-(1+\kappa x)^{-\lambda}\})^{-(\vartheta+1)}dx$$

$$E(x^r) = \frac{\vartheta\eta\kappa\lambda}{(1-(1+\eta)^{-\vartheta})}\int_0^\infty x^r(1+\kappa x)^{-(\lambda+1)}(1+\eta\{1-(1+\kappa x)^{-\lambda}\})^{-(\vartheta+1)}\,dx$$

Now , by using the series expansion

$$(1-z)^{-n} = \sum_{\delta=0}^{\infty}\frac{\Gamma(n+\delta)}{\delta!\Gamma(n)}z^{\delta} \qquad |z|<1,\ \ n>0\ , [6]$$

we get:

$$[1+\eta(1-\{1+\kappa x\}^{-\lambda})]^{-(\vartheta+1)} = [1+\eta-\eta\{1+\kappa x\}^{-\lambda}]^{-(\vartheta+1)}$$

$$[1+\eta(1-\{1+\kappa x\}^{-\lambda})]^{-(\vartheta+1)} = \sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)}{\delta!\Gamma(\vartheta+1)}\{\eta[1+\kappa x]^{-\lambda}-\eta\}^{\delta}$$

Thence

$$E(x^r)=\frac{\vartheta\eta\kappa\lambda}{[1-\{1+\eta\}^{-\vartheta}]}\int_0^\infty x^r\{1+\kappa x\}^{-(\lambda+1)}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)}{\delta!\Gamma(\vartheta+1)}\left[\eta\{1+\kappa x\}^{-\lambda}-\eta\right]^{\delta}dx$$

$$E(x^r)=\frac{\vartheta\eta\kappa\lambda}{[1-\{1+\eta\}^{-\vartheta}]}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)}{\delta!\Gamma(\vartheta+1)}\int_0^\infty x^r\{1+\kappa x\}^{-(\lambda+1)}\left[\eta\{1+\kappa x\}^{-\lambda}-\eta\right]^{\delta}dx$$

$$=\frac{\vartheta\eta\kappa\lambda}{[1-\{1+\eta\}^{-\vartheta}]}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)}{\delta!\Gamma(\vartheta+1)}\int_0^\infty x^r\{1+\kappa x\}^{-(\lambda+1)}[-\eta]^{\delta}\left(1-(1+\kappa x)^{-\lambda}\right)^{\delta}dx$$

Now , by applying

$$(1-z)^b=\sum_{m=0}^{\infty}(-1)^m\frac{\Gamma(b+1)}{m!\Gamma(b-i+1)}z^m,$$

$$|z|<1,\ b>0\quad ,[4]\qquad (2.5)$$

we get:

$$\left[1-\{1+\kappa x\}^{-\lambda}\right]^{\delta}=\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(\delta+1)}{m!\,\Gamma(\delta-m+1)}\left[\{1+\kappa x\}^{-\lambda}\right]^m$$

$$\left[1-\{1+\kappa x\}^{-\lambda}\right]^{\delta}=\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(\delta+1)}{m!\,\Gamma(\delta-m+1)}\{1+\kappa x\}^{-\lambda m}$$

Hence

$$E(x^r)=\frac{\vartheta\eta\kappa\lambda}{(1-(1+\eta)^{-\vartheta})}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)}{\delta!\Gamma(\vartheta+1)}(-\eta)^{\delta}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\int_0^\infty x^r(1+\kappa x)^{-(\lambda+1)}(1+\kappa x)^{-\lambda m}dx$$

$$=\frac{\vartheta\eta\kappa\lambda}{(1-(1+\eta)^{-\vartheta})}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)}{\delta!\Gamma(\vartheta+1)}(-\eta)^{\delta}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\int_0^\infty x^r\{1+\kappa x\}^{-(m+1)\lambda-1}dx$$

Let $y=1+\kappa x\rightarrow y-1=\kappa x\rightarrow\frac{y-1}{\kappa}=x\rightarrow dx=\frac{1}{\kappa}dy,$
Thus:

$$E(x^r)=\frac{\vartheta\eta\kappa\lambda}{[1-\{1+\eta\}^{-\vartheta}]}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)}{\delta!\,\Gamma(\vartheta+1)}[-\eta]^{\delta}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(\delta+1)}{m!\,\Gamma(\delta-m+1)}\frac{(-1)^r}{\kappa^{r+1}}\int_1^\infty y^{-(m+1)\lambda-1}\{1-y\}^r\,dy$$

$$E(x^r)=\frac{\vartheta\eta\lambda(-1)^r}{\kappa^r[1-\{1+\eta\}^{-\vartheta}]}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)}{\delta!\Gamma(\vartheta+1)}[-\eta]^{\delta}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\left(\int_0^\infty y^{-(m+1)\lambda-1}\{1-y\}^r dy-\int_0^1 y^{-(m+1)\lambda-1}\{1-y\}^r dy\right)\quad(2.6)$$

Let $I_1=\int_0^\infty y^{-(m+1)\lambda-1}(1-y)^r dy$,

$$I_2=\int_0^1 y^{-(m+1)\lambda-1}(1-y)^r dy$$

Now , finding the integrations of I_1 and I_2 , respectively , and substituting the results in equation (2.6) , we get

$$I_1=\frac{1}{2}\int_{-\infty}^{\infty}y^{-(m+1)\lambda-1}(1-y)^r dy$$

Now ,simplifying $(1-y)^r$ by using Eq (2.5)

we get :

$$I_1=\frac{1}{2}\int_{-\infty}^{\infty}y^{-(m+1)\lambda-1}\sum_{j=0}^{\infty}(-1)^j\frac{\Gamma(r+1)}{j!\,\Gamma(r-j+1)}y^j dy$$

$$I_1=\frac{1}{2}\sum_{j=0}^{\infty}(-1)^j\frac{\Gamma(r+1)}{j!\Gamma(r-j+1)}\int_{-\infty}^{\infty}y^{-(m+1)\lambda-1+j}dy=0$$

$$I_2=\int_0^1 y^{-(m+1)\lambda-1}(1-y)^{(r+1)-1}dy$$

$$I_2=B(-(m+1)\lambda,r+1)$$

$$I_2=\frac{\Gamma(-(m+1)\lambda)\,\Gamma(r+1)}{\Gamma(-(m+1)\lambda+r+1)}$$

$$E(x^r)=\frac{\vartheta\eta\lambda(-1)^{r+1}}{\kappa^r(1-(1+\eta)^{-\vartheta})}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)}{\delta!\,\Gamma(\vartheta+1)}(-\eta)^{\delta}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-(m+1)\lambda)\,\Gamma(r+1)}{\Gamma(-(m+1)\lambda+r+1)}\qquad(2.7)$$

The Characteristic function is given by:

$$\Psi_x(t)=E(e^{ixt})$$

$$\Psi_x(t)=E\left[\sum_{r=0}^{\infty}\frac{(it)^r}{r!}x^r\right]=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}E(x^r)$$

$$\Psi_x(t)=\frac{\vartheta\eta\lambda}{(1-(1+\eta)^{-\vartheta})}\sum_{r=0}^{\infty}\left(\frac{-it}{\kappa}\right)^r\left(\frac{-1}{r!}\right)\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-(m+1)\lambda)\,\Gamma(r+1)}{\Gamma(-(m+1)\lambda+r+1)}$$

The mean (μ) , variance (σ^2) , also skewness (s_k) and the kurtosis (k_r) , are presented by

$$\mu = E(x)$$

$$E(x) = \frac{\vartheta\eta\lambda}{\kappa\left(1-(1+\eta)^{-\vartheta}\right)}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+2)} \tag{2.8}$$

$$\sigma^2 = E(x^2) - \left(E(x)\right)^2$$

$$\sigma^2 = \frac{-2\vartheta\eta\lambda}{\kappa^2\left(1-(1+\eta)^{-\vartheta}\right)}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+3)} - \frac{\vartheta^2\eta^2\lambda^2}{\kappa^2\left[\left(1-(1+\eta)^{-\vartheta}\right)\right]^2}\left[\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+2)}\right]^2$$

$$\sigma^2 = \frac{-\vartheta\eta\lambda}{\kappa^2\left(1-(1+\eta)^{-\vartheta}\right)}\left\{2\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+3)} + \frac{\vartheta\eta\lambda}{\left(1-(1+\eta)^{-\vartheta}\right)}\left[\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+2)}\right]^2\right\} \tag{2.9}$$

$$s_k = \frac{\left(E(x^3)\right)^2}{\sqrt{\left(E(x^2)\right)^3}}$$

Then :

$$s_k = \frac{E(x^3) - 3E(x)E(x^2) + 2\left(E(x)\right)^3}{(\sigma^2)^{\frac{3}{2}}}$$

$$= \frac{\left[\begin{array}{l}\frac{6\vartheta\eta\lambda}{\kappa^3\left(1-(1+\eta)^{-\vartheta}\right)}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+4)}\\ -3\left(\frac{\vartheta\eta\lambda}{\kappa\left(1-(1+\eta)^{-\vartheta}\right)}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+2)}\right)\\ \left(\frac{-2\vartheta\eta\lambda}{\kappa^2\left(1-(1+\eta)^{-\vartheta}\right)}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+3)}\right)+\\ 2\left(\frac{\vartheta\eta\lambda}{\kappa\left(1-(1+\eta)^{-\vartheta}\right)}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+2)}\right)^3\end{array}\right]}{\left[\frac{-\vartheta\eta\lambda}{\kappa^2\left(1-(1+\eta)^{-\vartheta}\right)}\left\{\begin{array}{c}2\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+3)}\\ +\frac{\vartheta\eta\lambda}{\left(1-(1+\eta)^{-\vartheta}\right)}\\ \left(\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+2)}\right)^2\end{array}\right\}\right]^{\frac{3}{2}}} \tag{2.10}$$

$$k_r = \frac{E(x^3)}{\left(E(x^2)\right)^2} - 3$$

Then :

$$k_r = \frac{E(x^4) - 4E(x)E(x^3) + 6\left(E(x)\right)^2E(x^2) - 3\left(E(x)\right)^4}{(\sigma^2)^2} - 3$$

$$= \frac{\left[\begin{array}{l}\frac{-24\vartheta\eta\lambda}{\kappa^4\left(1-(1+\eta)^{-\vartheta}\right)}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+5)}\\ -4\left(\frac{\vartheta\eta\lambda}{\kappa\left(1-(1+\eta)^{-\vartheta}\right)}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+2)}\right)\\ \left(\frac{6\vartheta\eta\lambda}{\kappa^3\left(1-(1+\eta)^{-\vartheta}\right)}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+4)}\right)+\\ 6\left(\frac{\vartheta\eta\lambda}{\kappa\left(1-(1+\eta)^{-\vartheta}\right)}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+2)}\right)^2\\ \left(\frac{-2\vartheta\eta\lambda}{\kappa^2\left(1-(1+\eta)^{-\vartheta}\right)}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+3)}\right)-\\ 3\left(\frac{\vartheta\eta\lambda}{\kappa\left(1-(1+\eta)^{-\vartheta}\right)}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+2)}\right)^4\end{array}\right]}{\left[\begin{array}{c}\frac{-\vartheta\eta\lambda}{\left(1-(1+\eta)^{-\vartheta}\right)}\\ \left\{\begin{array}{c}2\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+3)}+\\ \frac{\vartheta\eta\lambda}{\left(1-(1+\eta)^{-\vartheta}\right)}\left[\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+2)}\right]^2\end{array}\right\}\end{array}\right]^2} - 3$$

$$= \frac{\left[\frac{\vartheta^4\eta^4\lambda^4}{\kappa^4\left(1-(1+\eta)^{-\vartheta}\right)^4}\left(\begin{array}{c}\frac{-24\left(1-(1+\eta)^{-\vartheta}\right)^3}{\vartheta^3\eta^3\lambda^3}\\ \sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+5)}\\ -\frac{4\left(1-(1+\eta)^{-\vartheta}\right)^2}{\vartheta^2\eta^2\lambda^2}\\ \left\{\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+2)}\right\}\\ \left\{6\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+4)}\right\}\\ +\frac{6\left(1-(1+\eta)^{-\vartheta}\right)}{\vartheta\eta\lambda}\\ \left\{\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+2)}\right\}^2\\ \left\{-2\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+3)}\right\}\\ -3\\ \left\{\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+2)}\right\}^4\end{array}\right)\right]}{\left[\frac{\vartheta^4\eta^4\lambda^4}{\kappa^4\left(1-(1+\eta)^{-\vartheta}\right)^4}\left(\begin{array}{c}2\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+3)}\\ +\frac{\vartheta\eta\lambda}{\left(1-(1+\eta)^{-\vartheta}\right)}\\ \left\{\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+2)}\right\}^2\end{array}\right)\right]^2} - 3$$

$$= \frac{\left(\begin{array}{c}\frac{-24\left(1-(1+\eta)^{-\vartheta}\right)^3}{\vartheta^3\eta^3\lambda^3}\\ \sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+5)}\\ -\frac{4\left(1-(1+\eta)^{-\vartheta}\right)^2}{\vartheta^2\eta^2\lambda^2}\\ \left\{\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+2)}\right\}\\ \left\{6\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+4)}\right\}\\ +\frac{6\left(1-(1+\eta)^{-\vartheta}\right)}{\vartheta\eta\lambda}\\ \left\{\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+2)}\right\}^2\\ \left\{-2\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+3)}\right\}\\ -3\\ \left\{\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+2)}\right\}^4\end{array}\right)}{\left(\begin{array}{c}\frac{2\left(1-(1+\eta)^{-\vartheta}\right)}{\eta\vartheta\lambda}\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+3)}\\ -\left\{\sum_{\delta=0}^{\infty}\frac{\Gamma(\vartheta+1+\delta)(-\eta)^{\delta}}{\delta!\Gamma(\vartheta+1)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(\delta+1)}{m!\Gamma(\delta-m+1)}\frac{\Gamma(-m\lambda-\lambda)}{\Gamma(-m\lambda-\lambda+2)}\right\}^2\end{array}\right)^2} - 3 \tag{2.11}$$

The mode and the median can be calculated respectively,

$$\frac{df(x)}{dx} = \frac{-\eta\vartheta\kappa\lambda}{(1-(1+\eta)^{-\vartheta})}.(\vartheta+1)(1+\kappa x)^{-2(\lambda+1)}\eta\lambda\kappa\left(1+\eta\{1-(1+\kappa x)^{-\lambda}\}\right)^{-(\vartheta+2)} + \frac{\eta\vartheta\kappa\lambda}{(1-(1+\eta)^{-\vartheta})}\kappa(-\lambda-1)(1+\kappa x)^{-(\lambda+2)}\left(1+\eta\{1-(1+\kappa x)^{-\lambda}\}\right)^{-(\vartheta+1)} = 0$$

$$\frac{-\eta\vartheta\kappa\lambda}{(1-(1+\eta)^{-\vartheta})}.(\vartheta+1)(1+\kappa x)^{-2(\lambda+1)}\eta\lambda\kappa\left(1+\eta\{1-(1+\kappa x)^{-\lambda}\}\right)^{-(\vartheta+2)}$$

$$= \frac{\eta\vartheta\kappa\lambda}{(1-(1+\eta)^{-\vartheta})}\kappa(-\lambda-1)(1+\kappa x)^{-(\lambda+2)}\left(1+\eta\{1-(1+\kappa x)^{-\lambda}\}\right)^{-(\vartheta+1)}$$

$$(\vartheta+1)(1+\kappa x)^{-\lambda}.\eta\lambda\kappa\left(1+\eta\{1-(1+\kappa x)^{-\lambda}\}\right)^{-1} = -\kappa\lambda-\kappa$$

$$\frac{(1+\kappa x)^{-\lambda}}{(1+\eta\{1-(1+\kappa x)^{-\lambda}\})} = \frac{(-\lambda-1)}{(\eta\lambda\vartheta+\eta\lambda)}$$

$$\eta\lambda\vartheta(1+\kappa x)^{-\lambda} + \eta\lambda(1+\kappa x)^{-\lambda} = -\lambda\left(1+\eta\{1-(1+\kappa x)^{-\lambda}\}\right) - \left(1+\eta\{1-(1+\kappa x)^{-\lambda}\}\right)$$

$$\eta\lambda\vartheta(1+\kappa x)^{-\lambda} - \eta(1+\kappa x)^{-\lambda} = -\lambda-\lambda\eta-1-\eta$$

$$(1+\kappa x)^{-\lambda} = \frac{-\lambda-\lambda\eta-1-\eta}{(\eta\lambda\vartheta-\eta)} = \frac{-(\lambda+\lambda\eta+1+\eta)}{\eta(\lambda\vartheta-1)} = \frac{-\{\lambda(1+\eta)+(1+\eta)\}}{\eta(\lambda\vartheta-1)}$$

$$(1+\kappa x)^{-\lambda} = \frac{-[(1+\eta)(\lambda+1)]}{\eta(\lambda\vartheta-1)}$$

$$x_{Mode} = \frac{\left[\left(\left(\frac{-[(1+\eta)(\lambda+1)]}{\eta(\lambda\vartheta-1)}\right)\right)^{\frac{-1}{\lambda}} - 1\right]}{\kappa} \quad (2.12)$$

$$F(x) = \frac{1}{2}$$

$$\frac{\left(1-\left[1+\eta\{1-(1+\kappa x)^{-\lambda}\}\right]^{-\vartheta}\right)}{(1-(1+\eta)^{-\vartheta})} = \frac{1}{2}$$

,

Then

$$1 - \frac{\{1-(1+\eta)^{-\vartheta}\}}{2} = \left(1+\eta\{1-(1+\kappa x)^{-\lambda}\}\right)^{-\vartheta}$$

$$\left[\frac{1+(1+\eta)^{-\vartheta}}{2}\right]^{-\frac{1}{\vartheta}} - 1 = \eta\{1-(1+\kappa x)^{-\lambda}\}$$

$$\frac{\left\{\left[\frac{1+(1+\eta)^{-\vartheta}}{2}\right]^{-\frac{1}{\vartheta}} - 1\right\}}{\eta} = 1-(1+\kappa x)^{-\lambda}$$

$$1+\kappa x = \left(1 - \frac{\left\{\left[\frac{1+(1+\eta)^{-\vartheta}}{2}\right]^{-\frac{1}{\vartheta}} - 1\right\}}{\eta}\right)^{\frac{-1}{\lambda}}$$

$$x_{median} = \frac{\left[\left(1-\frac{\left\{\left(\frac{[1+(1+\eta)^{-\vartheta}]}{2}\right)^{\frac{-1}{\vartheta}} - 1\right\}}{\eta}\right)^{\frac{-1}{\lambda}} - 1\right]}{\kappa} \quad (2.13)$$

The quantile function x_q can be find it as follows ,

$$q = P(X \le x_q) = F_x(x_q) = \frac{\left[1-\left(1+\eta\{1-(1+\kappa x)^{-\lambda}\}\right)^{-\vartheta}\right]}{[1-(1+\eta)^{-\vartheta}]}$$

$$0 < q < 1 \quad , x_q > 0$$

$$1-\left(1+\eta\{1-(1+\kappa x)^{-\lambda}\}\right)^{-\vartheta} = F_x(x_q).[1-(1+\eta)^{-\vartheta}]$$

$$\left(1+\eta\{1-(1+\kappa x)^{-\lambda}\}\right)^{-\vartheta} = 1-F_x(x_q)[1-(1+\eta)^{-\vartheta}]$$

$$\left(1-F_x(x_q)[1-(1+\eta)^{-\vartheta}]\right)^{\frac{-1}{\vartheta}} - 1 = \eta\{1-(1+\kappa x)^{-\lambda}\}$$

$$x_q = F^{-1}(q) = \frac{\left[\left\{1-\left(\frac{\left[\left(1-F_x(x_q).\left(1-(1+\eta)^{-\vartheta}\right)\right)^{\frac{-1}{\vartheta}} - 1\right]}{\eta}\right)\right\}^{\frac{-1}{\lambda}} - 1\right]}{\kappa} \quad (2.14)$$

By the inverse transform technique, we can create (r.v.) for ([0,1] TLLD) as follows,

$$x_p = \frac{\left[\left\{1-\left(\frac{\left[\left(1-U.\left(1-(1+\eta)^{-\vartheta}\right)\right)^{\frac{-1}{\vartheta}} - 1\right]}{\eta}\right)\right\}^{\frac{-1}{\lambda}} - 1\right]}{\kappa}$$

$$, 0 \le U \le 1$$

III. STRESS STRENGTH MODEL OF THE [0,1]TLLD DISTRIBUTION.

The stress strength is obtained through the formula as follows

$$R = P(y < x) = \int_0^\infty f_x(x) F_Y(x)\, dx$$

Now , Let X~ [0,1] TLLD $(\eta, \vartheta, \kappa, \lambda)$ and Y~ [0,1]TLLD $(\eta_1, \vartheta_1, \kappa_1, \lambda_1)$ respectively ,

$$R = P(y < x) = \int_0^\infty f_x(x) F_Y(x)\, dx$$

$$R = \int_0^\infty \frac{\vartheta\eta\kappa\lambda}{(1-(1+\eta)^{-\vartheta})} (1+\kappa x)^{-(\lambda+1)} \left(1+\eta\{1-(1+\kappa x)^{-\lambda}\}\right)^{-(\vartheta+1)} \frac{\left[1-\left(1+\eta_1\{1-(1+\kappa_1 x)^{-\lambda_1}\}\right)^{-\vartheta_1}\right]}{(1-(1+\eta_1)^{-\vartheta_1})}\, dx$$

Simplifying :

$\left(1+\eta\{1-(1+\kappa x)^{-\lambda}\}\right)^{-(\vartheta+1)}$ by using the series expansion ,

$(1-z)^{-k} = \sum_{s=0}^{\infty} \frac{\Gamma(k+s)}{s!\Gamma(k)} z^s \quad |z| < 1 , k > 0$, we obtain

$$\left(1+\eta\{1-(1+\kappa x)^{-\lambda}\}\right)^{-(\vartheta+1)} = \left(1+\eta-\eta(1+\kappa x)^{-\lambda}\right)^{-(\vartheta+1)}$$

$$\left(1-\{\eta(1+\kappa x)^{-\lambda}-\eta\}\right)^{-(\vartheta+1)} = \sum_{s=0}^{\infty} \frac{\Gamma(\vartheta+1+s)}{s!\Gamma(\vartheta+1)} \left(\eta(1+\kappa x)^{-\lambda}-\eta\right)^s$$

$$\left(1-\{\eta(1+\kappa x)^{-\lambda}-\eta\}\right)^{-(\vartheta+1)} = \sum_{s=0}^{\infty} \frac{\Gamma(\vartheta+1+s)}{s!\Gamma(\vartheta+1)} (-\eta)^s \left(1-(1+\kappa x)^{-\lambda}\right)^s$$

Then:

$$SR = \frac{\vartheta\eta\kappa\lambda}{(1-(1+\eta)^{-\vartheta})} \frac{1}{(1-(1+\eta_1)^{-\vartheta_1})} \int_0^\infty (1+\kappa x)^{-(\lambda+1)} \sum_{s=0}^{\infty} \frac{\Gamma(\vartheta+1+s)}{s!\Gamma(\vartheta+1)} (-\eta)^s \left(1-(1+\kappa x)^{-\lambda}\right)^s \left\{1-\left(1+\eta_1\{1-(1+\kappa_1 x)^{-\lambda_1}\}\right)^{-\vartheta_1}\right\} dx$$

$$R = \frac{\vartheta\eta\kappa\lambda}{[1-\{1+\eta\}^{-\vartheta}]} \frac{1}{[1-\{1+\eta_1\}^{-\vartheta_1}]} \sum_{s=0}^{\infty} \frac{\Gamma(\vartheta+1+s)}{s!\Gamma(\vartheta+1)} [-\eta]^s \int_0^\infty \{1+\kappa x\}^{-\lambda} \left[1-\{1+\kappa x\}^{-\lambda}\right]^s \left(1-\left\{1+\eta_1\left[1-\{1+\kappa_1 x\}^{-\lambda_1}\right]\right\}^{-\vartheta_1}\right) dx$$

Also, simplifying :

$\left(1-(1+\kappa x)^{-\lambda}\right)^s$ by using

$(1-z)^b = \sum_{u=0}^{\infty} (-1)^u \frac{\Gamma(b+1)}{u!\Gamma(b-u+1)} z^u$,

$|z| < 1 , b > 0$, we get

$$[1-\{1+\kappa x\}^{-\lambda}]^s = \sum_{u=0}^{\infty} \frac{(-1)^u \Gamma(s+1)}{u!\Gamma(s-u+1)} \left((1+\kappa x)\right)^{-\lambda u}$$

Therefore,

$$= \frac{\vartheta\eta\kappa\lambda}{(1-(1+\eta)^{-\vartheta})} \frac{1}{(1-(1+\eta_1)^{-\vartheta_1})} \sum_{s=0}^{\infty} \frac{\Gamma(\vartheta+1+s)}{s!\Gamma(\vartheta+1)} (-\eta)^s \sum_{u=0}^{\infty} (-1)^u \frac{\Gamma(s+1)}{u!\Gamma(s-u+1)}$$

$$\int_0^\infty (1+\kappa x)^{-(u+1)\lambda} \left[1-\left(1+\eta_1\{1-(1+\kappa_1 x)^{-\lambda_1}\}\right)^{-\vartheta_1}\right] dx$$

Using the same simplification approach:

$\left(1+\eta_1\{1-(1+\kappa_1 x)^{-\lambda_1}\}\right)^{-\vartheta_1}$ and using

$(1-z)^{-s} = \sum_{i=0}^{\infty} \frac{\Gamma(s+i)}{i!\Gamma(s)} z^i \quad |z| < 1 , s > 0$, gives

$$\left(1+\eta_1\{1-(1+\kappa_1 x)^{-\lambda_1}\}\right)^{-\vartheta_1} = \sum_{i=0}^{\infty} \frac{\Gamma(\vartheta_1+i)}{i!\,\Gamma(\vartheta_1)} \{\eta_1(1+\kappa_1 x)^{\lambda_1}-\eta_1\}^i$$

Then:

$$R = \frac{\vartheta\eta\kappa\lambda}{(1-(1+\eta)^{-\vartheta})} \frac{1}{(1-(1+\eta_1)^{-\vartheta_1})} \sum_{s=0}^{\infty} \frac{\Gamma(\vartheta+1+s)}{s!\Gamma(\vartheta+1)} (-\eta)^s \sum_{u=0}^{\infty} (-1)^u \frac{\Gamma(s+1)}{u!\Gamma(s-u+1)}$$

$$\int_0^\infty (1+\kappa x)^{-(u+1)\lambda} \left(1-\sum_{i=0}^{\infty} \frac{\Gamma(\vartheta_1+i)}{i!\,\Gamma(\vartheta_1)} \{\eta_1(1+\kappa_1 x)^{\lambda_1}-\eta_1\}^i\right) dx$$

$$= \frac{\vartheta\eta\kappa\lambda}{(1-(1+\eta)^{-\vartheta})} \frac{1}{(1-(1+\eta_1)^{-\vartheta_1})} \sum_{s=0}^{\infty} \frac{\Gamma(\vartheta+1+s)}{s!\Gamma(\vartheta+1)} (-\eta)^s \sum_{u=0}^{\infty} (-1)^u \frac{\Gamma(s+1)}{u!\Gamma(s-u+1)} \int_0^\infty (1+\kappa x)^{-(u+1)\lambda}\, dx \; - \frac{\vartheta\eta\kappa\lambda}{(1-(1+\eta)^{-\vartheta})} \frac{1}{(1-(1+\eta_1)^{-\vartheta_1})} \sum_{s=0}^{\infty} \frac{\Gamma(\vartheta+1+s)}{s!\Gamma(\vartheta+1)} (-\eta)^s \sum_{u=0}^{\infty} (-1)^u \frac{\Gamma(s+1)}{u!\Gamma(s-u+1)}$$

$$\int_0^\infty (1+\kappa x)^{-(u+1)\lambda} \left(\sum_{i=0}^{\infty} \frac{\Gamma(\vartheta_1+i)}{i!\Gamma(\vartheta_1)} \{\eta_1(1+\kappa_1 x)^{\lambda_1}-\eta_1\}^i\right) dx$$

$$= \frac{\vartheta\eta\kappa\lambda}{(1-(1+\eta)^{-\vartheta})} \frac{1}{(1-(1+\eta_1)^{-\vartheta_1})} \sum_{s=0}^{\infty} \frac{\Gamma(\vartheta+1+s)}{s!\Gamma(\vartheta+1)} (-\eta)^s \sum_{u=0}^{\infty} (-1)^u \frac{\Gamma(s+1)}{u!\Gamma(s-u+1)} \int_0^\infty (1+\kappa x)^{-(u+1)\lambda}\, dx \; - \frac{\vartheta\eta\kappa\lambda}{(1-(1+\eta)^{-\vartheta})} \frac{1}{(1-(1+\eta_1)^{-\vartheta_1})} \sum_{s=0}^{\infty} \frac{\Gamma(\vartheta+1+s)}{s!\Gamma(\vartheta+1)} (-\eta)^s \sum_{u=0}^{\infty} (-1)^u \frac{\Gamma(s+1)}{u!\Gamma(s-u+1)}$$

$$\sum_{i=0}^{\infty} \frac{\Gamma(\vartheta_1+i)}{i!\,\Gamma(\vartheta_1)} \int_0^\infty (1+\kappa x)^{-(u+1)\lambda} (-\eta_1)^i \{1-(1+\kappa_1 x)^{-\vartheta_1}\}^i\, dx$$

Simplifying $\{1-(1+\kappa_1 x)^{-\vartheta_1}\}^i$ by using $(1-z)^m = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(m+1)}{n!\Gamma(m-n+1)} z^m$,

$|z| < 1 , b > 0$, gives

$$\{1-(1+\kappa_1 x)^{-\vartheta_1}\}^i = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(i+1)}{i!\Gamma(i-n+1)} (1+\kappa_1 x)^{-\lambda_1 i}$$

$$= \frac{\vartheta\eta\kappa\lambda}{(1-(1+\eta)^{-\vartheta})} \frac{1}{(1-(1+\eta_1)^{-\vartheta_1})} \sum_{s=0}^{\infty} \frac{\Gamma(\vartheta+1+s)}{s!\Gamma(\vartheta+1)} (-\eta)^s$$

$$\sum_{u=0}^{\infty} (-1)^u \frac{\Gamma(s+1)}{u!\Gamma(s-u+1)} \int_0^\infty (1+\kappa x)^{-(u+1)\lambda}\, dx$$

$$-\frac{\vartheta\eta\kappa\lambda}{(1-(1+\eta)^{-\vartheta})}\frac{1}{(1-(1+\eta_1)^{-\vartheta_1})}\sum_{s=0}^{\infty}\frac{\Gamma(\vartheta+1+s)}{s!\Gamma(\vartheta+1)}(-\eta)^s$$

$$\sum_{u=0}^{\infty}(-1)^u\frac{\Gamma(s+1)}{u!\Gamma(s-u+1)}\sum_{i=0}^{\infty}\frac{\Gamma(\vartheta_1+i)}{i!\Gamma(\vartheta_1)}$$

$$\int_0^{\infty}(1+\kappa x)^{-(u+1)\lambda}(-\eta_1)^i\ \sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(i+1)}{n!\Gamma(i-n+1)}(1+\kappa_1 x)^{-\lambda_1 i}dx$$

Since $(1+\kappa_1 x)^{-\lambda_1 i}=\left(1-(-\kappa_1 x)\right)^{-\lambda_1 i}$ by using

$$(1-z)^{-k}=\sum_{v=0}^{\infty}\frac{\Gamma(k+v)}{v!\Gamma(k)}z^v \qquad |z|<1, k>0,$$

gives,

$$\left(1-(-\kappa_1 x)\right)^{-\lambda_1 i}=\sum_{v=0}^{\infty}\frac{\Gamma(\lambda_1 i+v)}{v!\,\Gamma(\lambda_1 i)}(-\kappa_1 x)^v$$
$$=\sum_{v=0}^{\infty}\frac{\Gamma(\lambda_1 i+v)}{v!\,\Gamma(\lambda_1 i)}(-\kappa_1)^v x^v$$

Then:

$$R=\frac{\vartheta\eta\kappa\lambda}{(1-(1+\eta)^{-\vartheta})}\frac{1}{(1-(1+\eta_1)^{-\vartheta_1})}\sum_{s=0}^{\infty}\frac{\Gamma(\vartheta+1+s)}{s!\Gamma(\vartheta+1)}(-\eta)^s\sum_{u=0}^{\infty}(-1)^u\frac{\Gamma(s+1)}{u!\Gamma(s-u+1)}$$

$$\int_0^{\infty}(1+\kappa x)^{-(u+1)\lambda}dx-$$
$$\frac{\vartheta\eta\kappa\lambda}{(1-(1+\eta)^{-\vartheta})}\frac{1}{(1-(1+\eta_1)^{-\vartheta_1})}\sum_{s=0}^{\infty}\frac{\Gamma(\vartheta+1+s)}{s!\Gamma(\vartheta+1)}(-\eta)^s\sum_{u=0}^{\infty}(-1)^u\frac{\Gamma(s+1)}{u!\Gamma(s-u+1)}$$

$$\sum_{i=0}^{\infty}\frac{\Gamma(\vartheta_1+i)}{i!\Gamma(\vartheta_1)}(-\eta_1)^i\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(i+1)}{n!\Gamma(i-n+1)}\sum_{v=0}^{\infty}\frac{\Gamma(\lambda_1 i+v)}{v!\Gamma(\lambda_1 i)}(-\kappa_1)^v\int_0^{\infty}x^v(1+\kappa x)^{-(u+1)\lambda}dx \qquad (3.1)$$

Let $\quad I_1=\int_0^{\infty}(1+\kappa x)^{-(u+1)\lambda}dx$ and

$$I_2=\int_0^{\infty}x^v(1+\kappa x)^{-(u+1)\lambda}dx$$

Now , finding the integrations of I_1 and I_2 , respectively , then replacing the results in (3.1) , gives

Let $y=1+\kappa x\to y-1=\kappa x\to\frac{y-1}{\kappa}=x\to\frac{1}{\kappa}dy=dx$, therefore,

$$I_1=\int_1^{\infty}y^{-(u+1)\lambda}\frac{1}{\kappa}dy=\frac{1}{\kappa}\int_1^{\infty}y^{-(u+1)\lambda}dy=\frac{-1}{\kappa}$$

$$I_2=\int_1^{\infty}\left(\frac{y-1}{\kappa}\right)^v y^{-(u+1)\lambda}\frac{1}{\kappa}dy=\frac{1}{\kappa^{v+1}}\int_1^{\infty}y^{-(u+1)\lambda}(y-1)^v dy$$

$$I_2=\frac{1}{\kappa^{v+1}}\left\{\int_0^{\infty}y^{-(u+1)\lambda}(y-1)^v dy-\int_0^{1}y^{-(u+1)\lambda}(y-1)^v dy\right\}$$

(3.2)

Now , Let

$$A_1=\int_0^{\infty}y^{-(u+1)\lambda}(y-1)^v dy \text{ and}$$

$$A_2=\int_0^{1}y^{-(u+1)\lambda}(y-1)^v dy$$

Again, finding the integrations of A_1 and A_2 ,respectively , then replacing the results in (3.2) gives

$$A_1=\int_0^{\infty}y^{-(u+1)\lambda}(y-1)^v dy$$
$$=\frac{1}{2}\int_{-\infty}^{\infty}y^{-(u+1)\lambda}(y-1)^v dy$$

Using the same way , simplifying $(1-y)^v$ by using Eq.(2.5)

we obtain

$$A_1=\frac{1}{2}\int_{-\infty}^{\infty}y^{-(u+1)\lambda}\sum_{m=0}^{\infty}(-1)^m\frac{\Gamma(v+1)}{m!\,\Gamma(v-m+1)}y^m dy$$

$$A_1=\frac{1}{2}\sum_{m=0}^{\infty}(-1)^m\frac{\Gamma(v+1)}{m!\,\Gamma(v-m+1)}\int_{-\infty}^{\infty}y^{-(u+1)\lambda+m}dy$$
$$=0$$

$$A_2=\int_0^{1}y^{(-(u+1)\lambda+1)-1}(1-y)^{(v+1)-1}dy$$

$$A_2=B(-u\lambda-\lambda+1,v+1)=\frac{\Gamma(-u\lambda-\lambda+1)\,\Gamma(v+1)}{\Gamma(-u\lambda-\lambda+v+2)}$$

Therefore $I_2=\frac{-1}{\kappa^{v+1}}\frac{\Gamma(-u\lambda-\lambda+1)\,\Gamma(v+1)}{\Gamma(-u\lambda-\lambda+v+2)}$

Hence,

$$R=$$

$$\frac{-\vartheta\eta\lambda}{(1-(1+\eta)^{-\vartheta})}\frac{1}{(1-(1+\eta_1)^{-\vartheta_1})}\sum_{s=0}^{\infty}\frac{\Gamma(\vartheta+1+s)}{s!\,\Gamma(\vartheta+1)}(-\eta)^s\sum_{u=0}^{\infty}(-1)^u\frac{\Gamma(s+1)}{u!\,\Gamma(s-u+1)}$$
$$+\frac{\vartheta\eta\lambda}{(1-(1+\eta)^{-\vartheta})}\frac{1}{(1-(1+\eta_1)^{-\vartheta_1})}\sum_{s=0}^{\infty}\frac{\Gamma(\vartheta+1+s)}{s!\,\Gamma(\vartheta+1)}(-\eta)^s\sum_{u=0}^{\infty}(-1)^u\frac{\Gamma(s+1)}{u!\,\Gamma(s-u+1)}$$

$$\sum_{i=0}^{\infty}\frac{\Gamma(\vartheta_1+i)}{i!\,\Gamma(\vartheta_1)}(-\eta_1)^i\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(i+1)}{n!\,\Gamma(i-n+1)}\sum_{v=0}^{\infty}\frac{\Gamma(\lambda_1 i+v)}{v!\,\Gamma(\lambda_1 i)}$$

$$\frac{(-\kappa_1)^v}{\kappa^v}\frac{\Gamma(-u\lambda-\lambda+1)\,\Gamma(v+1)}{\Gamma(-u\lambda-\lambda+v+2)} \qquad (3.3)$$

IV. Conclusions

In reality , we suggested ([0,1] TLLD) distribution dependent on ([0,1] TLD) distribution and It's mathematical properties have been obtained and also , stress-strength reliability.

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